MA 396 Lesson 10 - Board Problems

For problems 1 – 3, solve the differential equations. You may use *Mathematica* if you desire. Verify your result satisfies the initial condition and satisfies the differential equation.

1. $y' = \dfrac{4t^3 y}{1+t^4}$, $0 \le t \le 1$, $y(0) = 1$.

2. $y' + 3y = t + 1$, $y(0) = 2$.

For problems 3 – 4:
(*i*) Does f satisfy a Lipschitz condition on $D = \{(t, y) | 0 \le t \le 1, -\infty < y < \infty\}$?
(*ii*) Can Theorem 5.6 be used to show that the initial value problem $y' = f(t, y)$, $0 \le t \le 1$, $y(0) = 1$ is well-posed?

3. $f(t, y) = ty$

4. $f(t, y) = -ty + \dfrac{4t}{y}$

5. You're trying to reinvent Coca-Cola. Develop the differential equation that models the rate of change of sugar in a mixing tank described below:

The tank has a flow of clear water coming in at a rate of 3 gal/sec and a flow of solution coming in at a rate of 2 gal/sec. The concentration of solution entering the tank is ½ lb/gal. The flow rate out of the 150-gallon tank is 5 gal/sec. Assume pumps are turned on when the tank is full of clear water.

Define your variables! Solve your model. What is the long-term behavior of the sugar in the tank?

1. $\int \frac{dy}{y} = \int \frac{4t^3}{1+t^4}\, dt$ $\qquad u = 1+t^4$, $du = 4t\, dt$

$\ln y = \int \frac{du}{u} = \ln(1+t^4) + c$

$y = (1+t^4)e^c \qquad c = 0$

$\underline{\underline{y = 1+t^4}} \qquad y(0) = 1+0 = 1$ ✓

$y' = 4t^3$

$4t^3 = \frac{4t^3\cancel{(1+t^4)}}{\cancel{1+t^4}} = 4t^3$ ✓

2. USE A INTEGRATING FACTOR TO SOLVE.

$y' + 3y = t+1 \quad \Rightarrow \mu(t) = e^{\int 3 dt} = e^{3t}$

$y(t) = \frac{1}{\mu(t)} \int \mu(t) r(t)\, dt$

$y(t) = \frac{1}{e^{3t}} \int e^{3t}(t+1)\, dt$

$e^{3t} y = \underbrace{\int e^{3t} t\, dt}_{\text{INTEGRATION BY PARTS}} + \int e^{3t}\, dt$

$u = t \qquad dv = e^{3t}\, dt$

$du = dt \qquad v = \frac{1}{3}e^{3t}$

$$e^{3t}y = uv - \int v\,du + \frac{1}{3}e^{3t} + c$$

$$e^{3t}y = \frac{1}{3}e^{3t}t - \int \frac{1}{3}e^{3t}\,dt + \frac{1}{3}e^{3t} + c$$

$$e^{3t}y = \frac{1}{3}e^{3t}t - \frac{1}{9}e^{3t} + \frac{1}{3}e^{3t} + c$$

$$y = \frac{1}{3}t - \frac{1}{9} + \frac{1}{3} + ce^{-3t}$$

$$y = \frac{2}{9} + \frac{1}{3}t + ce^{-3t}$$

$$y(0) = 2 = \frac{2}{9} + c \Rightarrow c = \frac{16}{9}$$

$$\underline{y = \frac{2}{9} + \frac{1}{3}t + \frac{16}{9}e^{-3t}}$$

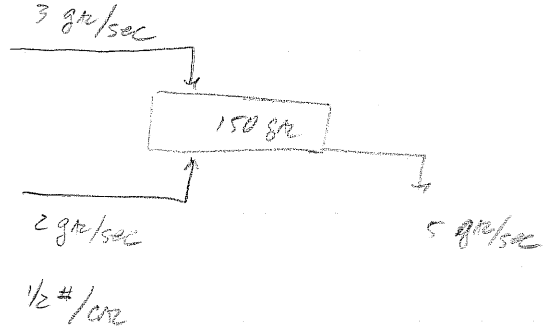
ANS.

3. $\frac{\partial f}{\partial y} = t \leq 1$ & CONTINUOUS ✓

4. $\frac{\partial f}{\partial y} = -t - \frac{4t}{y^2}$ ∴ NOT DEFINED @ $y = 0$.

THM 5.6 CAN NOT BE USED!

5.



LET S = CONCENTRATION OF SUGAR IN TANK

$S' = \text{IN} - \text{OUT}$

$S' = \frac{.5\#}{gal}\,\frac{2\,gal}{sec} - \frac{S}{150\,gal}\left(\frac{5\,gal}{sec}\right.$

$S' = 1 - \frac{S}{30}$

FROM MATHEMATICA - $S(t) = 30 - 30e^{-t/30}$ $\quad t \to \infty,\ S = 30$

Lesson 10 Boards

1.

```
sol = DSolve[{y'[t] == (4*t^3*y[t]) / (1 + t^4), y[0] == 1}, y, t]
```

$\{\{y \to \text{Function}[\{t\}, 1 + t^4]\}\}$

```
y[0] /. sol
```

{1}

```
y'[t] == (4*t^3*y[t]) / (1 + t^4) /. sol
```

{True}

2.

```
sol1 = DSolve[{y'[t] + 3*y[t] == t + 1, y[0] == 2}, y, t]
```

$\left\{\left\{y \to \text{Function}\left[\{t\}, \frac{1}{9} e^{-3t} (16 + 2 e^{3t} + 3 e^{3t} t)\right]\right\}\right\}$

```
y[0] /. sol1
```

{2}

```
y'[t] + 3*y[t] == t + 1 /. sol1
```

$\left\{\frac{1}{9} e^{-3t} (9 e^{3t} + 9 e^{3t} t) == 1 + t\right\}$

5.

```
In[1]:= sol1 = DSolve[{s'[t] == 1 - s[t] / 30, s[0] == 0}, s, t]
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Out[1]= $\{\{s \to \text{Function}[\{t\}, 30\, e^{-t/30} (-1 + e^{t/30})]\}\}$