Surface Normal Vectors

two ways to represent a surface:

① $\left.\begin{array}{l} z = f(x,y) \\ x = f(y,z) \\ \underbrace{y = f(x,z)}_{\text{use most convenient}} \end{array}\right\} \Rightarrow \underbrace{g(x,y,z) = 0}_{\text{recall level curves!}}$

Ex: a sphere:

$$x^2 + y^2 + z^2 - a^2 = 0 \quad \rightarrow \quad g(x,y,z) = 0$$

or

$$z = \sqrt{a^2 - x^2 - y^2} \quad \rightarrow \quad z = f(x,y)$$

② Parameterization of a surface:
⇒ similar to parameterization of a curve, except we now need two variables

$$\vec{r}(\hat{u}, \hat{v}) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

For our sphere:

let $\left.\begin{array}{l} x = u \\ y = v \\ z = \sqrt{a^2 - u^2 - v^2} \end{array}\right\}$ is there a better ~~parametrizat~~ parameterization?

Finding The Normal:

for a surface represented as ①, ~~the Norma~~ a normal is:

$$\vec{N} = \vec{\nabla} g(x,y,z)$$

ex: $y = x^2 - z^3 \rightarrow g(x,y,z) = x^2 - z^3 - y = 0$

$$\vec{N} = \vec{\nabla} g = \langle 2x, -1, -3z^2 \rangle$$

the unit $\hat{n}$ormal is

$$\vec{n} = \vec{N}/|\vec{N}| = \vec{\nabla} g / |\vec{\nabla} g|$$

$$= \frac{\langle 2x, -1, -3z^2 \rangle}{\sqrt{4x^2 + 1 + 9z^4}}$$

For a parameterized surface,
$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
a normal is:

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

recall the symbology,
$\vec{r}_u$ refers to $\frac{\partial \vec{r}}{\partial u}$, &
$\vec{r}_v$ refers to $\frac{\partial \vec{r}}{\partial v}$

Flux Integrals:

What is flux? Flux $\frac{\text{amt}}{\text{unit area} \cdot \text{unit time}} \rightarrow \frac{kg}{m^2 s}$ one example

Imagine we want to know the amount of material flowing through a surface per unit time → how would we find that?

Flux integral!

$$\text{Flux} = \rho \cdot v = \frac{kg}{m^3} \cdot \frac{m}{s} = \frac{kg}{m^2 s}$$

by integrating flux over a surface we end up with kg/s

But how do we do a surface integral?

setup: $\iint_S \vec{F} \cdot \vec{n} \, dS$

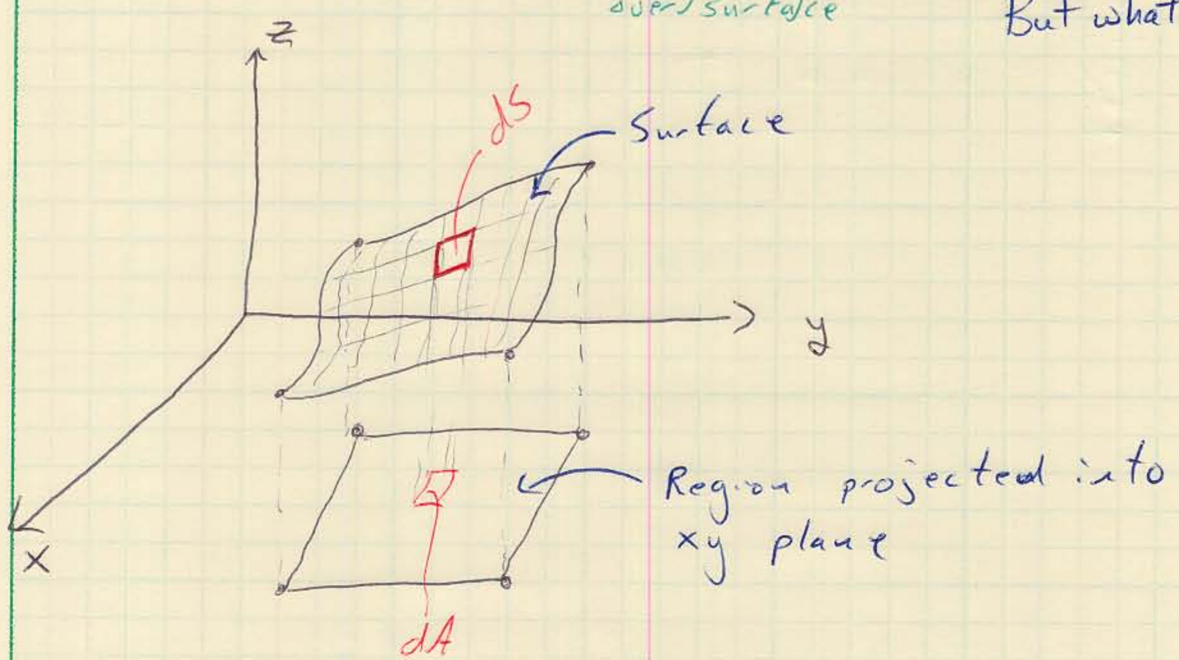
unit normal to surface

incremental piece of surface

But what is it?

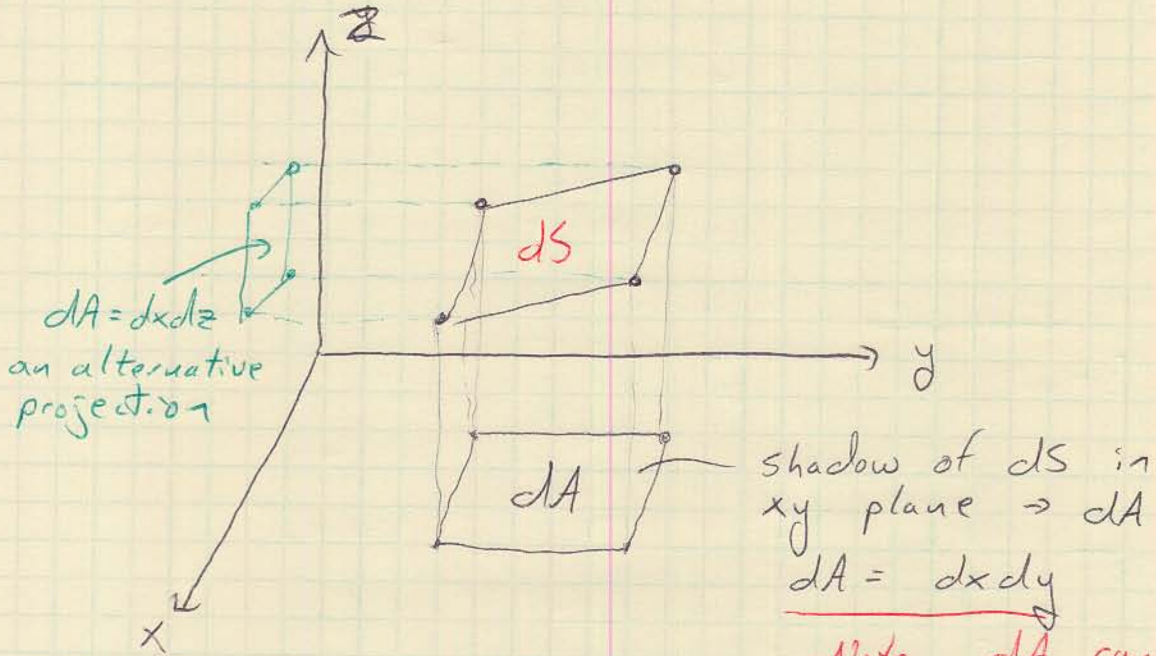
vector field integrating over surface

S → Surface



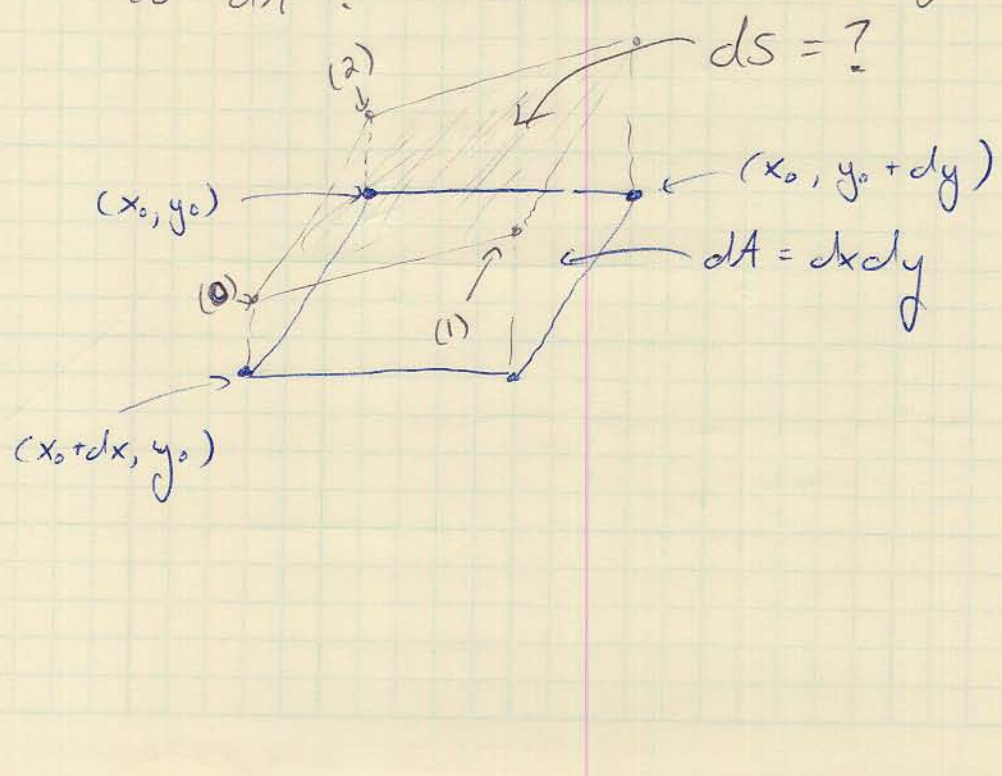
We know how to integrate over dA, so can we relate dS to dA?

Lets zoom in on our dS:



Note dA can also be $dy\,dz$ if projecting the surface onto yz plane, or $dx\,dz$ if projecting into xz plane

We've zoomed in so much that dS is a small flat rectangle at an ~~angt~~ angle to dA:



We know how to integrate over dA, so can we relate dS to dA?

let $\vec{u}$ ~~point~~ be the vector from point 0 to point 2:

$$\vec{u} = \langle dx, 0, \underbrace{a\,dx}\rangle$$

change in z, linear since we zoomed in so much

let $\vec{v}$ be the vector from 0 to point 1:

$$\vec{V} = \langle 0, dy, \underbrace{b\,dy}\rangle$$

change in z

the area of $dS = \vec{u} \times \vec{v} = \sqrt{1+a^2+b^2}\,dx\,dy$

Finally, what are a & b?

change in x, change in z

$$a\,dx = dz$$

$$a = \frac{dz}{dx},$$ so if our surface is in the form of

$$z = f(x, y),$$

then:

$$a = \frac{\partial f(x,y)}{\partial x} \text{ or } f_x$$

$$b\,dy = dz, \quad \text{so } b = f_y$$

Now:

$$dS = \sqrt{1+f_x^2+f_y^2}\,\overbrace{dx\,dy}^{dA}$$

Similar for different projections & surface reps.

$$dS = \sqrt{1+f_y^2+f_z^2}\,dy\,dz, \quad x = f(y,z)$$

$$dS = \sqrt{1+f_x^2+f_z^2}\,dx\,dz, \quad y = f(x,z)$$

Now: putting it together:

$$\iint_S \vec{F} \cdot \vec{n} \, dS$$

$$= \iint_R \vec{F} \cdot \vec{n} \sqrt{1+f_x^2+f_y^2} \, dx\,dy \quad \text{or}$$

$$\iint_R \vec{F} \cdot \vec{n} \sqrt{1+f_y^2+f_z^2} \, dy\,dz \quad \text{or}$$

$$\underbrace{\iint_R \vec{F} \cdot \vec{n} \sqrt{1+f_x^2+f_z^2} \, dx\,dz}$$

depending on surface representation

another form if using parameterized surface:

$$\iint_{R_{uv}} \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v) \, du\,dv$$

What about $\vec{n}$?

$\vec{n} = \vec{N}/|\vec{N}|$ and if:

the surface can be represented as

$z = f(x,y)$ ~~or $x = f(y,z)$ or $y = f(x,z)$~~

similar if $x = f(y,z)$ or $y = f(x,z)$

then put in the form:

$$g = z - f(x,y) = 0$$

$$\& \quad \vec{n} = \frac{\vec{\nabla} g}{|\vec{\nabla} g|} = \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{1 + f_x^2 + f_y^2}}$$

So: $\iint_S \vec{F} \cdot \vec{n} \, dS$

$$= \iint_R \vec{F} \cdot \overbrace{\frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+f_x^2+f_y^2}}}^{\vec{n}} \overbrace{\sqrt{1+f_x^2+f_y^2} \; \underbrace{dx\,dy}_{dA}}^{dS}$$

$$\boxed{\begin{aligned} &= \iint_R \vec{F} \cdot \langle -f_x, -f_y, 1\rangle \, dx\,dy \quad \text{if} \quad z = f(x,y) \\ &= \iint_R \vec{F} \cdot \langle -f_x, 1, -f_z\rangle \, dx\,dz \quad \text{if} \quad y = f(x,z) \\ &= \iint_R \vec{F} \cdot \langle 1, -f_y, -f_z\rangle \, dy\,dz \quad \text{if} \quad x = f(y,z) \end{aligned}}$$

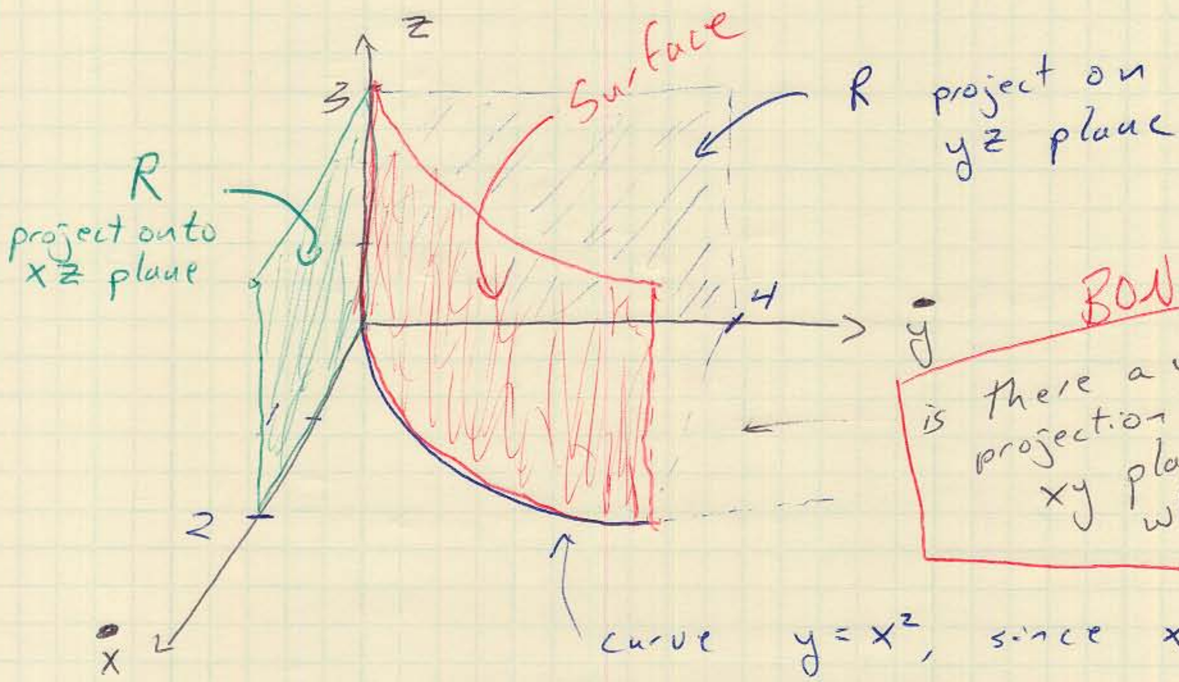
You must be able to use these & understand these!

Lets look at an example from the book:

Given $\vec{F} = \langle 3z^2, 6, 6xz\rangle$ → flux
& S: $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$

Find The amount of material flowing through the surface

① Plot the surface!



BONUS
is there a usable projection on xy plane? why or why Not?

curve $y = x^2$, since x goes from 0 to 2, y goes from 0 to 4

what about z?

since z goes from 0 to 3, imagine the curve raising out of the ground up to $z=3$

why did we project onto xz plane?

what form is our surface in?

$z = f(x,y)$

$x = f(y,z)$

$\boxed{y = f(x,z)}$ ← since in this form, project onto xz plane

~~Now~~ ② Setup integral:

a. $\iint_S \vec{F} \cdot \hat{n}\, dS$ ← all surface integrals start this way

b. Find Components

$\vec{F} = \langle 3z^2, 6, 6xz \rangle$ ← Given

$\vec{n} = ?$ given $y = x^2$ & defines surface

$y = f(x, z)$

$\underbrace{y - f(x, y)}_{g(x,y,z)} = 0$

$$\vec{n} = \frac{\vec{\nabla} g}{|\vec{\nabla} g|} = \frac{\langle -f_x, 1, -f_z \rangle}{\sqrt{1 + f_x^2 + f_y^2}}$$

$$-f_x = -\frac{\partial f(x,z)}{\partial x} = \frac{-\partial (x^2)}{\partial x} = -2x$$

$$-f_z = -\frac{\partial f(x,z)}{\partial z} = \frac{-\partial (x^2)}{\partial z} = 0$$

$$\vec{n} = \frac{\langle -2x, 1, 0 \rangle}{\sqrt{1 + 4x^2}}$$

$$dS = \sqrt{1 + f_x^2 + f_y^2}\ dA$$

$$= \sqrt{1 + 4x^2}\ dx\,dz$$

so

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \cancel{\langle 3z^2, 6, 6xz \rangle \cdot \langle -2x, 1,}$$

$$\iint_R \langle 3z^2, 6, 6xz \rangle \cdot \frac{\langle -2x, 1, 0 \rangle}{\cancel{\sqrt{1+4x^2}}} \cancel{\sqrt{1+4x^2}}\ dx\,dz$$

c. set up limits of integration:

what does x vary from?

Nice rectangular region

$0 \le x \le 2$

$0 \le z \le 3$

$$\int_0^3\int_0^2 \langle 3z^2, 6, 6xz\rangle \cdot \langle -2x, 1, 0\rangle \, dx\, dz$$

$$= \int_0^3\int_0^2 -6xz^2 + 6 \; dx\, dz$$

$$= \int_0^3 \left[-3x^2z^2 + 6x\right]\Big|_0^2 \, dz$$

$$= \int_0^3 12z^2 - 12 \; dz = \frac{12z^3}{3} - 12z\Big|_0^3 = \underline{\underline{72}}$$

Bonus: Set up the integral ~~as~~ by projecting the surface onto the yz plane instead of the xz plane

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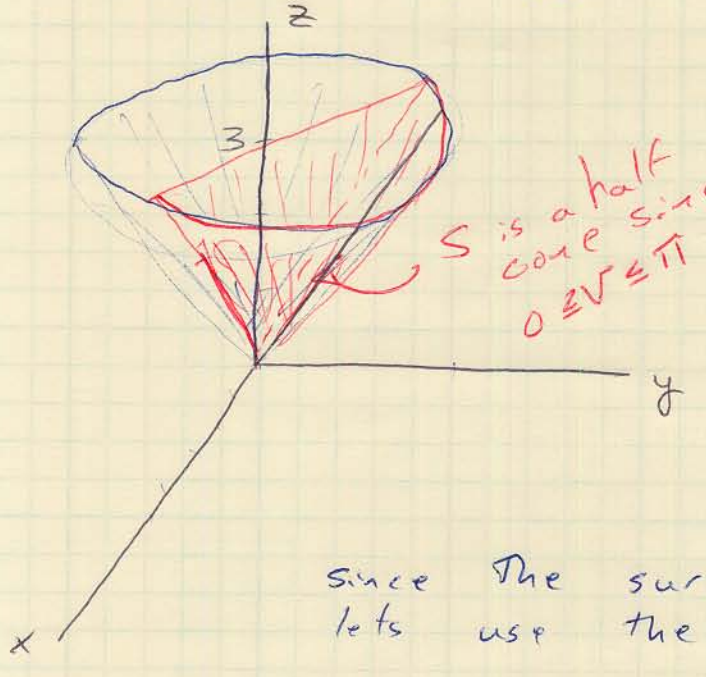
Given: $\vec{F} = \langle x-z,\ y-x,\ z-y \rangle$

$S: \vec{r} = \langle u\cos v,\ u\sin v,\ u \rangle$

$0 \le u \le 3$
$0 \le v \le \pi$

Find the flux integral

① Draw The Surface!!



what does
$\vec{r}(t) = \langle \cos t, \sin t \rangle$
give → a circle,
now
$\vec{r}(u,v) = \langle u\cos v, u\sin v, u \rangle$ circle
what happens to the radius of our circle as u increases?

Since The surface is parameterized, lets use the book method:

② Setup:

$$\iint_S \vec{F} \cdot \vec{n}\ dS$$

$$= \iint_{R_{uv}} \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\ du\ dv$$

only need to find $\vec{N}(u,v)$

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \langle \cos v, \sin v, 1 \rangle$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial \vec{v}} = \langle -u\sin v, u\cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= \vec{i}\,(-u\cos v) - \vec{j}\,(u\sin v) + \vec{k}\,(u\cos^2 v + u\sin^2 v)$$

$$= \langle -u\cos v, -u\sin v, u \rangle$$

$$\vec{F}(\vec{r}(u,v)) = \ ?$$

$$\vec{r}(u,v) = \langle \overbrace{u\cos v}^{x}, \overbrace{u\sin v}^{y}, \overbrace{u}^{z} \rangle$$

$$\vec{F}(x,y,z) = \langle x-z, y-x, z-y \rangle$$

substitute in $x(u,v)$ for x, etc

$$\vec{F}(\vec{r}(u,v)) = \langle u\cos v - u, u\sin v - u\cos v, u - u\sin v \rangle$$

$$\iint_{R_{uv}} \langle u\cos v - u, u\sin v - u\cos v, u - u\sin v \rangle \cdot \langle -u\cos v, -u\sin v, u \rangle \, du\, dv$$

$$-u^2\cos^2 v + u^2\cos v + -u^2\sin^2 v + u^2\cos v\sin v + u^2 - u^2\sin v$$

$$-u^2 + u^2 + u^2(\cos v - \sin v) + u^2\cos v\sin v$$

$$= \iint_{R_{uv}} u^2(\cos v - \sin v + \cos v\sin v)\, du\, dv$$

c. Setup limits & evaluate

$$\left.\begin{array}{l} 0 \le u \le 3 \\ 0 \le v \le \pi \end{array}\right\} \text{Given}$$

$$\int_0^{\pi}\int_0^3 u^2(\cos v - \sin v + \cos v \sin v)\,du\,dv$$

$$\frac{u^3}{3}\Big|_0^3 = 9$$

$$= \int_0^{\pi} 9(\cos v - \sin v + \cos v \sin v)\,dv$$

$$= 9\left(\sin v + \cos v + \frac{\sin^2 v}{2}\right)\Big|_0^{\pi}$$

$$= \underline{\underline{-18}}$$