MA 371 Homework 1. Selected solutions

Problem 1 (c), (d). Any vector, $\mathbf{w} = (w_1, w_2, w_3, w_4, w_5)$ which is orthogonal to $\mathbf{v}_1 = [0, 2, 1, 0, 2], \mathbf{v}_2 = [1, 0, 0, 2, 0], \mathbf{v}_3 = [0, 1, 2, 0, 1]$, and $\mathbf{v}_4 = [2, 0, 0, 1, 0]$ must satisfy $\mathbf{w} \cdot \mathbf{v}_i = 0$ for $i = 1, 2, 3, 4$. This can be formulated in matrix form as

$$\begin{pmatrix} 0 & 2 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = 0.$$

The RREF of A is $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, from which is easy to see that any scalar multiple of the vector $\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$.

To show that there does not exist a non-zero vector in $\mathcal{R}^5$ which is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, and also to $(0, -1, 0, 0, 1)$, we proceed as above and find the RREF form of a matrix the rows of which comprise these five vectors. If

$$B = \begin{pmatrix} 0 & 2 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix},$$

then the RREF of B is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

from which we see that the only solution to $B\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = (0, 0, 0, 0, 0)$. Thus, there is no non-zero vector orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, and also to $(0, -1, 0, 0, 1)$.

2. (a). If $\mathbf{x} = k\mathbf{y}$ for some scalar, k, then using Theorem 1.2.2 (c) we have

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\| &= \|k\mathbf{y} + \mathbf{y}\| \\ &= \|(k+1)\mathbf{y}\| \\ &= |k+1|\|\mathbf{y}\|. \end{aligned}$$

On the other hand, again using Theorem 1.2.2 (c),

$$\begin{aligned} \|\mathbf{x}\| + \|\mathbf{y}\| &= \|k\mathbf{y}\| + \|\mathbf{y}\| \\ &= |k|\|\mathbf{y}\| + \|\mathbf{y}\| \\ &= (|k| + 1)\|\mathbf{y}\|. \end{aligned}$$

We see that if $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\|+\|\mathbf{y}\|$ for arbitrary $\mathbf{x}$ and $\mathbf{y}$, we must have $|k+1| = |k|+1$, which is true if and only if $k \geq 0$. Thus, the statement is not true for all scalars k, but *is* true for $k \geq 0$.

2. (b) Using Theorem 1.2.6 (c) the left-hand side of Cauchy-Schwarz is

$$\begin{aligned} |\mathbf{x} \cdot \mathbf{y}| &= |k\mathbf{y} \cdot \mathbf{y}| \\ &= |k||\mathbf{y} \cdot \mathbf{y}| \\ &= |k|\|\mathbf{y}\|^2, \end{aligned}$$

while the right-hand side of Cauchy-Schwarz gives, using Theorem 1.2.2 (c),

$$\begin{aligned} \|\mathbf{x}\|\|\mathbf{y}\| &= \|k\mathbf{y}\|\|\mathbf{y}\| \\ &= |k|\|\mathbf{y}\|\|\mathbf{y}\| \\ &= |k|\|\|\mathbf{y}\|^2. \end{aligned}$$

Thus the statement is true.

3.(a) As suggested in the hints, write

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2. \end{aligned}$$

So if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, we see that $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$. This implies that $\mathbf{u} \cdot \mathbf{v} = 0$, that is, that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. So the statement is true.

3 (b) was done in class, and also in the hints. (It's true.)

3. (c) This one is false. The easiest way to see this is to to draw a picture.

5. (a)(First system) The augmented matrix for the first system is $\begin{pmatrix} 2 & 3 & -1 & 1 \\ 4 & 6 & -2 & 0 \end{pmatrix}$, which has RREF $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so the system has no solution. Geometrically, this corresponds to the fact that the planes $2x + 3y - z = 1$ and $4x + 6y - 2z = 0$ do not intersect.

5 (b) (First system)

The coefficient matrix for the first system is $\begin{pmatrix} 2 & 3 & -1 \\ 4 & 6 & -2 \end{pmatrix}$. The matrix comprises the following column vectors in $\mathcal{R}^2$: $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$. The left-hand side of the system can be written as a linear combination of these vectors:

$$x \begin{pmatrix} 2 \\ 4 \end{pmatrix} + y \begin{pmatrix} 3 \\ 6 \end{pmatrix} + z \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

But note that these vectors all lie on the same line, since they are multiples of each other. Any linear combination of the these must also lie on the same line. Geometrically from this perpective, the system having no solution corresponds to the fact that the "target" vector $(1, 0)^T$ does *not* lie on the same line.

5. (a) (Second system) The RREF for the augmented matrix of the second system is $\begin{pmatrix} 1 & 0 & -1/2 & -1/10 \\ 0 & 1 & 0 & 2/5 \end{pmatrix}$. The variable z is free, so the system has infinite solutions. (Why?). Geometrically, this corresponds to the fact that the planes $2x + 3y - z = 1$ and $4x + y - 2z = 0$ *do* intersect; two planes intersect in a line (an infinite number of points). This time column vectors for the matrix are *not* multiples of each other (though the first and the third are). So the system would have infinite solutions given *any* target vector. (Why?)

Problems 6 and 7 done in Mathematica.