Lesson 9

Section 6.3 Unit Step, t-Shifting

So far, everything we've done in Laplace, we can easily accomplish using classical approaches, but in this section we are going to start discussing discontinuous functions, ones that don't start immediately at $t=0$, and may last only a short period of time.

Unit Step Function → think of it as an on or off switch

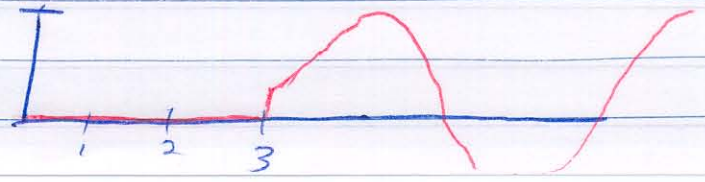
$$u(t-a) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t>a \end{cases}$$

How would I use this to turn on a forcing function, say $\cos \omega t$, at $t=3$?

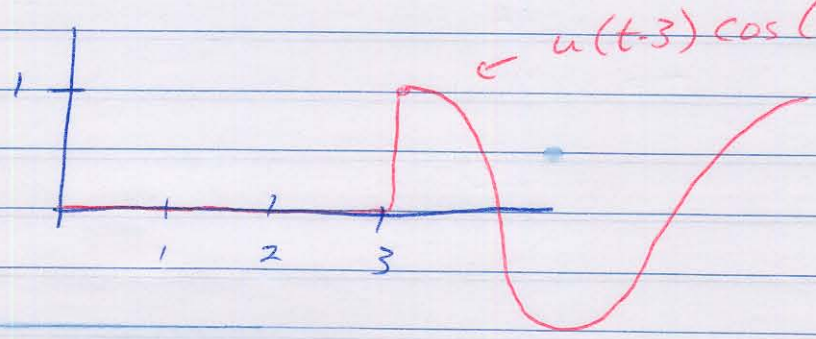
if I multiply $u(t-3)$ by $\cos \omega t$,

$u(t-3), \cos \omega t$

then I get:



So the function turned on at $t=3$, but notice it doesn't necessarily start at a value of 1 ($\cos(0)=1$), how do we fix?



Bonus: For the 10th, 11th, & 12th response, how would I create a ~~cos(t)~~ ~~cos(πt)~~ cos(2t) that turns on at $t=4\pi$ and off at $t=8\pi$, and at $t=4\pi$, the value should be 1 (equivalent to cos(0))

Lets Look @ $\mathcal{L}\{u(t-a)\}$

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt$$

since $u(t-a)=1$ for $t>a$

remember, the integral is linear!

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

0 ← since $u(t-a)=0$ for $t<a$

e^{-as}/s

$$\mathcal{L}\{u(t-a)\} = e^{-as}/s$$

Now lets look at a second shifting Theorem:

t-shifting

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

or

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

Lets try to use these:

Ex 1: $y'' + 8y' + 25y = 5 + u(t-2)(-5)$, $y(0)=0$, $y'(0)=0$

$y'' + 8y' + 25y$ defines our system; $5 + u(t-2)(-5)$ inputs; $y(0)=0$, $y'(0)=0$ initial conditions

(1) $\mathcal{L}\{y'' + 8y' + 25y = 5 + u(t-2)(-5)\}$

$= s^2Y(s) - sy(0) - y'(0) + 8sY(s) - y(0) + 25Y(s)$ (system)

$=$

$\mathcal{L}\{5\} + \mathcal{L}\{u(t-2)(-5)\}$ — $u(t-a)$, $f(t-a)$ — Note, since -5 is a const function it is not in $t-a$

$\frac{5}{s} + e^{-2s}F(s)$

$\frac{5}{s} + \frac{-5}{s}e^{-2s}$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$F(s) = \mathcal{L}\{f(t)\}$

So, we have:

$$s^2 Y(s) + 8s\,Y(s) + 25\,Y(s) = \frac{5}{s} - \frac{5}{s}e^{-2s}$$

② Solve For $Y(s)$

$$Y(s)\,(s^2 + 8s + 25) = \frac{5}{s} - \frac{5}{s}e^{-2s}$$

$$Y(s) = \underbrace{\left[\frac{5}{s} - \frac{5e^{-2s}}{s}\right]}_{R(s)}\underbrace{\left[\frac{1}{s^2 + 8s + 25}\right]}_{Q(s)}$$

$R(s)$ ↑ input $Q(s)$ ↑ transfer-function

③ Inverse Transform $Y(s)$

$$\mathcal{L}^{-1}\{Y(s)\} = \underbrace{\frac{5}{5(s^2+8s+25)}}_{? \text{ no third order polynomials in tables, need to use partial Fractions'}} - \frac{5e^{-2s}}{5(s^2+8s+25)}$$

$\downarrow$ $y(t)$

$$\frac{5}{5(s^2+8s+25)} = \frac{A}{s} + \frac{Bs+C}{(s^2+8s+25)}$$

multiply by s & let $s = 0$

$$\frac{5}{25} = A, \qquad A = \frac{1}{5}$$

mult by $s(s^2+8s+25)$

$$5 = A(s^2+8s+25) + (Bs+C)s$$

$A = 1/5$

$$5 = \tfrac{1}{5}s^2 + \tfrac{8}{5}s + 5 + Bs^2 + Cs$$

now, equate equivalent powers of s

$5 = 5$ ✓

$0s = \tfrac{8}{5}s + Cs$ $\quad C = -\tfrac{8}{5}$ ✓

$0s^2 = \tfrac{1}{5}s^2 + Bs^2$ $\quad B = -\tfrac{1}{5}$ ✓

? Do I need to do it again for

$$\frac{5e^{-2s}}{s(s^2+8s+25)}\ ??$$

$$N_0 \to e^{-2s}\left(\frac{5}{s(s^2+8s+25)}\right)$$

we now have the partial fraction for this piece.

Let's see what we have so far:

$$\mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{s^2+8s+25} - e^{-2s}\left[\frac{A}{s} + \frac{Bs+C}{s^2+8s+25}\right]\right\}$$

we know A, B, C

$$\mathcal{L}^{-1}\left\{\underset{①}{\tfrac{1}{5}\cdot\tfrac{1}{5}} + \underset{②}{-\tfrac{1}{5}\cdot\frac{s}{s^2+8s+25}} \underset{③}{-\tfrac{8}{5}\frac{1}{s^2+8s+25}}\right.$$

$$\left. - e^{-2s}\left[\underset{④}{\tfrac{1}{5}\cdot\tfrac{1}{5}} - \underset{⑤}{\tfrac{1}{5}\frac{s}{s^2+8s+25}} - \underset{⑥}{\tfrac{8}{5}\frac{1}{s^2+8s+25}}\right]\right\}$$

→ # each piece to do to

That's a lot, let's focus on each piece:

① $\mathcal{L}^{-1}\left\{\frac{1}{5}\cdot\frac{1}{5}\right\} = \frac{1}{5}$

② $\mathcal{L}^{-1}\left\{-\frac{1}{5}\cdot\frac{5}{s^2+8s+25}\right\}$

↳ still needs work. Complete the Square!

$s^2+8s+25 = \underbrace{(s+4)^2}_{s^2+8s+16} + 9$

$s^2+8s+16 + 9 = s^2+8s+25$ ✓

$\mathcal{L}^{-1}\left\{-\frac{1}{5}\left(\frac{5}{(s+4)^2+9}\right)\right\}$

starting to look like #11 in table but still needs more, we need $s+4$ in the numerator ... how?

Add 0!

$\frac{5}{(s+4)^2+9} = \frac{5}{(s+4)^2+9} + \underbrace{\frac{4}{(s+4)^2+9} + \frac{-4}{(s+4)^2+9}}_{=0}$

$= \frac{s+4}{(s+4)^2+9} - \frac{4}{(s+4)^2+9}$

$\mathcal{L}^{-1}\left\{-\frac{1}{5}\cdot\frac{s+4}{(s+4)^2+9} \cdot \left(-\frac{1}{5}\left(\frac{-4}{(s+4)^2+9}\right)\right)\right\}$ → we will group this w/ piece #③

$\mathcal{L}^{-1}\left\{-\frac{1}{5}\cdot\frac{s+4}{(s+4)^2+9}\right\} = -\frac{1}{5}e^{-4t}\cdot\cos 3t$

③ $\mathcal{L}^{-1}\left\{-\frac{8}{5}\cdot\frac{1}{(s+4)^2+9} + \frac{4}{5}\cdot\frac{1}{(s+4)^2+9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{-4}{5}\cdot\left(\frac{1}{(s+4)^2+9}\right)\right\}$

looks like #12 in table, but off by a constant!

③ $\mathcal{L}^{-1}\left\{ -\frac{4}{5}\cdot\left(\frac{1}{3}\right)\cdot\frac{3}{(s+4)^2+9}\right\} = -\frac{4}{15}e^{-4t}\sin 3t$

So, for pieces 1, 2, 3, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{5}\cdot\frac{1}{s} + \frac{-1}{5}\left(\frac{s}{s^2+8s+25}\right) + \frac{-8}{5}\cdot\left(\frac{1}{s^2+8s+25}\right)\right\}$$

$$= \underset{①}{\frac{1}{5}} \quad \underset{②}{-\frac{1}{5}e^{-4t}\cos 3t} \quad \underset{②\&③}{-\frac{4}{15}e^{-4t}\sin 3t}$$

what's left?

$$\mathcal{L}^{-1}\left\{ \underbrace{e^{-2s}}_{e^{-as}} \underbrace{\left[\frac{1}{5}\cdot\frac{1}{s} + \frac{-1}{5}\left(\frac{s}{s^2+8s+25}\right) + \frac{-8}{5}\left(\frac{1}{s^2+8s+25}\right)\right]}_{F(s)} \right\}$$

are we going to have to do a lot more work? No!

t-shifting

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

we found $f(t)$ from pieces ①, ②, & ③

(remember, $f(t) = \mathcal{L}^{-1}\{F(s)\}$)

so $f(t) = \frac{1}{5} - \frac{1}{5}e^{-4t}\cos 3t - \frac{4}{15}e^{-4t}\sin 3t$

to get $f(t-a)$, just replace every t with $t-a$

e^{-as} $F(s)$

$$\mathcal{L}^{-1}\left\{ e^{-2s}\left[\frac{1}{5}\cdot\frac{1}{s} + \frac{-1}{5}\left(\frac{s}{s^2+8s+25}\right) - \frac{8}{5}\left(\frac{1}{s^2+8s+25}\right)\right]\right\}$$

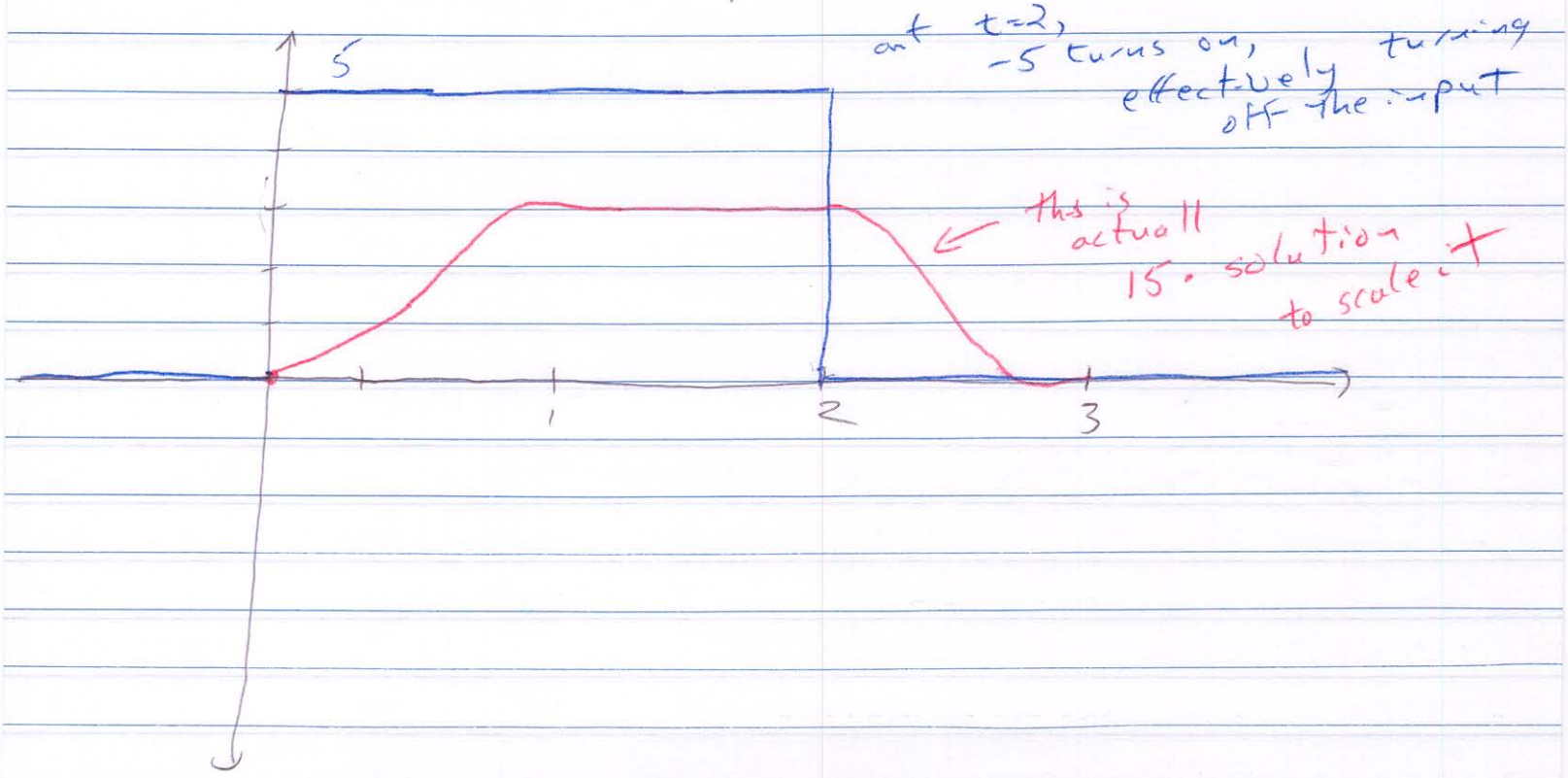
$a = 2$

replace t with $t-2$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,F(t-a)$$

$$= -u(t-2)\cdot\left[\frac{1}{5} - \frac{1}{5}e^{-4(t-2)}\cos(3(t-2)) - \frac{4}{15}e^{-4(t-2)}\sin(3(t-2))\right]$$

Now lets put it all together:

$$y(t) = \frac{1}{5} - \frac{1}{5}e^{-4t}\cos 3t - \frac{4}{15}e^{-4t}\sin 3t$$
$$-u(t-2)\left[\frac{1}{5} - \frac{1}{5}e^{-4(t-2)}\cos(3(t-2)) - \frac{4}{15}e^{-4(t-2)}\sin(3(t-2))\right]$$

Lets plot our input vs the solution: