Lesson 4 Homogeneous 2^{nd} Order Differential Equations

Homogeneous 2^{nd} Order:

$$y'' + p(x)\,y' + q(x)\,y = 0$$

We will discuss Linear ODEs, & The principle of super position or The linearity principle

Superposition:

if y_1 & y_2 are linearly independent solutions to a linear, 2^{nd} order diff eq, then $C_1 y_1 + C_2 y_2$ ~~are~~ is a solution

Superposition ex:

$$y'' + y = 0$$

$$\left.\begin{aligned} y_1 &= \cos x \\ y_2 &= \sin x \end{aligned}\right\} \text{ verify on your own!}$$

is $C_1 \cos x + C_2 \sin x$ a solution?

$$y = C_1 \cos x + C_2 \sin x$$
$$y'' = -C_1 \cos x - C_2 \sin x$$

$y + y'' = 0$ ✓ so the linear combo is a solution

in fact the sol $y = C_1 \cos x + C_2 \sin x$, covers all possible solutions & is called a **Basis**

Basis

A basis of solutions of a linear ODE on an open interval I is a set of linearly independent solutions of the ODE on I.

Constant Coefficient Linear ODEs

$$y'' + ay' + by = 0 \qquad \text{From MA205!}$$

when we had

$y' + ky = 0$, we had a sol

$y = ce^{-kx}$

will this work for the 2^d order case?

Let $y = e^{\lambda x}$;

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$, then

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

since $e^{\lambda x} > 0$ for all x,

$$\lambda^2 + a\lambda + b = 0$$

solve for λ:

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Three Cases!

Case I: Two real roots, $a^2 - 4b > 0$

then: the Basis is:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

Case II: Repeated roots $a^2 - 4b = 0$

(through reduction of order, see pg 55)

the Basis is

$$y = (c_1 + C_2 x) e^{-ax/2}$$

Case III: Two imaginary roots

$$\lambda_{1,2} = -\tfrac{1}{2}a \pm i\omega, \qquad \omega^2 = b - \tfrac{1}{4}a^2$$

so our solution is

$$e^{(-\frac{1}{2}a + i\omega)x} + e^{(-\frac{1}{2}a - i\omega)x}$$

which is

$$e^{-\frac{1}{2}ax} e^{i\omega x} + e^{-\frac{1}{2}ax} e^{-i\omega x}$$

$$= e^{-\frac{1}{2}ax}(e^{i\omega x} + e^{-i\omega x})$$

but what is $e^{i\omega x}$?

$$e^{i\omega x} = \cos \omega x + i \sin \omega x$$
$$e^{-i\omega x} = + \cos \omega x - i \sin \omega x$$

} two solutions but we want real

Eulers

$$y_1 = \frac{1}{2} C_1 (2 \cos \omega x)$$

$$y_1 = C_1 \cos \omega x \text{ or } A \cos \omega x$$

$$\cos \omega x + i \sin \omega x$$
$$- \quad \cos \omega x - i \sin \omega x$$

$$y_2 = -\frac{1}{2i} C_2 (-2i \sin \omega x)$$

$$y_2 = C_2 \sin \omega x \text{ or } B \sin \omega x$$

the Basis is:

$$y = e^{-\frac{ax}{2}} (A \cos \omega x + B \sin \omega x)$$