Lesson 3 — 1st Order Differential Equations

Review: Seperable Equations
Direction Fields (slope Fields)
New: Exact & Integrating Factors

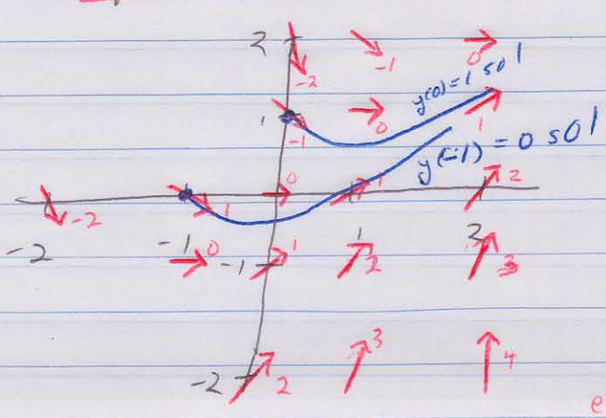
Geometric Meaning of $y' = f(x,y)$

what is y'? $\frac{dy}{dx}$, slope, derivative, rate of change

so when have $y' = f(x,y)$, instead of having a value for the function $y(x)$ at any point x, we have the value of the slope at a point (x,y).

How do we use this?
Given $y' = x - y$, plot the slope at each point

(1)



x	y	y'
0	0	0
1	0	1
2	0	2
1	1	0
2	1	1
0	1	-1
etc. 0	2	-2

both similar to Euler's Method but not exactly

Next: Given $y(0) = 1$, plot solution, following the slope lines

Seperable:

When the differential equation can be manipulated into the following form:

$$g(y)\,y' = f(x)$$

The equation is seperable and is solved by replacing y' with $\frac{dy}{dx}$ as follows:

$$g(y)\frac{dy}{dx} = f(x),$$

then we treat $\frac{dy}{dx}$ as a fraction & seperate by multiplying the equation through by dx:

$$dx\left(g(y)\frac{dy}{dx} = f(x)\right)$$

$$\Rightarrow \quad g(y)\,dy = f(x)\,dx$$

With the equation now in this form, we integrate both sides:

$$\int g(y)\,dy = \int f(x)\,dx$$

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$$L\frac{dI}{dt} + RI = 0, \qquad I(0) = I_0$$

What is my dependent variable? I (y)
What is my independent variable? t (x)

so:

$$L\frac{dI}{dt} = -RI$$

$$L\,dI = -RI\,dt$$ Are we done? No!

$$\frac{dI}{I} = -\frac{R}{L}\,dt$$ Now integrate

$$\int \frac{dI}{I} = \int -\frac{R}{L}\,dt$$

$$\ln I = -\frac{R}{L}t + C$$ Are we done? No! we want an <u>explicit</u> solution if possible

$$I = e^{-\frac{R}{L}t + C}$$
$$= e^{-\frac{R}{L}t}\underbrace{e^{C}}_{\text{just another constant}}$$

$$I(t) = Ce^{-\frac{R}{L}t}$$ what about the initial condition?

$$I(0) = Ce^{-\frac{R}{L}(0)} = I_0$$
$$= C(1) = I_0$$

so

$$C = I_0$$

and then

$$I(t) = I_0 e^{-\frac{R}{L}t}$$

Now we're done!

★ Special Form of Seperable:

if $y' = F\left(\frac{y}{x}\right)$

then let $y = ux$,

and $y' = u'x + u$ (chain rule)

Now substitute:

$$u'x + u = F\left(\frac{ux}{x}\right)$$

$$\downarrow$$

$$\frac{du}{dx}x + u = F(u)$$

$$\frac{du}{dx}x = F(u) - u$$

then,

$$\frac{du}{F(u) - u} = \frac{dx}{x}$$

finally integrate & solve!

Exact: Another Special Form:

if a differential equation can be reduced to the following form:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

and,

Use ① to solve!

$$u = \int M\,dx + k(y)$$

$$= \int (e^y - ye^x)\,dx + k(y)$$

$$= xe^y - ye^x + k(y)$$

Now find $k(y)$ be comparing $\partial u/\partial y$ with N (N should equal $\partial u/\partial y$)!

$$\frac{\partial u}{\partial y} = xe^y - e^x + \underbrace{k'(y)}_{\text{don't forget this piece!}} = N$$

$$xe^y - e^x + k'(y) = xe^y - e^x$$

solve for $k'(y)$:

$$k'(y) = 0$$

integrate

$$k(y) = C$$

so, u is:

$$u(x,y) = xe^y - ye^x + C$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then the equation is exact & the solution is found either by

① $u = \int M\,dx + k(y)$

② $u = \int N\,dy + l(x)$

to find either $k(y)$ or $l(x)$ we find $\frac{\partial u}{\partial y}$ from ① and use N to get $\frac{dk}{dy}$, then integrate!

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$$\underbrace{(e^y - ye^x)}_{M}dx + \underbrace{(xe^y - e^x)}_{N}dy = 0$$

? $\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial y} = e^y - e^x$

$\frac{\partial N}{\partial x} = e^y - e^x$

so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation is exact!

Use a given initial condition to solve for C.

What form of solution is this?

<u>Implicit!</u>

When we cannot solve for y in terms of x explicitly, then it is an implicit solution!

* What if its Not exact? :(

Integrating Factor → A function, that when multiplied to a differential equation makes it exact!

Example:

is $-y\,dx + x\,dy = 0$ exact?

$$\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x} \Rightarrow -1 \neq 1$$

Not exact

But, what if I multiply the equation by:

$$\frac{1}{x^2}\ ?$$

$$\frac{1}{x^2}(-y\,dx + x\,dy) = 0$$

$$\frac{-y}{x^2}\,dx + \frac{x}{x^2}\,dy = 0$$

$$-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = 0$$

Is it exact?

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\ ?$$

$$\downarrow$$

$$-\frac{1}{x^2} = -\frac{1}{x^2}\ \checkmark$$

Equation is exact!

Can solve using earlier technique!

How do I find integrating factor?

if an equation is not exact, let it be represented as

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

where:

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$ so not exact!

To make exact, Multiple equation by:

$$F(x) = e^{\int \left(\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right) dx}$$

or by

$$F^*(y) = e^{\int \left(\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right) dy}$$

Integrating Factors

then solve as an exact equation
(Note: to ensure your integrating factor is correct, check for exactness!)