(a) In which system does the prey reproduce more quickly when there are no predators (when $y = 0$) and equal numbers of prey?

(b) In which system are the predators more successful at catching prey? In other words, if the number of predators and prey are equal for the two systems, in which system do the predators have a greater effect on the rate of change of the prey?

(c) Which system requires more prey for the predators to achieve a given growth rate (assuming identical numbers of predators in both cases)?

20. The system

$$\frac{dx}{dt} = ax - by\sqrt{x}$$
$$\frac{dy}{dt} = cy\sqrt{x}$$

has been proposed as a model for a predator-prey system of two particular species of microorganisms (where a, b, and c are positive parameters).

(a) Which variable, x or y, represents the predator population? Which variable represents the prey population?

(b) What happens to the predator population if the prey is extinct?

21. The following systems are models of the populations of pairs of species that either *compete* for resources (an increase in one species decreases the growth rate of the other) or *cooperate* (an increase in one species increases the growth rate of the other). For each system, identify the variables (independent and dependent) and the parameters (carrying capacity, measures of interaction between species, etc.) Do the species compete or cooperate? (Assume all parameters are positive.)

(a)
$$\frac{dx}{dt} = \alpha x - \alpha\frac{x^2}{N} + \beta xy$$
$$\frac{dy}{dt} = \gamma y + \delta xy$$

(b)
$$\frac{dx}{dt} = -\gamma x - \delta xy$$
$$\frac{dy}{dt} = \alpha y - \beta xy$$

1.2 ANALYTIC TECHNIQUE: SEPARATION OF VARIABLES

What Is a Differential Equation and What Is a Solution?

A first-order differential equation is an equation for an unknown function in terms of its derivative. As we saw in the previous section, there are three types of "variables" in differential equations — the independent variable (almost always t for time in our examples), one or more dependent variables (which are functions of the independent variable), and the parameters. This terminology is standard but a bit confusing. The dependent variable is actually a function, so technically it should be called the dependent function.

The standard form for a first-order differential equation is

$$\frac{dy}{dt} = f(t, y).$$

Here the right-hand side typically depends on both the dependent and independent variables, although we often encounter cases where either t or y is missing.

A **solution** of the differential equation is a function of the independent variable that, when substituted into the equation as the dependent variable, satisfies the equation for all values of the independent variable. That is, a function $y(t)$ is a solution if it satisfies $dy/dt = y'(t) = f(t, y(t))$. This terminology doesn't tell us how to find solutions, but it does tell us how to check whether a candidate function is or is not a solution. For example, consider the simple differential equation

$$\frac{dy}{dt} = y.$$

We can easily check that the function $y_1(t) = 3e^t$ is a solution, whereas $y_2(t) = \sin t$ is not a solution. The function $y_1(t)$ is a solution because

$$\frac{dy_1}{dt} = \frac{d(3e^t)}{dt} = 3e^t = y_1 \quad \text{for all } t.$$

On the other hand, $y_2(t)$ is not a solution since

$$\frac{dy_2}{dt} = \frac{d(\sin t)}{dt} = \cos t,$$

and certainly the function $\cos t$ is not the same function as $y_2(t) = \sin t$.

Checking that a given function is a solution to a given equation

If we look at a more complicated equation such as

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t},$$

then we have considerably more trouble finding a solution. On the other hand, if somebody hands us a function $y(t)$, then we know how to check whether or not it is a solution.

For example, suppose we meet three differential equations textbook authors — say Paul, Bob, and Glen — at our local espresso bar, and we ask them to find solutions of this differential equation. After a few minutes of furious calculation, Paul says that

$$y_1(t) = 1 + t$$

is a solution. Glen then says that

$$y_2(t) = 1 + 2t$$

is a solution. After several more minutes, Bob says that

$$y_3(t) = 1$$

is a solution. Which of these functions is a solution? Let's see who is right by substituting each function into the differential equation.

First we test Paul's function. We compute the left-hand side by differentiating $y_1(t)$. We have

$$\frac{dy_1}{dt} = \frac{d(1+t)}{dt} = 1.$$

Substituting $y_1(t)$ into the right-hand side, we find

$$\frac{(y_1(t))^2 - 1}{t^2 + 2t} = \frac{(1+t)^2 - 1}{t^2 + 2t} = \frac{t^2 + 2t}{t^2 + 2t} = 1.$$

The left-hand side and the right-hand side of the differential equation are identical, so Paul is correct.

To check Glen's function, we again compute the derivative

$$\frac{dy_2}{dt} = \frac{d(1+2t)}{dt} = 2.$$

With $y_2(t)$, the right-hand side of the differential equation is

$$\frac{(y_2(t))^2 - 1}{t^2 + 2t} = \frac{(1+2t)^2 - 1}{t^2 + 2t} = \frac{4t^2 + 4t}{t^2 + 2t} = \frac{4(t+1)}{t+2}.$$

The left-hand side of the differential equation does not equal the right-hand side for all t since the right-hand side is not the constant function 2. Glen's function is *not* a solution.

Finally, we check Bob's function the same way. The left-hand side is

$$\frac{dy_3}{dt} = \frac{d(1)}{dt} = 0$$

because $y_3(t) = 1$ is a constant. The right-hand side is

$$\frac{y_3(t)^2 - 1}{t^2 + t} = \frac{1 - 1}{t^2 + t} = 0.$$

Both the left-hand side and the right-hand side of the differential equation vanish for all t. Hence, Bob's function *is* a solution of the differential equation.

The lessons we learn from this example are that a differential equation may have solutions that look very different from each other algebraically and that (of course) not every function is a solution. Given a function, we can test to see whether it is a solution by just substituting it into the differential equation and checking to see whether the left-hand side is identical to the right-hand side. This is a very nice aspect of differential equations: *We can always check our answers.* So we should never be wrong.

Initial-Value Problems and the General Solution

When we encounter differential equations in practice, they often come with **initial conditions**. We seek a solution of the given equation that assumes a given value at a particular time. A differential equation along with an initial condition is called an **initial-value problem**. Thus the usual form of an initial-value problem is

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

Here we are looking for a function $y(t)$ that is a solution of the differential equation *and* assumes the value y_0 at time t_0. Often, the particular time in question is $t = 0$ (hence the name *initial condition*), but any other time could be specified.

For example,

$$\frac{dy}{dt} = t^3 - 2\sin t, \quad y(0) = 3.$$

is an initial-value problem. To solve this problem, note that the right-hand side of the differential equation depends only on t, not on y. We are looking for a function whose derivative is $t^3 - 2\sin t$. This is a typical antidifferentiation problem from calculus, so all we need to do is to integrate this expression. We find

$$\int (t^3 - 2\sin t)\,dt = \frac{t^4}{4} + 2\cos t + c,$$

where c is a constant of integration. Thus the solution of the differential equation must be of the form

$$y(t) = \frac{t^4}{4} + 2\cos t + c.$$

We now use the initial condition $y(0) = 3$ to determine c by

$$3 = y(0) = \frac{0^4}{4} + 2\cos 0 + c = 0 + 2\cdot 1 + c = 2 + c.$$

Thus $c = 1$, and the solution to this initial-value problem is

$$y(t) = \frac{t^4}{4} + 2\cos t + 1.$$

The expression

$$y(t) = \frac{t^4}{4} + 2\cos t + c$$

is called the **general solution** of the differential equation because we can use it to solve any initial-value problem whatsoever. For example, if the initial condition is $y(0) = \pi$, then we would choose $c = \pi - 2$ to solve the initial-value problem $dy/dt = t^3 - 2\sin t$, $y(0) = \pi$.

Separable Equations

Now that we know how to check that a given function is a solution to a differential equation, the question is: How can we get our hands on a solution in the first place? Unfortunately, it is rarely the case that we can find explicit solutions of a differential equation. Many differential equations have solutions that cannot be expressed in terms of known functions such as polynomials, exponentials, or trigonometric functions. However, there are a few special types of differential equations for which we can derive explicit solutions, and in this section we discuss one of these types of differential equations.

The typical first-order differential equation is given in the form

$$\frac{dy}{dt} = f(t, y).$$