Stokes Theorem

Greens' Theorem: $\iint_A \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x}\right) \cdot \vec{k} \, dA = \oint_C \vec{F} \cdot d\vec{r}$

Stokes Theorem: $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \oint_C F \cdot r'(t) \, dt$

extension of Green's Theorem in 3-D

relates the surface integral of the curl of a vector field

to a ~~line~~ closed loop line integral

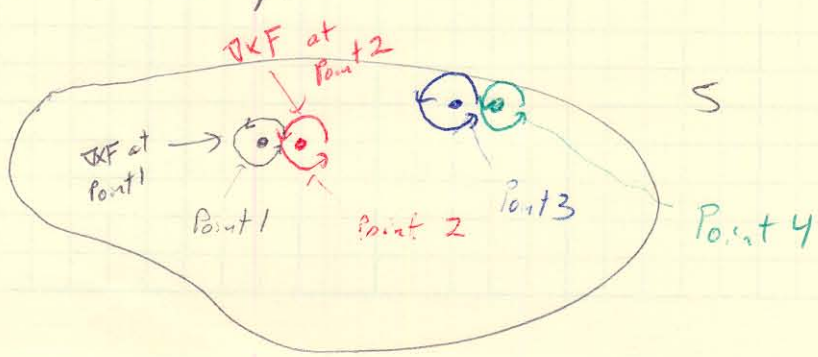
S = piecewise oriented surface

C = Boundary curve of the surface

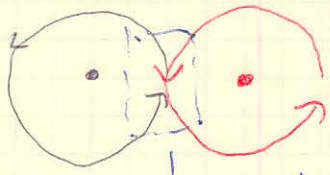
Let's examine this:

1st: Based on orientation of surface, the direction of the curve changes.

2nd: What is curl? → the measurement of rotation about a point, so if I integrate, I sum up all the rotations at each point:



Looking at Point 1 & 2,



↳ what's the net curl here?

$-\text{curl} + +\text{curl} = 0$

so the curls sum to 0 inside curve

Looking at Points 2 & 3, where are the only points that the curls don't cancel?

Along the surface Border, C!

This is a short graphical explanation, not a true proof, we will not go into a proof.

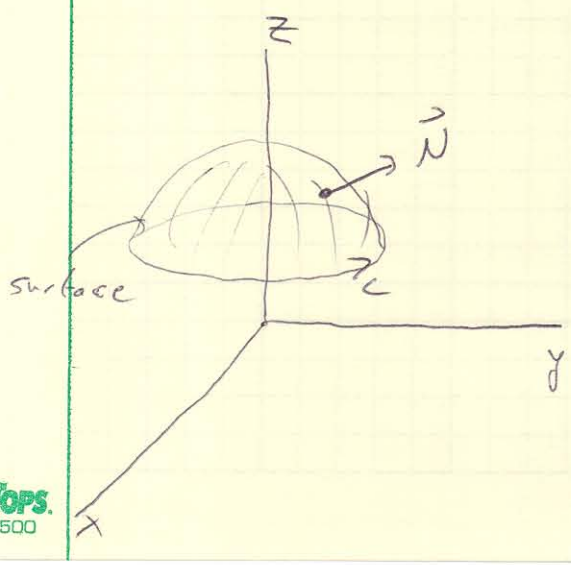
→ Now let's try a few examples.

Ex 1

Given: $z = 1-(x^2+y^2) = f(x,y)$ surface

$z \geq 0$

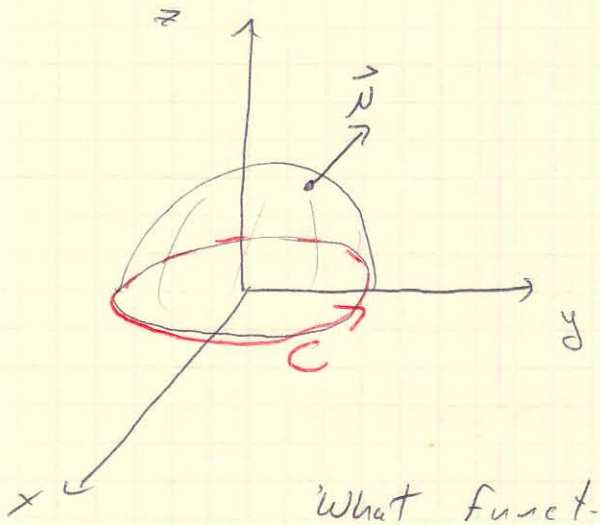
$\vec{F} = \langle y, z, x \rangle$ vector field



Find: $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$

two ways to solve, actually do the surface integral over the spherical surface, or use Stokes Theorem to do a line integral - which is easier?

First, Find C, the border of our surface:



Note direction → right hand rule

What function defines C?

$$z = 1-(x^2+y^2), \quad z \geq 0$$

if $z=0$, base of surface

$$0 = 1-(x^2+y^2)$$

$$x^2+y^2 = 1 \Rightarrow C$$

Since C is a relatively simple curve to parameterize & $\vec{F}$ is a fairly simple vector field, so lets try the line integral:

ensure direction is correct

$$C \Rightarrow \vec{r}(t) = \langle \cos t, \sin t, 0 \rangle, \quad \boxed{0 \leq t \leq 2\pi}$$

$$\vec{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$$

Do not forget this piece!

$$\vec{F}(\vec{r}(t)) = \langle \sin t, \overset{0}{\cancel{\cos t}}, \cos t \rangle$$

$$\oint_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle \sin t, 0, \cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \underbrace{\int_0^{2\pi} -\sin^2 t \, dt}_{\text{From FE}}$$

$$= -\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t \Big|_0^{2\pi}\right) = -\pi$$

→ Now, lets look @ Green's Theorem as a special case of Stoke's Theorem pg 471

Example 3: Evaluate $\int_C \vec{F} \cdot d\vec{r}$

Given: $C \Rightarrow x^2+y^2=4,\ z=-3$ oriented ccw

$\vec{F} = \langle y, xz^3, -zy^3 \rangle$

① As a line integral:

parameterize C: $\vec{r}(t) = \langle 2\cos t, 2\sin t, -3 \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, -3 \rangle$

Find $\vec{F}(\vec{r}(t))$: $\vec{F}(r(t)) = \langle 2\sin t, 2\cos t(-27), 3\cdot(2\sin t)^3 \rangle$

$$\int_0^{2\pi} \langle 2\sin t, -54\cos t, 24\sin^3 t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

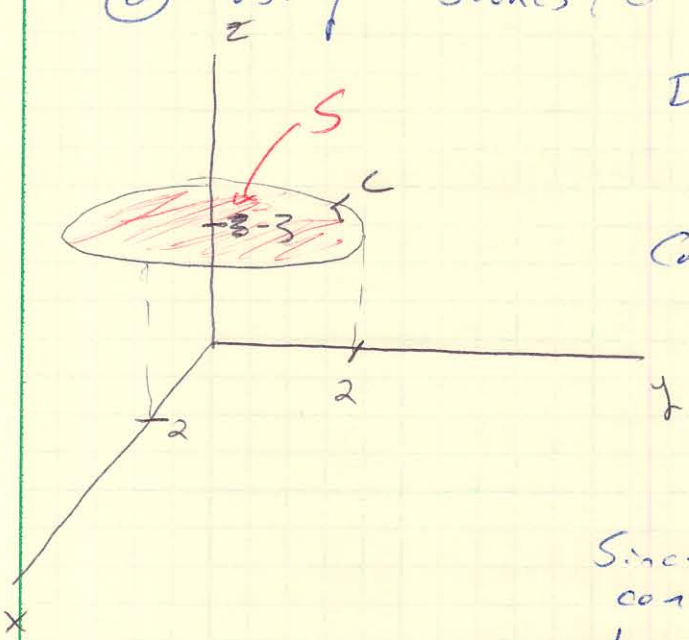
$$= \int_0^{2\pi} -4\sin^2 t - 108\cos^2 t \, dt$$

$$= -4\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right)\Big|_0^{2\pi} - 108\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$$

$$-4\pi - 108\pi = \underline{\underline{-112\pi}}$$

? → Not quite, the vector field isn't 2-D, but will be similar

② using Stokes (Greens) Theorem



Do we have a given surface?
No!

Can we make one?
Yes → fill in the hole,

$z = -3$

Since no surface given, we can choose any surface bounded by the curve, a plane is the easiest surface, thus we choose

$$z = -3$$

using Stokes Theorem:

$$\oint \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

what is $\vec{n}$?

what is a normal vector to the $z = -3$ surface?

Gradient!

based on orientation of C

$$\vec{n} = \langle 0, 0, +1 \rangle$$

$$= \vec{k}$$

$$\vec{\nabla} \times \vec{F} : \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & +z^3 & -zy^3 \end{vmatrix} = \vec{i}(y^3 - z^3) - \vec{j}(0) + \vec{k}(z^3 - 1)$$

$$\vec{\nabla} \times \vec{F} = \langle y^3 - z^3, 0, z^3 - 1 \rangle$$

What is z in this problem?

$z = -3$,

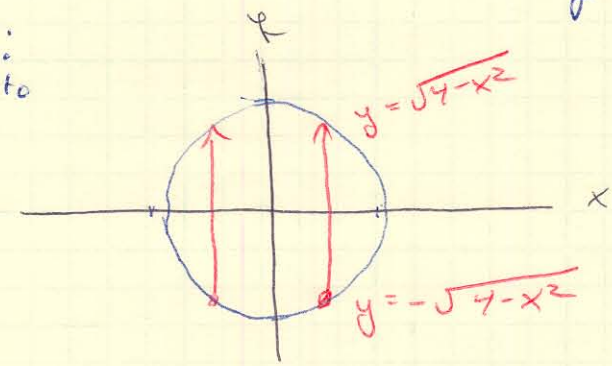
so $\vec{\nabla} \times \vec{F} = \langle y^3 + 27, 0, -28 \rangle$

~~as~~ setting up the surface integral:

$$\iint_S \langle y^3 + 27, 0, -28 \rangle \cdot \langle 0, 0, 1 \rangle \, \underline{dS}$$

What is dS? Since dS is the plane $z = -3$, $dS = dA = dx\,dy$

and S: (projected into xy plane)



so we have:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} -28 \, dy\,dx$$

can we make easier?

Polar Coordinates:

$$\int_0^{2\pi} \int_0^2 -28 \, r\,dr\,d\theta = \underline{\underline{-112\pi}}$$

One last example:

Pr Sec 10.9, #4

Find: $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, dS$

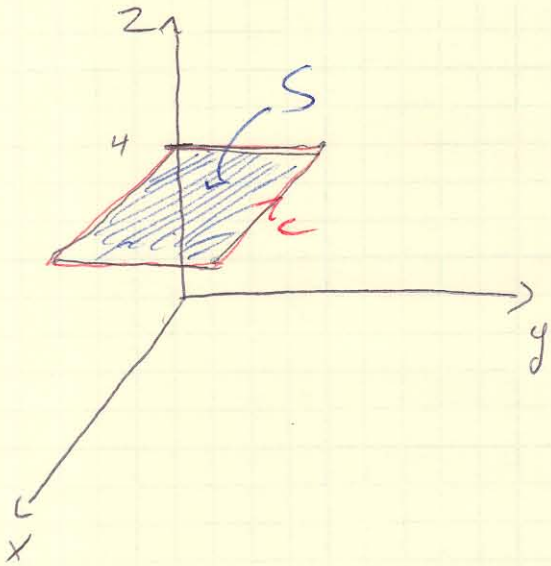
Given: $\vec{F} = \langle 3\cos y, \cosh z, x \rangle$

S: square $0 \le x \le 2$, $0 \le y \le 2$, $z = 4$

what are we missing?

Orientation → Assume counter clockwise

Picture!



$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3\cos y & \cosh z & x \end{vmatrix}$$

$$= \vec{i}(-\sinh z) - \vec{j}(1) + \vec{k}(3\sin y)$$

$$= \langle -\sinh z, -1, 3\sin y \rangle$$

$\vec{n} = ?$ since surface is parallel w/ xy plane,

$\vec{n} = \langle 0, 0, 1 \rangle$

$dS = dx\,dy$

So: $\int_0^2 \int_0^2 \langle -\sinh z, -1, 3\sin y \rangle \cdot \langle 0, 0, 1 \rangle \, dx\,dy$

$$= \int_0^2 \int_0^2 3\sin y \, dx\,dy = -(2)(3)\cos y \Big|_0^2$$

$$= \underline{-6(\cos 2 - 1)}$$