Problem Set 6
MA364, AY09-1
Differential Vector Calculus

Due Beginning of Lesson 30 (12 November)

1. (10 pts) Pg. 425, #10
2. (7 pts) Pg. 432, #8
3. (8 pts) a. Pg. 432, #14
4. (10 pts) Pg. 444, #8
5. (10 pts) Pg. 456, #2
6. (10 pts) Pg. 456, #10
7. (10 pts) Pg. 463, #18
8. (10 pts) Pg. 473, #4

Problem Set 6 A409-1

#1 p.425, #10

Calculate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle x, -z, 2y \rangle$

from $(0,0,0)$ straight to $(1,1,0)$ then to $(1,1,1)$ and back to $(0,0,0)$.

C_1: $\vec{r}_1(t) = \langle t, t, 0 \rangle$, $t \in [0,1]$;

$d\vec{r}_1 = \langle 1,1,0 \rangle dt$

so $\int_{C_1} \vec{F}\cdot d\vec{r}_1 = \int_0^1 \langle t, 0, 2t \rangle \circ \langle 1,1,0 \rangle dt = \int_0^1 t\, dt = 1/2$

C_2: $\vec{r}_2(t) = \langle 1,1,0 \rangle + t\langle 0,0,1 \rangle$, $t \in [0,1]$ $d\vec{r}_2 = \langle 0,0,1 \rangle dt$

$\Rightarrow \int_{C_2} \vec{F}\cdot d\vec{r}_2 = \int_0^1 \langle 1, -t, 2 \rangle \circ \langle 0,0,1 \rangle dt = \underline{\underline{2}}$

C_3: $\vec{r}_3(t) = \langle 1-t, 1-t, 1-t \rangle$; $d\vec{r}_3 = \langle -1,-1,-1 \rangle dt$

so $\int_{C_3} \vec{F}\cdot d\vec{r}_3 = \int_0^1 \langle 1-t, t-1, 2-2t \rangle \circ \langle -1,-1,-1 \rangle dt$

$= \int_0^1 2t-2\, dt = t^2 - 2t \Big|_0^1 = -1$

$\therefore \oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r}_1 + \int_{C_2} \vec{F}\cdot d\vec{r}_2 + \int_{C_3} \vec{F}\cdot d\vec{r}_3 = \underline{\underline{3/2}}$ ans

#2 p. 432, #8

Show that the integrand is exact, and evaluate the integral.

$$\int_{(2,0,1)}^{(4,4,0)} \langle 2x(y^3-z^3),\ 3x^2y^2,\ -3x^2z^2\rangle \cdot \langle dx, dy, dz\rangle = \int_C \vec{F}\cdot d\vec{r}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x(y^3-z^3) & 3x^2y^2 & -3x^2z^2 \end{vmatrix} = \langle 0, -(-6xz^2-6xz^2),\ 6xy-6xy\rangle = \vec{0}$$

Note that $\vec{F} = \langle 2x(y^3-z^3),\ 3x^2y^2,\ -3x^2z^2\rangle = \vec{\nabla} f$

where $f(x,y,z) = x^2(y^3-z^3) + C$

So $\int_C \vec{F}\cdot d\vec{r} = f(4,4,0) - f(2,0,1)$

$= 4^2(4^3-0) - 2^2(0-1)$

$= 1024 + 4 = \underline{1028}$ ans

#3 p. 432, #14

Is $\vec{F} = \langle 2x \sin y, x^2 \cos y, y^2 \rangle$ conservative?

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x\sin y & x^2 \cos y & y^2 \end{vmatrix} = \langle 2y, 0, 0 \rangle \neq \vec{0}$$

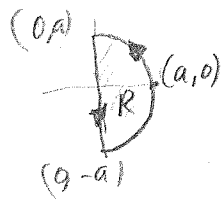
So the line integral connecting $(0,0,0)$ to (a,b,c) is not path independent.

#4 p. 444, #8

Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle e^x \cos y, -e^x \sin y \rangle = \langle P, Q \rangle$

R is the semi disk $x^2+y^2 \le a^2$

$x \ge 0$



Green's theorem applies:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (Q_x - P_y)\, dA$$

$$= \iint_R -e^x \sin y - (-e^x \sin y)\, dA = \underline{\underline{0}} \text{ ans}$$

Note that $\vec{F} = \vec{\nabla} f$ where $f = e^x \cos y$

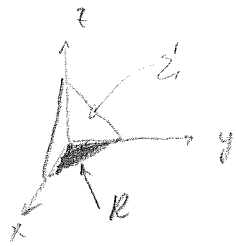
and $\oint_C \vec{\nabla} f \cdot d\vec{r} = 0.$

#5 P.456, #2

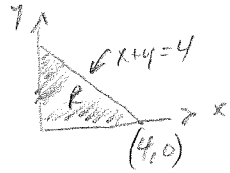
Evaluate the surface integral

$$\vec{F} = \langle x^2, y^2, z^2 \rangle$$

S: $x+y+z=4$ in 1st octant



$$\hat{n} = \frac{\langle 1,1,1 \rangle}{\sqrt{3}} \qquad dS = \sqrt{3}\, dA$$



$$\iint_S \vec{F} \cdot \hat{n}\, dS = \iint_R \vec{F} \cdot \frac{\langle 1,1,1 \rangle}{\sqrt{3}} \sqrt{3}\, dA$$

$$= \int_0^4 \int_0^{4-x} x^2 + y^2 + z^2 \, dy\, dx \qquad z = (4-x-y)$$

$$\int_0^4 \int_0^{4-x} \left[x^2 + y^2 + (4-x-y)^2 \right] dy\, dx$$

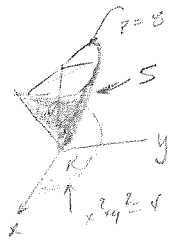
$$= \underline{\underline{64}} \text{ ans}$$

#6 p. 456, #10

$(x^2+y^2)^{1/2} = 2$

$x^2+y^2 = 4$

$\vec{F} = \langle y^2, x^2, z^4 \rangle$



$S: \; z = 4(x^2+y^2)^{1/2} \qquad 0 \le z \le 8 \qquad y \ge 0$

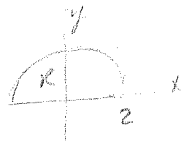
$$\hat{n} = \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{f_x^2+f_y^2+1}} \Rightarrow \quad f_x = 4(\tfrac{1}{2})(\;)^{-1/2} \cdot 2x$$

$$= 4x(\;)^{-1/2} = \frac{4x}{(\;)^{1/2}}$$

$$f_y = \frac{4y}{(\;)^{1/2}}$$

$$\hat{n} = \frac{\left\langle \frac{-4x}{(x^2+y^2)^{1/2}}, \frac{-4y}{(x^2+y^2)^{1/2}}, 1 \right\rangle}{\frac{(4x)^2+(4y)^2}{x^2+y^2}+1}$$

$$= \iint_R \langle y^2, x^2, z^4 \rangle \cdot \left\langle \frac{-4x}{(\;)^{1/2}}, \frac{-4y}{(\;)}, 1 \right\rangle dA$$



$$= \iint -4xy^2 - 4yx^2 + (4(x^2+y^2)^{1/2})^4 \cdot dA$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{\pi}\int_0^{2} \left(-4(r\cos\theta)(r\sin\theta)^2 - 4(r\sin\theta)(r\cos\theta)^2 + 256r^4\right) r\,dr\,d\theta$$

$$= \frac{32}{3}(256\pi - 1) = \underline{\underline{8567.98}}$$ ANS

#7 p. 463, #18

Use the divergence theorem to evaluate:

$$\oiint_{\Sigma} \vec{F}\cdot\hat{n}\, dS = \iiint_{M} \vec{\nabla}\cdot\vec{F}\, dV \quad \text{where } M \text{ is enclosed by } \Sigma.$$

$\vec{F} = \langle 4x, 3z, 5y \rangle$, S: Surface of cone $x^2+y^2 \le z^2$, $0 \le z \le 2$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial 4x}{\partial x} + \frac{\partial(3z)}{\partial y} + \frac{\partial(5y)}{\partial z} = 4$$

$$\text{So } \oiint_{\Sigma} \vec{F}\cdot\hat{n}\, dS = \iiint 4\, dV = 4\iiint dV$$

$= 4 \times$ volume of cone

$$= 4 \times \tfrac{1}{3}\pi r^2 h$$

$$= \tfrac{4}{3}\pi (2)^2\, 2$$

$$= \frac{32\pi}{3}$$

ans

divergence in cylindrical coords: (r, θ, z) $\qquad F = \langle 4x, 3z, 5y \rangle$

$$\vec{F} = \langle 4r\cos\theta,\ 3z,\ 5r\sin\theta \rangle$$

$$\operatorname{div}\vec{F} = \frac{1}{r}\frac{\partial}{\partial r}\underbrace{\left(r \cdot 4r\cos\theta\right)}_{4r^2\cos\theta} + \frac{1}{r}\frac{\partial(3z)}{\partial\theta} + \frac{\partial\, 5r\sin\theta}{\partial z}$$

$\frac{\cos\theta}{r}$ $\quad 8r = 8\cos\theta$

$$F = 4r\cos\theta\,\hat{\imath} + 3z\,\hat{\jmath} + 5r\sin\theta\,\hat{k} =$$

Note: unit vectors map to each

$$\hat{\imath} = \hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta$$
$$\hat{\jmath} = \hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta$$
$$\hat{k} = \hat{e}_z$$

$$\Rightarrow \vec{F} = 4r\cos\theta\left(\hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta\right) + 3z\left(\hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta\right) + 5r\sin\theta\,\hat{e}_z$$

$$4r\cos^2\theta\,\hat{e}_r - 4r\cos\theta\sin\theta\,\hat{e}_\theta + 3z\sin\theta\,\hat{e}_r + 3z\cos\theta\,\hat{e}_\theta + 5r\sin\theta\,\hat{e}_z$$

thus: $\vec{F}(r,\theta,z) = \langle (4r\cos^2\theta + 3z\sin\theta),\ -4r\cos\theta\sin\theta + 3z\cos\theta,\ 5r\sin\theta \rangle$

so $\operatorname{div}\vec{F}(r,\theta,z) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(4r\cos^2\theta + 3z\sin\theta\right)\right] + \frac{1}{r}\frac{\partial(-4r\cos\theta\sin\theta + 3z\cos\theta)}{\partial\theta} + \frac{\partial\, 5r\sin\theta}{\partial z}$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(4r^2\cos^2\theta + 3rz\sin\theta\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(-4r\cos\theta\sin\theta + 3z\cos\theta\right) + 0$$

$$= \frac{1}{r}\left(8r\cos^2\theta + \cancel{3z\sin\theta}\right) + \frac{1}{r}(-4r)\left(\cos^2\theta - \sin^2\theta\right) - \cancel{\frac{3z\sin\theta}{r}}$$

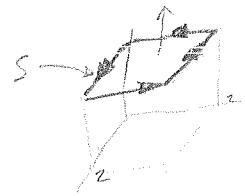
$$8\cos^2\theta - 4\cos^2\theta + 4\sin^2\theta$$

$$= \underline{\underline{4}}\ !!$$

so we have $\iiint 4\,dV = 4 \times \text{vol of cone } r=2, h=2 = \frac{4}{3}\pi r^2 h = \underline{\underline{\frac{32\pi}{3}}}$ ans

#8, p.473, #4

Evaluate $\iint_{\Sigma_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, dA$ for the square $0 \le x \le 2$, $0 \le y \le 2$, $z = 4$



$\vec{F} = \langle 3\cos y, \cosh z, x \rangle$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3\cos y & \cosh z & x \end{vmatrix} = \langle \sinh z, -1, +3\sin y \rangle$$

note that on $z = 2$, $\hat{n} = \hat{k}$

so $(\vec{\nabla} \times \vec{F}) \cdot \hat{n} = \langle \sinh z, -1, 3\sin y \rangle \cdot \langle 0, 0, 1 \rangle$

$= 3\sin y$

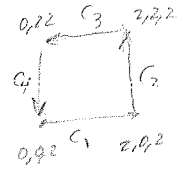
So $\iint_{\Sigma_1} \vec{\nabla} \times \vec{F} \cdot \hat{n} \, dA = \int_0^2 \int_0^2 3\sin y \, dy \, dx$

$= 6 \; -\cos y \Big|_0^2$

$= \underline{\underline{-6(\cos 2 - 1)}}$ ANS

Comment:

note that $\iint_{\Sigma} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS = \oint_{C} \vec{F} \cdot d\vec{r}$



$\vec{r}_1(t) = \langle 2t, 0, 2 \rangle$; $d\vec{r}_1 = \langle 2, 0, 0 \rangle dt$

$\vec{r}_2(t) = \langle \overset{x}{2}, \overset{y}{2t}, \overset{z}{2} \rangle$; $d\vec{r}_2 = \langle 0, 2, 0 \rangle dt$

$\vec{r}_3(t) = \langle 2-2t, 2, 2 \rangle$ $d\vec{r}_3 = \langle -2, 0, 0 \rangle dt$

$\vec{r}_4(t) = \langle 0, 2-2t, 2 \rangle$ $d\vec{r}_4 = \langle 0, -2, 0 \rangle dt$

$$\boxed{\vec{F} = \langle 3\cos y, \cosh z, x \rangle}$$

C_1: $\int_{C_1} \vec{F} \cdot d\vec{r}_1 = \int_0^1 \langle 3, \cosh 2, 0 \rangle \cdot \langle 2, 0, 0 \rangle dt = 6$

C_2: $\int_{C_2} \vec{F} \cdot d\vec{r}_2 = \int_0^1 \langle 3\cos(2t), \cosh 2, 2 \rangle \cdot \langle 0, 2, 0 \rangle dt = 2\cosh 2$

C_3: $\int_{C_3} \vec{F} \cdot d\vec{r}_3 = \int_0^1 \langle 3\cos 2, \cosh 2, 2-2t \rangle \cdot \langle -2, 0, 0 \rangle dt = -6\cos 2$

C_4: $\int_{C_4} \vec{F} \cdot d\vec{r}_4 = \int_0^1 \langle 3\cos(2-2t), \cosh 2, 0 \rangle \cdot \langle 0, -2, 0 \rangle dt = -2\cosh 2$

$\Rightarrow \oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = \underline{\underline{6 - 6\cos 2}}$ (as on previous page)