Lesson 5 Non homogeneous 2^d Order ODEs

2.7: Method of Undetermined Coefficients
2.8: Forced Oscillations

} Virtually Identical Material!

2.7: Method of Undetermined Coefficients

Given: $\underbrace{y'' + p(x)y' + q(x)y}_{\text{Linear!}} = r(x) \neq 0$

$$y(x) = y_h(x) + y_p(x)$$

n → I will use subscript n to refer to non-homogeneous solution

solution to homogeneous equation:
$y'' + p(x)y' + q(x)y = 0$
$\downarrow$
$y_h(x) = C_1 y_1 + C_2 y_2$

solution to non-homogeneous piece
book calls this the 'particular' - but then uses the word again to mean something else

then the "Particular Solution" is

$$y_p(x) = C_1 y_1 + C_2 y_2 + y_h(x)$$

Where C_1 & C_2 are known through application of Boundary Conditions

How can we just add the two solutions?

Linearity!

Let $L[y] = y'' + p(x)y' + q(x)y$

L is our differential equation, and since our equation is linear,

$$L[y_h + y_n] = L[y_h] + L[y_n]$$

$L[y_h] = 0$ Definition of homogeneous solution

$L[y_n] = r[x]$

so

$$L[y_h + y_n] = L[y_h] + L[y_n] = 0 + r(x) = r(x)$$

Linearity!

→ Let's Solve Some Non-homogeneous equations!

Given:

$$y'' + ay' + by = r(x)$$

→ classify it: 2nd Order, Linear, Non-homog, Cnst Coeff

How Do we solve homogeneous part?

$$y'' + ay' + by = 0$$

3 Cases:

review from Lesson 4

Case I: $a^2 - 4b > 0$ $\quad \lambda_{1,2} = \dfrac{-a \pm \sqrt{a^2-4b}}{2}$

$$y_h = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

Case II: $a^2 - 4b = 0$ $\quad \lambda = \dfrac{-a}{2}$

$$y_h = (C_1 x + C_2) e^{\lambda x}$$

Case III: $a^2 - 4b < 0$ $\quad \omega = \dfrac{\sqrt{4b - a^2}}{2}$

$$y_h = e^{(-a/2)x} (A \cos \omega x + B \sin \omega x)$$

Now, Then non homogeneous solution!

Given

$y'' + ay' + by = r(x)$, find $y_n(x)$

how? By Guessing

if $r(x)$	then guess
$Ke^{\gamma x}$	$Ce^{\gamma x}$
Kx^n	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_0$
$K \cos \omega x$, $K \sin \omega x$	$K \cos \omega x + M \sin \omega x$

More guesses in book, even more in FE.

Lets try it:

$$y'' + 2y' + y = 3e^{-x}$$

1. Find homogeneous solution:

$$\lambda = \frac{-2 \pm \sqrt{4-4}}{2} = -1, \text{ Case II}$$

so,

$$y_h = (C_1 x + C_2)e^{-x} \quad \checkmark$$

2. Find non homogeneous solution by guess!

$r(x) = 3e^{-x}$ so by table, we guess: $y_n = Ce^{-x}$

wait, thats the solution to ~~non~~-homog piece, what next?

Modification rule:
if solution repeated, multiply x

so: $y_n = Cxe^{-x}$

still no good

keep multiplying by x:

$$y_n = Cx^2e^{-x}$$

③ How do we determine the C?

plug the solution into the equation

$$y_n = Cx^2e^{-x}$$
$$y_n' = 2xCe^{-x} - Cx^2e^{-x}$$
$$y_n'' = 2Ce^{-x} - 2xCe^{-x} - 2xCe^{-x} + Cx^2e^{-x}$$

So

$$2C\cancel{e^{-x}} - 2xC\cancel{e^{-x}} - 2xC\cancel{e^{-x}} + Cx^2\cancel{e^{-x}} + 2(2xC\cancel{e^{-x}} - Cx^2\cancel{e^{-x}}) + Cx^2\cancel{e^{-x}} = 3\cancel{e^{-x}}$$

Divide Through by e^{-x}

$$2C - 2xC - 2xC + Cx^2 + 4xC - 2Cx^2 + Cx^2 = 3$$

$$2C = 3$$
$$C = 1.5$$

remember, I said 80% of this class is Algebra!

④ Put it all Together:

$$y_g(x) = y_h(x) + y_n(x)$$
$$= (C_1x + C_2)e^{-x} + 1.5x^2e^{-x}$$

Now Check solution by plugging it all into the Diff Equation!

⑤ What Do we need for a Paticular Solution?

Initial Conditions!!

undamped

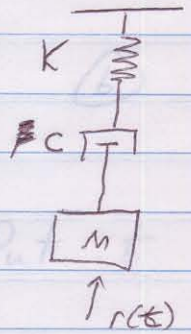
Next: Forced Oscillations

Guess what → Same solution technique, we're just going to point out interesting phenomena

We will ~~Derive~~ → The book did the deriving look at what happens when you force with a sinusoidal.

↳ why is this important over others? wait until Fourier ☺

Setup: Mass Spring System (just as easy → Electrical System)



$$my'' + cy' + Ky = F_0 \cos \omega t$$

First, let $c = 0$, then

$$my'' + Ky = F_0 \cos \omega t$$ → Undamped Forced Oscillators

① $y_h(t) = A \cos \omega_0 t + B \sin \omega_0 t$, $\omega_0 = \sqrt{K/m}$

same as we've done before, just different names for the constants

$C \cos(\omega_0 t - \delta)$ $\quad C = \sqrt{A^2 + B^2}$, $\delta = \tan^{-1}\left(\frac{B}{A}\right)$

just another way of writing it

undamped

2 $y_n(t) = \underbrace{C_1 \cos(\omega t) + C_2 \sin(\omega t)}_{\text{or}}$

3 Solve for Constants: from the book (it did the algebra)

a $y_n(t) = \dfrac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$

? What happens if $\omega_0 = \omega$?
then we need a new guess for y_n:

$y_n = t(C_1 \cos \omega_0 t + C_2 \sin \omega_0 t)$

from modification rule

or (from the book)

b $y_n = \dfrac{F_0}{2m\omega_0} t \sin \omega_0 t$ ← Resonance

4 Put it together:

$y_a(t) = C\cos(\omega_0 t - \delta) + \dfrac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$ a

$C\cos(\omega_0 t - \delta) + \dfrac{F_0}{2m\omega_0} t \sin \omega_0 t$ b

can cause beats

resonance

Let's Look @ <u>Steady State Response</u>

↳ how the system behaves after a long period of time

<u>Beats:</u> For $\omega \neq \omega_0$:

$$y_c(t) = \underbrace{C\cos(\omega_0 t - \delta)}_{A\sin\omega_0 t + B\cos\omega_0 t} + \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t$$

Let's Look at a particular piece of the solution:

$$\underbrace{\frac{F_0}{m(\omega_0^2-\omega^2)}}_{A}\underbrace{(\cos\omega t - \cos\omega_0 t)}_{\text{using trig IDs:}}$$

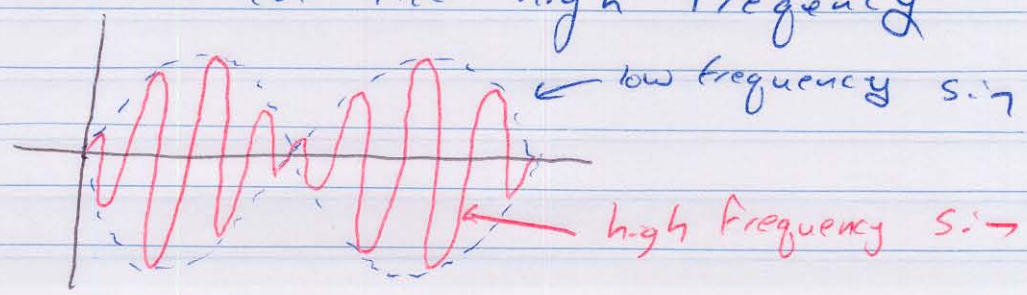
$$2A\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)$$

since ω & ω_0 are close

$\sin\left(\frac{\omega_0+\omega}{2}t\right) \rightarrow$ high freq

$\sin\left(\frac{\omega_0-\omega}{2}t\right) \rightarrow$ low freq

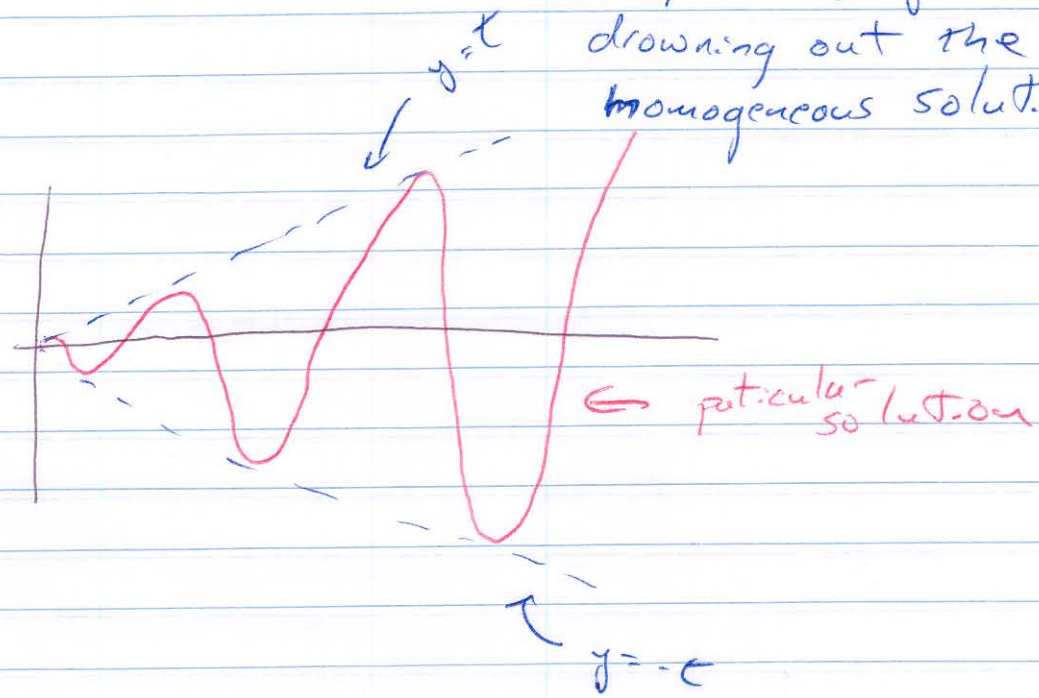
the low frequency acts like an amplitude function for the high frequency



Resonance:

$$y_c(t) = C\cos(\omega_0 t - \delta) + \underbrace{\frac{F_0}{2m\omega_0} t \sin \omega_0 t}$$

as t increases, so does the amplitude, quickly drowning out the homogeneous solution



Why is this bad?

Damped Oscillation

For a damped oscillation does the homogeneous solution affect steady State?

No, since there is no driving the ~~steady~~ homogeneous response goes to zero.

So lets Look @ the non-homogeneous solution for Damped Oscillation:

$$y_n(t) = a\cos\omega t + b\sin\omega t$$

$$a = F_0 \frac{m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2} \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}$$

or

$$y_n(t) = C\cos(\omega t - \eta)$$

amplitude

$$C = \sqrt{a^2+b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}}$$

$$\eta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\omega c}{m(\omega_0^2-\omega^2)}\right)$$

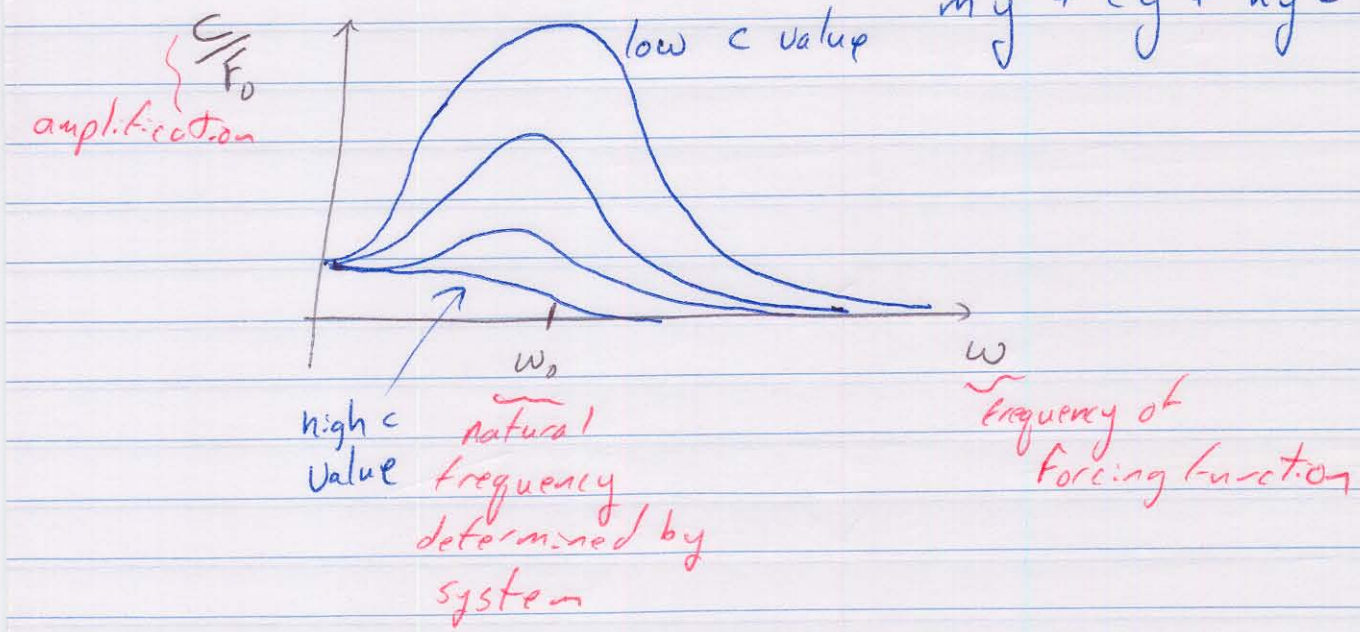
phase angle

Often we find it more interesting to look at amplitude (C) verses frequency & phase angle (η) verses frequency.

Why? think Tacoma Bridge
Shattering a glass } neither are undamped but close

Lets look at the amplification of the input signal (~~[illegible]~~ amplitude F_0) by examing the output amplitude (C) over the input amplitude (F_0) or C/F_0, vs frequency for different damping coefficients (c)

$$my'' + cy' + ky = F_0 \cos(\omega t)$$



It's often helpful to look @ η_E as a function of ω also ... especially in control systems

Lets look @ an aexample

Problem #3 pg 90

$$y'' + y' + 4.25y = 221 \cos 4.5t$$

① Lets determine what all our constants are so we can match them to the equations:

$$my'' + cy' + 4.25y = 221 \cos 4.5t$$

$m = 1$ $F_0 = 221$

$c = 1$ $\omega = 4.5$

$k = 4.25$

② Next, lets find ω_0:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\frac{1}{4}}{1}} = \sqrt{\frac{17}{4}} = \frac{\sqrt{17}}{2}$$

Do we need to worry about beats or resonance? No → This is damped oscillation

③ Lets determine our Steady State Solution:

From pg 85, eq (3)
the non-homogeneous solution (which will be our steady state solution) is

$$y_A(t) = a\cos\omega t + b\sin\omega t$$

what are a & b?

pg 85, eq (15)

$$a = F_0 \frac{m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}$$

$$b = F_0 \frac{\omega c}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}$$

Now, plug in the values for m, ω_0, ω, c, & F_0

$$a = \underbrace{221}_{F_0} \cdot \frac{\overbrace{1}^{m}\left(\overbrace{\frac{17}{4}}^{\omega_0^2} - \overbrace{\frac{81}{4}}^{\omega^2}\right)}{\underbrace{1^2}_{m^2}\left(\underbrace{\frac{17}{4}}_{\omega_0^2} - \underbrace{\frac{81}{4}}_{\omega^2}\right)^2 + \underbrace{\frac{81}{4}}_{\omega^2}\cdot\underbrace{1}_{c^2}}$$

$$= 221 \cdot \frac{-\frac{64}{4}}{\left(-\frac{64}{4}\right)^2 + \frac{81}{4}} = -\frac{64}{5}$$

$$b = 221 \cdot \frac{\frac{81}{4}\cdot 1}{\left(-\frac{64}{4}\right)^2 + \frac{81}{4}} = \frac{18}{5}$$

Denominators the same for a & b

Put it together:

$$y_{ss}(t) = \underline{\underline{-\frac{64}{5}\cos\left(\frac{9}{2}t\right) + \frac{18}{5}\sin\left(\frac{9}{2}t\right)}}$$

steady state

that's it

Lets extend this problem:

$$y'' + y' + 4.25y = \underbrace{\overset{F_0}{\cancel{22}}\cos \overset{\omega}{\cancel{4.5}}t}$$

lets look at the response for a general ω

$m = 1$ $\quad F_0 = F_0$

$c = 1$ $\quad \omega = \omega$

$k = 4.25$

what changes?

Lets find the amplification, c/F_0 as a function of ω

what does C equal?

pg 88 (14)

$$C(\omega) = \sqrt{a^2+b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + \omega^2 c^2}}$$

so $$\underbrace{C(\omega)/F_0}_{A(\omega)} = \frac{1}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}}$$

$$A(\omega) = \frac{1}{\sqrt{1^2(17/4-\omega^2)^2+\omega^2 1^2}}$$

$$= \frac{1}{\sqrt{(17/4-\omega^2)^2 + \omega^2}}$$

Lets look @ when is $A(\omega)$ at its Max?

→ how do we find the max of a function?

1st Derivative test &
2nd Derivative test

Details in book, but result is

$$\omega_{max}^2 = \omega_0^2 - \frac{c^2}{2m^2} = 17/4 - 1/2$$

$$= 15/4 = \omega_{max}^2$$

the frequency where A is maximum

What happens to ω_{max} as $c \to 0$,

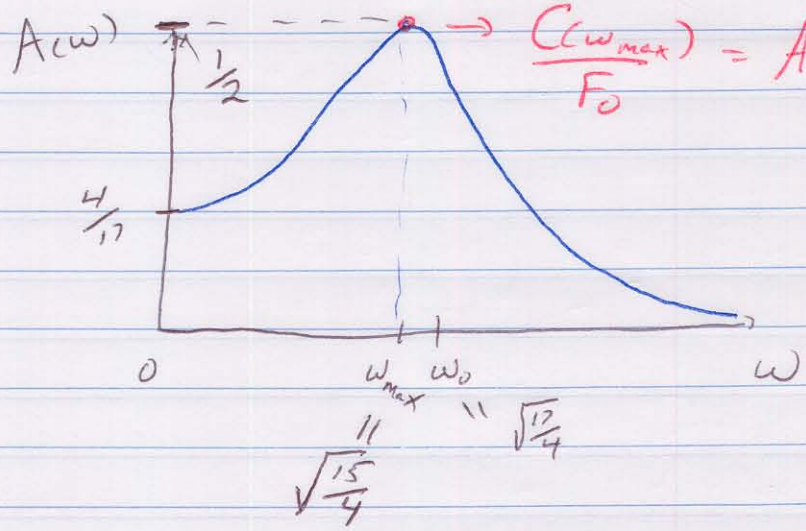
ω_{max} Approaches ω_0

what if $c = 0$,

Then we have undamped oscillations & use a different set of equations!

So, lets answer our ~~equat~~ question:

What is the general response



$$\frac{C(\omega_{max})}{F_0} = A(\omega_{max}) = \frac{2m}{c\sqrt{4m^2\omega_0^2 - c^2}}$$

$$= \frac{2}{\sqrt{4 \cdot \frac{17}{4} - 1}}$$

DO THE PROBLEMS :)