Curl

First, lets Do an example problem from previous:

Given to: PE between two objects is equal

$$PE(x,y,z) = \frac{k}{\sqrt{x^2+y^2+z^2}}$$

Find the gravitational Field between the objects:

to get a ~~field~~ vector field from a potential function we need to take the gradient:

$$\nabla PE = \langle \frac{\partial}{\partial x} PE, \frac{\partial}{\partial y} PE, \frac{\partial}{\partial z} PE \rangle$$

$$\frac{\partial}{\partial x}\left(\frac{k}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left(k(x^2+y^2+z^2)^{-\frac{1}{2}}\right)$$

$$= \frac{-kx}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial}{\partial y}\left(\frac{k}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-ky}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial}{\partial z}\left(\frac{k}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-kz}{(x^2+y^2+z^2)^{3/2}}$$

so

$$\nabla PE = \langle \frac{-kx}{(x^2+y^2+z^2)^{3/2}}, \frac{-ky}{(x^2+y^2+z^2)^{3/2}}, \frac{-kz}{(x^2+y^2+z^2)^{3/2}}$$

Now, lets take the Divergence of our vector Field:

$$\vec{\nabla} \cdot \left\langle \frac{-kx}{(x^2+y^2+z^2)^{3/2}}, \frac{-ky}{(x^2+y^2+z^2)^{3/2}}, \frac{-kz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$= \frac{\partial}{\partial x}\left(\frac{-kx}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{-ky}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{-kz}{(x^2+y^2+z^2)^{3/2}}\right)$$

Fortunately due to symmetry, only need to do one

$$\frac{\partial}{\partial x}\left(-kx\,(x^2+y^2+z^2)^{-\frac{3}{2}}\right)$$

$$= \frac{-k}{(x^2+y^2+z^2)^{3/2}} + \frac{3}{2}kx(x^2+y^2+z^2)^{-5/2} \cdot 2x$$

$$= \frac{-k}{(x^2+y^2+z^2)^{3/2}} + \frac{3kx^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial}{\partial y}\left(-ky(x^2+y^2+z^2)^{-\frac{3}{2}}\right) = \frac{-k}{(x^2+y^2+z^2)^{3/2}} + \frac{3ky^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial}{\partial z}\left(-kz\,(x^2+y^2+z^2)^{-\frac{3}{2}}\right) = \frac{-k}{(x^2+y^2+z^2)^{3/2}} + \frac{3kz^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\boxed{\vec{\nabla} \cdot (\vec{\nabla} PE) = \frac{-3k}{(x^2+y^2+z^2)^{3/2}} + \frac{3k(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = \underline{0}}$$

$\vec{\nabla} \cdot \text{grad } F$ is called the Laplacian and is denoted $\nabla^2 F$

$$\boxed{\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0}$$

Laplaces equation

↳ solutions to this function describe the behavior of electric, gravitational & fluid potentials!!

Fields derived from these Potential Functions ~~are~~ by taking the gradient are called conservative, because energy is conserved.

Conservative Fields → work independent of path

examples: Magnetic Fields
Electric Fields
Gravitational Fields

Non conservative Forces: Friction

→ Now onto Curl:

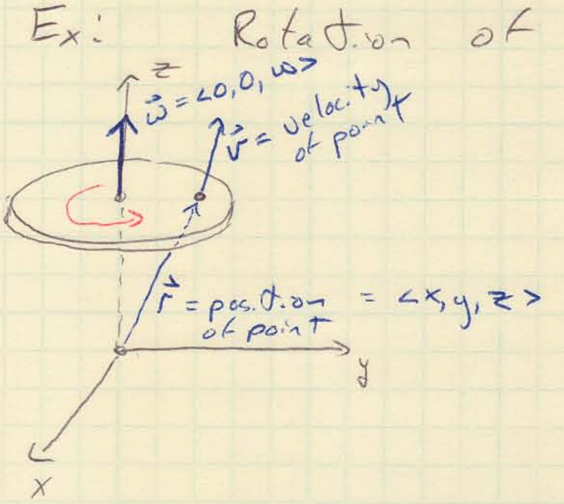
Mechanical Definition:

The Curl $\vec{F} = \vec{\nabla} \times \vec{F}$

vector × vector = vector

so Gradient operates on scalar & gives vector
Divergence operates on vector & gives scalar
Curl operates on vector & gives vector

Ex: Rotation of a Rigid Body



the $\vec{v}$ of a point (x, y, z) is equal to the angular velocity, $\vec{\omega}$ crossed w/ its position, $\vec{r}$

$\vec{v} = \vec{\omega} \times \vec{r}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \cancel{0+0} + \cancel{\omega 0}$$

tempting but don't, for curl, use top row expand!!

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & \omega \\ y & z \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & \omega \\ x & z \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 0 \\ x & y \end{vmatrix}$$

$$= -\omega y\,\vec{i} + \omega x\,\vec{j}$$

$$|\vec{v}| = \sqrt{\omega^2 y^2 + \omega^2 x^2}$$

$$= \omega\sqrt{x^2+y^2}$$

what does $x^2+y^2 = ?$

since rotating,

$$x^2+y^2 = r^2$$

$$|\vec{v}| = \omega r$$

What if we took the curl of $\vec{v}$?

$$\text{Curl } \vec{v} = \vec{\nabla} \times \vec{v} = \underbrace{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}_{\vec{\nabla}} \times \underbrace{\langle -\omega y, \omega x, 0 \rangle}_{\vec{v}}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 2\omega\vec{k}$$

$$\vec{\nabla} \times \vec{v} = \langle 0, 0, 2\omega \rangle$$

$$\vec{\omega} = \langle 0, 0, \omega \rangle$$

so $\vec{\nabla} \times \vec{v} = 2\vec{\omega}$

Important Facts:

$$\nabla \times \nabla F = 0$$
$$\nabla \cdot (\nabla \times \vec{v}) = 0$$

BONUS → Prove one or both!!

Ex: #9 pg 416

Let $\vec{v}$ be the velocity vector of a steady fluid flow. Is the flow irrotational? is it Incompressible? Find the Streamlines.

$\vec{v} = \langle 0, z^2, 0 \rangle$

a. Irrotational - ?? What does this mean?

$\hookrightarrow \vec{\nabla} \times \vec{v} = 0?$

we'll plot it out to visualize when we do streamlines

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z^2 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ z^2 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 0 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & z^2 \end{vmatrix}$$

$$= -2z\,\vec{i} = \langle -2z, 0, 0 \rangle$$

since $\vec{\nabla} \times \vec{v} \neq 0$, the field is not irrotational

b. Incompressible - ??

$\hookrightarrow \vec{\nabla} \cdot \vec{v} = 0?$

$$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle 0, z^2, 0 \rangle = 0 + 0 + 0 = 0$$

so it is Incompressible

c. Streamlines → ?? What are these?

Streamlines map the path of particles in the fluid flow.

What do we Know?

$\vec{v}(t) = \langle 0, z^2, 0 \rangle$, but

$\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

so

$\langle 0, z^2, 0 \rangle = \langle x'(t), y'(t), z'(t) \rangle$

or

$$\left.\begin{aligned} 0 &= x'(t) \\ z^2 &= y'(t) \\ 0 &= z'(t) \end{aligned}\right\}$$

A system of Differential Equations!!

Fortunately, we can solve this system independently:

$x'(t) = 0, \quad x(t) = c_1$

$z'(t) = 0, \quad z(t) = c_3$

$y'(t) = (z(t))^2 = c_3^{\,2}$,

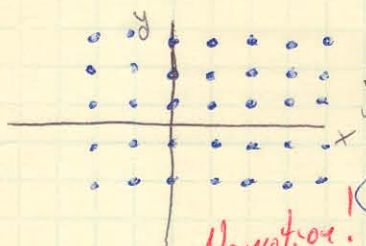
thus $\quad y(t) = c_3^{\,2}\, t + c_2$

but what does this mean?

1st lets look @ the xy plane for different z's

~~with $c_1 = c_2 = 0$~~

$z = 0$



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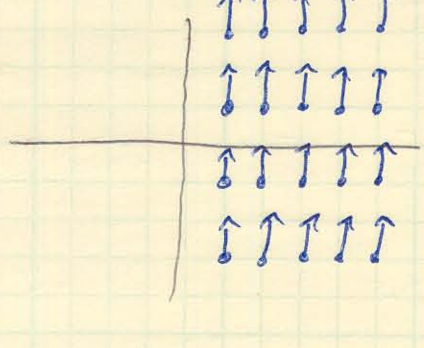
$x'(t) = 0$
$y'(t) = 0$
represents the blue dots

No motion!

$z = 1$



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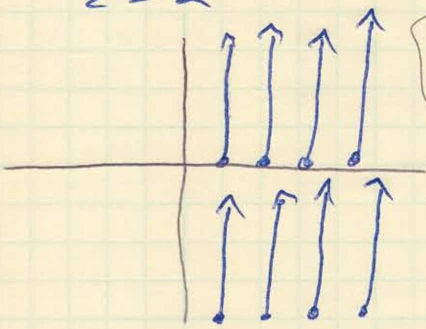
$y(t) = z^2 t$
$= t$

$x'(t) = 0$
$y'(t) = 1$
blue arrows indicate direction of particle flow

$z = 2$

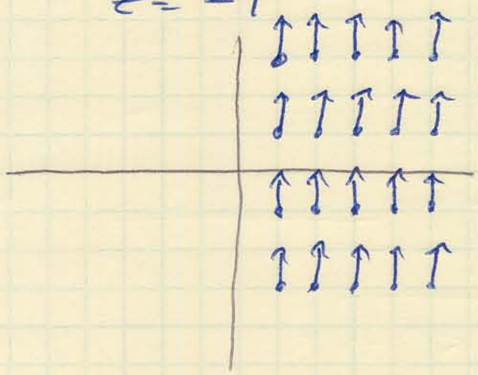


position

$$x(t) = c$$
$$y(t) = z^2 t + c$$

$$x'(t) = 0$$
$$y'(t) = 4$$
velocity

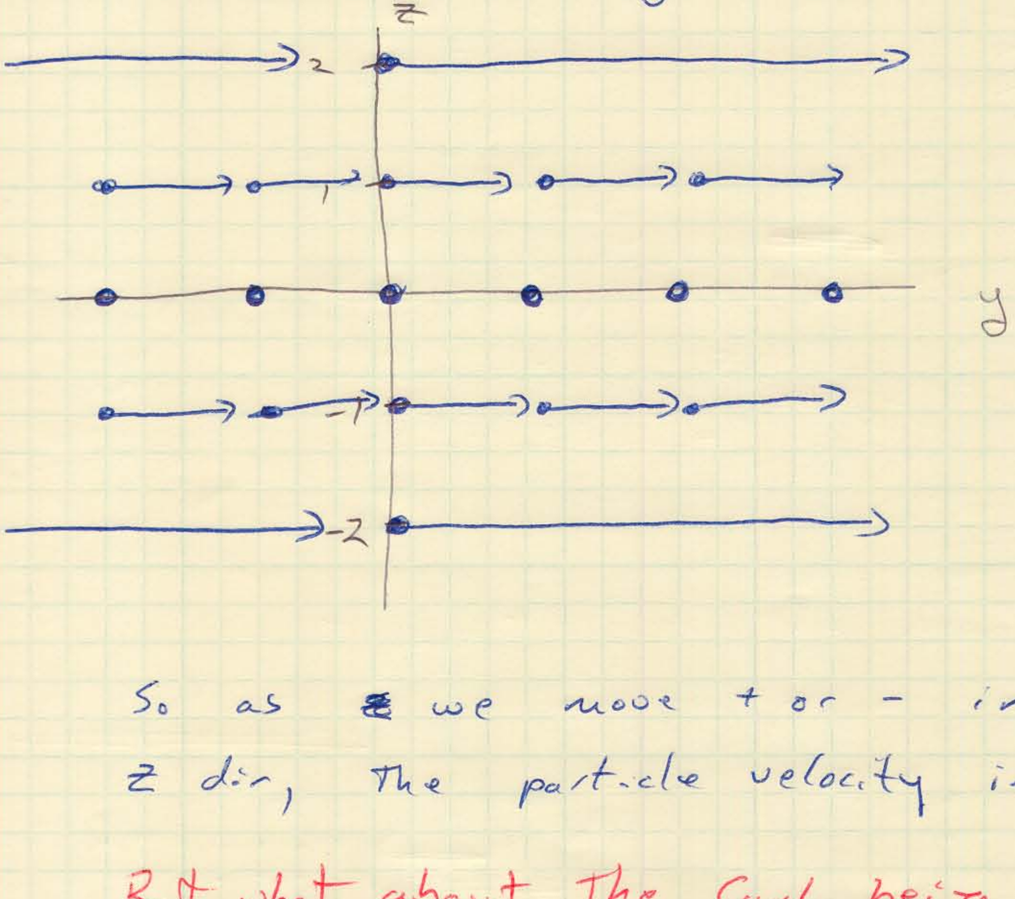
$z = -1$



$x(t) = c$
$y(t) = (-1)^2 t + c$

$x'(t) = 0$
$y'(t) = 1$

Since there is no motion in the x-dir, lets look @ the y-z plane



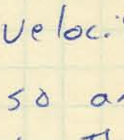
particles @ $z=0$, have no motion

$$y'(t) = z^2$$

particles @ $z = 1, -1$ have velocity $y' = 1$, so are moving in the $+y$ dir

So as ~~z~~ we move + or - in the z dir, the particle velocity increases.

But what about the Curl being non-zero? Does it look like there is rotation in that vector field?

Imagine if you put a pin wheel () @ the 1,1 position, would it spin?