MA396 BOARD PROBLEMS: Lesson 3

Modeling with Differential Equations

1. Newton's Law of Cooling says the rate of change of an object's temperature is proportional to the difference between the object's temperature and the temperature of the surrounding medium. When the casing for a claymore mine is removed from the baking oven, its temperature is measured at 300°F. Three minutes later, its temperature is measured at 200°F. How long will it take the mine casing to cool off to within one degree of the room temperature of 70°F? $\rightarrow$ 28.59 minutes.

$$\frac{dT}{dt} = k(T-70) \qquad T - 70 = C_2 e^{kt}$$

$$\int \frac{dT}{T-70} = k\,dt \qquad T = 70 + C_2 e^{kt} \qquad T(0) = 300$$

$$\underline{T = 70 + 230\, e^{-.1902t}} \qquad T(3) = 200$$

$$\ln|T-70| = kt + c_1 \qquad T(\underline{28.59}) = 71$$

2. Suppose a plebe returns from a pass to USMA carrying a flu virus. It is assumed that the rate at which the virus spreads is proportional not only to the number of infected cadets but also to the number of cadets not infected. After 4 days, 50 cadets are infected. Model the spread (rate of change) of the infected population.

T = CORPS $\qquad T(0) = 1$

$T(4) = 50$

$$\frac{dS}{dt} = kS(T-S)$$

$$\underline{\frac{dS}{dt} = kS(T-S)}$$

ANS

3. The rate of decay of carbon-14 in a dead organism is assumed proportional to the amount of carbon-14 present. Model the amount of carbon-14 with a differential equation if the half-life is 4166.6667 years. Assume the initial amount of carbon-14 is 2000 kg. Find the general and particular solution. Determine when 25% of the carbon-14 is left. What are the units of the decay rate parameter?

$$\frac{dc}{dt} = \lambda c \qquad \frac{1}{2} = e^{-\lambda(4166.6667)} \Rightarrow \lambda = \underline{.000166}$$

$$c(t) = C_0 e^{-\lambda t} \qquad c(t) = 2000\, e^{-.000166t}$$

$$\frac{1}{4} = e^{-.000166t} \Rightarrow t = \underline{8333.33}$$

UNITS OF DECAY RATE PAR = $\frac{1}{\text{YEARS}}$

4. The rate of change in the population of New York State over one year is proportional to the product of the population and the difference between 25 million and the population. On January 1, 1980, the population was 10.5 million. On January 1, 1990, the population was 19.8 million. Model the population growth of New York State continuously using a differential equation. Based on the differential equation, what is the assumed maximum population of New York State? What will be the population of the state on January 1, 2003?

$$\frac{dP}{dt} = k\,P(25-P) \qquad P(0) = 10.5 \qquad P(10) = 19.8$$

SEE ATTACHED MATHEMATICA NOTEBOOK

In[2]:= **f = 1 / (P * (25 - P))**

Out[2]= $\frac{1}{(25-P)\ P}$

In[4]:= **a1 = Integrate[f, P]**

Out[4]= $-\frac{1}{25}\ \text{Log}[-25+P]+\frac{\text{Log}[P]}{25}$

In[5]:= **a2 = Integrate[K, t]**

Out[5]= $K\ t$

In[10]:= **ans = Exp[a1] == C Exp[a2]**

Out[10]= $\frac{P^{1/25}}{(-25+P)^{1/25}} == C\ e^{K\ t}$

In[11]:= **Solve[ans, P]**

Out[11]= $\left\{\left\{P\to\frac{25\ C^{25}\ e^{25\ K\ t}}{-1+C^{25}\ e^{25\ K\ t}}\right\}\right\}$

Solve for C

In[14]:= $\mathbf{10.5 == \frac{25\ C^{25}}{-1+C^{25}}}$

Out[14]= $10.5 == \frac{25\ C^{25}}{-1+C^{25}}$

In[15]:= **Solve[%, C]**

Out[15]=
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{{C → -0.987172}, {C → -0.956158 - 0.2455 i}, {C → -0.956158 + 0.2455 i},
 {C → -0.865065 - 0.475574 i}, {C → -0.865065 + 0.475574 i},
 {C → -0.719617 - 0.675766 i}, {C → -0.719617 + 0.675766 i},
 {C → -0.528953 - 0.833497 i}, {C → -0.528953 + 0.833497 i}, {C → -0.305053 - 0.938856 i},
 {C → -0.305053 + 0.938856 i}, {C → -0.061985 - 0.985224 i}, {C → -0.061985 + 0.985224 i},
 {C → 0.184978 - 0.969687 i}, {C → 0.184978 + 0.969687 i}, {C → 0.420317 - 0.89322 i},
 {C → 0.420317 + 0.89322 i}, {C → 0.629247 - 0.760629 i}, {C → 0.629247 + 0.760629 i},
 {C → 0.798639 - 0.580245 i}, {C → 0.798639 + 0.580245 i}, {C → 0.917849 - 0.363402 i},
 {C → 0.917849 + 0.363402 i}, {C → 0.979388 - 0.123725 i}, {C → 0.979388 + 0.123725 i}}
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In[16]:= $\mathbf{P[t\_] = \frac{25\ (-0.9871720529022844)^{25}\ e^{25\ K\ t}}{-1+(-0.9871720529022844)^{25}\ e^{25\ K\ t}}}$

Out[16]= $-\frac{18.1034\ e^{25\ K\ t}}{-1-0.724138\ e^{25\ K\ t}}$

In[17]:= **P[10]**

Out[17]= $-\frac{18.1034\ e^{250\ K}}{-1-0.724138\ e^{250\ K}}$

In[18]:= **Solve[% == 19.8, K]**

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Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More...
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Out[18]= {{K → 0.00663919}}

In[19]:= **Sol[t_] =** $$\frac{25\,(-0.9871720529022844)^{25}\,e^{25*0.0066391868175046395\,t}}{-1+(-0.9871720529022844)^{25}\,e^{25*0.0066391868175046395\,t}}$$ ← Particular Solution

Out[19]= $$-\frac{18.1034\,e^{0.16598\,t}}{-1-0.724138\,e^{0.16598\,t}}$$

▪ Double Check Answer

In[20]:= **Sol[0]**

Out[20]= 10.5

In[21]:= **Sol[10]**

Out[21]= 19.8

What will population be in 100 years?

In[22]:= **Sol[100]**

Out[22]= 25.

What will population be in 23 years?

In[24]:= **Sol[23]**

Out[24]= 24.2635 ← Population @ 2008.