Lesson #1 - Calculus Review

1. Assignment: Sec 1.1, pp 1-17.

2. Objectives:
   a. To become familiar with the notation used in the text.
   b. To quickly review calculus and locate applicable theorems that will be used in the course.
   c. To be introduced to the concept of truncation error that will be dealt with later.
   d. To become familiar with the software package *Mathematica*.

3. Suggested Problems: Sec 1.1: 1a, 3c, 5, 11 and the following review problem:
Given $f(x) = e^x + 5$:
   (1) Show that $f(x)$ is continuous at $x_0 = 0$.
   (2) Is $f(x)$ differentiable at $x_0 = 0$?

4. Computer Experiment: Use *Mathematica* and try to integrate the function $\frac{1}{2}\int_{-1}^{1} \frac{\sin(x)}{x}\,dx$. What does *Mathematica* return as a result? Why is this function difficult to integrate? Can you think of a better way to approximate this integral? Using a sixth-degree Taylor approximation for $\sin(x)$ about the point $x_0 = 0$, attempt the integration again. What problem did you eliminate by using the Taylor approximation?

Lesson #2 - Round-Off Errors and Computer Arithmetic; Algorithms and Convergence

1. Assignment: Sec1.2 – 1.3, pp 18-39. Review *equilibrium values* from MA103.

2. Objectives:
   a. To know the difference between absolute error and relative error.
   b. To understand the different effects of rounding and chopping numbers.
   c. To obtain bounds in terms of the quantity $\left|1 - \frac{y}{x}\right|$, which is a convenient measure for the *closeness* of x and y. This aids us in determining how many significant digits are lost in the subtraction of x and y.
   d. To determine if a numerical process is stable or unstable and how sensitive the solution of a problem may be to small changes in the input data.
   e. To understand what linear and exponential error growth and their roles in determining whether an algorithm is stable or unstable.

3. Suggested Problems: Sec 1.2: 1a, 2a, 3a; Sec 1.3: 6a, 7c.

Lesson #3 – Differential Equations Models

1. Assignment: BDH: Sec 1.1, pgs 1 - 19

2. Objectives:
   a. Understand the terminology used:
      1) Independent variable, dependent variable, and parameters.
      2) Ordinary differential equation to include classification.
      3) Initial Value Problem (IVP).
      4) Equilibrium solution.
   b. Model physical phenomenon using differential equations. These phenomena include:
      1) Proportional growth/decay problems
      2) Nonlinear population models (including the logistic equation)
      3) Newton's Law of Cooling
      4) Mixing Problems
   c. Verify a solution to a differential equation by substituting the solution into the differential equation.

3. Suggested Problems: BDH: Sec 1.1: 9, 13, 19

Lesson #4 – Introduction to *Mathematica*

1. Assignment: Instructor handout.

2. Objective: Use *Mathematica* to numerically and graphically solve ordinary differential equations.

Lesson #5 – Project #1 In-Progress-Review

This is an optional IPR.

Lesson #6 – Numerical Differentiation

1. Assignment: Sec 4.1, pp166-178.

2. Objectives:

a. To understand the derivation of the differentiation formulas (equations 4.1, 4.4-4.8), the assumptions used, and how to apply them to a differentiation problem.

b. To understand the difference between a forward-difference formula and a backward-difference formula.

c. To be able to find an error bound for a differentiation problem.

d. To understand the relationship between h, round-off error, and truncation error.

e. To be able to compute an optimum h for a differentiation problem.

f. To understand the instability associated with numerical differentiation problems.

g. To understand the derivation of the formulas for higher-order differentiation.

3. Suggested Problems: Sec 4.1: 1a, 2a, and 5a.

Lesson #7 - Richardson's Extrapolation

1. Assignment: Sec 4.2, pp 178-186.

2. Objectives:

a. To understand the process for generating higher-order formulas by using low-order formulas.

b. To be able to generate higher-order formulas using Taylor series polynomials for $f(x_0+h)$ and $f(x_0-h)$.

3. Suggested Problems: Sec 4.2: 1a, b and the following problem:

Using Taylor's theorem for $f(x_0+h)$ and $f(x_0-h)$, establish the important formula for second derivatives

$$f''(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2} - \frac{h^2}{12} f^{(4)}(\xi)$$

Lesson #8 - Elements of Numerical Integration

1. Assignment: Sec 4.3, pp 186-196 and Sec 4.4, pp 196-207.

2. Objectives:

a. To understand the derivation of the Trapezoidal Rule and Simpson's Rule from Lagrange polynomials.

b. To understand the derivation of Simpson's Rule from Taylor polynomials, and why we do it that way instead of Lagrange polynomials.

c. To understand the difference between, and the relative advantages and disadvantages of the closed and open Newton-Cotes formulas.

d. To understand the derivation of the error terms for all the formulas.

e. To understand the concept of Composite Numerical Integration and why it is used.

f. To be able to apply the Composite Simpson's Algorithm.

g. To be able to apply the Composite Trapezoidal Algorithm.

3. Suggested Problems: Sec 4.3:1c, 15f, and Sec 4.4:1a, 2a.

Lesson #9 – Project #2 In-Progress-Review

This is an optional IPR.

Lesson #10 – Theory of Initial Value Problems

1. Assignment: Sec 5.1, pgs 249-256. *BDH*: Sec 1.2, pgs 19-22.

2. Objectives:
   a. Understand Definition 5.1, Lipschitz condition.
   b. Be able to use Theorem 5.3, pg 255, to determine if a differential equation satisfies a Lipschitz condition and to determine the Lipschitz constant.
   c. Be able to determine if a differential equation is well-posed by applying Definition 5.5.

3. Suggested Problems: Sec 5.1: 1b, 2a.

Lesson #11 – Euler's Method

1. Assignment: Sec 5.2, pgs 256-266

2. Objectives:
   a. Understand the visual/graphical representation of Euler's method.
   b. Know the relationship of Euler's method to the Taylor series expansion and the insight it provides regarding the error of the method.
   c. Understand and be able to apply Theorem 5.9 to determine the errors for Euler's method.
   d. Understand the definition of mesh points and step size.
   e. Be able to apply Eqn 5.14 to determine the minimal value of E(h), the error bound for Euler's method as well as the optimal step size h.
   f. Understand the difference between local error and global error.

3. Suggested Problems: 5.2: 1a, 1b, 2a, 2b. Work by hand and then try the *Mathematica* algorithm.

4. Computer Experiment: Try running Euler's method with decreasing step sizes on a problem that you know the answer to (refer to *BDH*). See how long it takes to start generating large errors and compare the step size at which the problems start occurring to the minimal h you compute from Eqn 5.14. (Use either the algorithm or *Excel*.)

Lesson #12 - Higher Order Taylor Methods

1. Assignment: Sec 5.3, pgs 266-272

2. Objectives:
   a. Understand the derivation of the higher-order Taylor methods.
   b. Be able to determine the local truncation error for a Taylor method.
   c. Understand that all higher-order Taylor methods have local truncation error $O(h^n)$, where "n" is the order of the Taylor polynomial.

d. Be able to use equation 5.17, pg 267, to solve a differential equation using a Taylor method of order n.

3. Suggested Problems: Sec 5.3: 3b and 7.

Lesson #13 – Runge-Kutta Methods

1. Assignment: Sec 5.4, pgs 272-282

2. Objectives:

a. Understand the derivation and graphical representation of the Runge-Kutta method of order 4.

b. Understand the advantages of the RK methods over the higher-order Taylor methods.

c. Be able to solve a differential equation by using the RK method of order 4 (by computer or by hand).

d. Understand that the RK method of order 4 has a local truncation error of $O(h^4)$, provided the solution has five continuous derivatives.

3. Suggested Problems: Sec 5.4: 10a.

4. Computer Experiment: Try solving a differential equation by using different methods. First, solve by Euler's method with one step-size, then with Modified Euler with half the step-size. Compare your results and see if you can explain them. See the discussion with example 4 on page 280.

Lesson #14 – Error Control and the Runge-Kutta-Fehlberg Method

1. Assignment: Sec 5.5, pgs 282-289

2. Objectives:

a. Understand that the local error estimate comes from comparing the answers for an $O(h^n)$ method to an $O(h^{n+1})$ method.

b. Understand that local truncation error is tied to the global error. Therefore, if we can control the local truncation error, we can control the global error of our method.

c. Realize that using RKF method results in fewer functional evaluations than using two different methods to arrive at the same accuracy.

d. Be able to apply a RKF method to solve a differential equation (either by hand or by computer).

e. Understand the algorithm required for adapting step-sizes based on the error estimates for the two methods used.

3. Suggested Problems: Sec 5.5:1a, 1b, and the following problem:

Given: $\frac{dy}{dt} = t^2 + y$, $0 \le t \le 2$, $y(0) = 1$, perform 3 , successful iterations of an adaptive 1st order single step method. Let your initial step size = 0.5 and required tolerance = 0.01.

Lesson #15 – Multistep Methods

1. Assignment: Sec 5.6, pgs 289-301

2. Objectives:
   a. Be able to apply a multistep method to a differential equation problem (by hand or by computer).
   b. Understand the difference between implicit (Adam-Moulton) and explicit (Adams-Bashforth) methods and how and when to apply them.
   c. Understand why we use predictor-corrector methods, which are a combination of explicit and implicit methods.

3. Suggested Problems: Sec 5.6: 1a, 2a, 4a.

Lesson #16 – Variable Step Size Multistep Methods

1. Assignment: Sec 5.7, pgs 301-306

2. Objectives:
   a. Understand why we use adaptive methods.
   b. Be able to apply a variable step-size multistep method to a differential equation problem.

3. Suggested Problems: Sec 5.7: 2a, 2b, 5.

Lesson #17 – Project 3 In-Progress-Review

This is a lab day to work on Project 2, which is due during Lesson 15.

Lesson #18 – Extrapolation Methods

1. Assignment: Sec 5.8, pgs 307-312

2. Objectives:
   a. Be able to apply Algorithm 5.6 (Extrapolation) to solve a differential equation (by hand and by computer).

b. Understand the derivation of the extrapolation algorithm.
c. Understand what stopping criterion are used for this method.

3. Suggested Problems: Sec 5.8: 1a, 1b

Lesson #19 – Theory of Second Order Initial Value Problems

1. Assignment: *BDH*: Sec 3.6, pgs 297-309.

2. Objectives:
a. Be able to find the solution of a linear, constant coefficient, homogeneous differential equation using the auxiliary (characteristic) equation.
b. Know and be able to state the following definitions as related to systems: solution vector, superposition principle, linear independence, fundamental set, and general solution.
c. Be able to use standard notation or matrix-vector notation to write systems of ordinary differential equations.
d. Be able to write the characteristic equation for a given system and find the corresponding eigenvalues and eigenvectors.
e. Be able to solve a homogeneous 2 x 2 linear system with constant coefficients when the characteristic equation has distinct real roots.

3. Suggested Problems: *BDH*: Sec 1.2: 25, 29, 33, Sec 3.6: 1

Lesson #20 – Higher-Order Equations and Systems of Differential Equations

1. Assignment: Sec 5.9, pgs 313-324

2. Objectives:
a. Understand that all of the methods we have learned so far for solving IVP's can be extended to solve systems of ODEs.
b. Be able to transform a higher-order differential equation to a system of first-order differential equations.
c. Be able to apply Algorithm 5.7 (Runge-Kutta for Systems of Differential Equations) to solve a higher-order differential equation.
d. Be able to apply Theorem 5.17 to determine if a system of first-order DE's has a unique solution.

3. Suggested Problems: Sec 5.9: 1a, 2a.

Lesson #21 – Stability

1. Assignment: Sec 5.10, pgs 324-334

2. Objectives:

a. Understand the definitions of consistent, convergent and stable.

b. Understand the relationship between the stability of an equation and a well-posed equation.

c. *Understand*: A method is convergent if the solution to the difference equation approaches the solution to the differential equation as the step size approaches zero.

d. *Understand*: A method is consistent if the difference equation approaches the differential equation as the step size approaches zero.

e. Be able to use the characteristic polynomial of a method to determine if the method is strongly stable, weakly stable or unstable.

3. Suggested Problems: Sec 5.10: 3, 5.

Lesson #22 – Stiff Differential Equations

1. Assignment: Sec 5.11, pgs 334-341

2. Objectives:

a. Understand what a stiff differential system is. (The notion is somewhat imprecise, but it "embodies the idea that the solution has various components, some of which change much more rapidly than others".)

b. Understand the definitions of region R of absolute stability and an A-stable numerical method.

3. Suggested Problems: Sec 5.11: 1a, 4a.

Lesson #23 - Review

Review Lessons 1 through 22.

Lesson #24 - WPR

The WPR will cover Lessons 1 through 22.

Lesson #25 – Project 4 In-Progress-Review

This is an optional IPR.

Lesson #26 – The Linear Shooting Method

1. Assignment: Sec 11.1, pgs 645-653

2. Objectives:

a. Understand when to apply Theorem 11.1 and when to apply Corollary 11.2 to a boundary-value problem in order to determine if a unique solution exists.

b. Be able to transform a 2nd order differential equation boundary-value problem into a system of 2nd order initial-value problems.

c. Understand the derivation of Eqn 11.5 on pg 647 and how it works.

d. Be able to determine if a boundary value problem has a unique solution.

3. Suggested Problems: Sec 11.1: 1a, 10

Lesson #27 – The Shooting Method for Nonlinear Problems

1. Assignment: Sec 11.2, pgs 653-660

2. Objectives:

a. Understand that the solution to nonlinear problems cannot simply be a linear combination of the solution to two other initial value problems as it was in the case of solving linear problems.

b. Understand the derivation of the equations for Newton's method of solving nonlinear boundary value problems.

c. Understand the graphical representation of the Shooting Method.

d. Be able to apply Algorithm 11.2, Nonlinear Shooting Method, (by hand or by computer) to solve nonlinear boundary value problems.

3. Suggested Problems: Sec 11.2: 1, 2.

4. Computer Experiment: For a given choice of t, it will take a certain number of iterations to get a good solution. Try using different, rational choices of t and see if you can improve the number of iterations it takes to get a solution.

Lesson #28 – Matrix Factorization

1. Assignment: Sec 6.5, pgs 388-398 RAN: Sec 6.6, pgs 398-413

2. Objectives:

a. Understand the terminology and methods used for factorization of matrices.

b. Specifically understand and be able to apply Algorithm 6.7 (Crout Factorization for Tridiagonal Linear Systems) to solve an $n \times n$ linear system.

3. Suggested Problems: Sec 6.6: 6a

Lesson #29 – Finite-Difference Method for Linear Problems

1. Assignment: Sec 11.3, pgs 660-666

2. Objectives:

a. Understand the derivation of the finite-difference methods.

b. Be able to use Theorem 11.3 to determine if the tridiagonal linear system described by equation 11.19 has a unique solution.

c. Understand why we must use Theorem 11.3 to determine if we have a unique solution.

d. Be able to derive equation 11.19.

e. Be able to apply Algorithm 11.3 by hand and by computer to solve a linear boundary-value problem.

3. Suggested Problems: Sec 11.3: 1

Given: $y'' = -3y' + 2y + 2x + 3,\ 0 \le x \le 1,\ y(0) = 2,\ y(1) = 1,\ h = 0.2$. Approximate the solutions to the boundary value problem - by hand!

4. Computer Experiment: Take Example 1 from page 663, decrease the step size and see if the answer gets more accurate. How far can you decrease the step size and still make this algorithm work?

Lesson #30 – Finite-Difference Method for Nonlinear Problems

1. Assignment: Sec 11.4, pgs 667-672. RAN: Sec 10.2, pgs 611-620

2. Objectives:

a. Be able to set up and use Algorithm 11.4, Nonlinear Finite-Difference Method, to solve a nonlinear boundary-value problem using finite-differences.

b. Be able to solve a nonlinear boundary-value problem using Algorithm 11.4, by hand or by computer.

3. Suggested Problems:

Given: $y'' = 3yy' - y^2 \sin(x),\ 0 \le x \le 1,\ y(0) = 1,\ y(1) = 4,\ h = 0.25$.

Required: Using nonlinear finite differences,

(1) derive the system of equations to be solved, and

(2) calculate the Jacobian matrix for this system.

Lesson #31 - The Rayleigh-Ritz Method

1. Assignment: Sec 11.5, pg 672 through Example 1, pg 680.

2. Objectives: Understand and apply the piecewise linear Rayleigh-Ritz method of solving a boundary-value problem.

3. Suggested Problems: Sec 11.5: 1

Lesson #32 – Project 5 In-Progress-Review

This is an optional IPR.

Lesson #33 – Physical Problems Involving Partial Differential Equations

Lesson #34 – Elliptic Partial-Differential Equations

1. Assignment: Sec 12.1, pgs 691-704

2. Objectives:
   a. Understand the difference between elliptic, parabolic and hyperbolic Partial Differential Equations.
   b. Be able to apply Algorithm 12.1, Poisson Equation Finite-Difference, to solve Partial-Differential Equations.

3. Suggested Problems: Sec 12.1: 1.

Lesson #35 – Parabolic Partial-Differential Equations I

1. Assignment: Sec 12.2, pg 704 through Example 2, pg 710

2. Objectives:
   a. Be able to derive equation 12.11, pg 705 (finite difference scheme using forward differences).
   b. Understand the difference between the forward and backward difference methods.
   c. Understand and be able to determine conditional stability and unconditional stability.
   d. Be able to derive the backward difference method.

3. Suggested Problems: Sec 12.2: 1a (by hand, using finite-differences)

Lesson #36 – Parabolic Partial-Differential Equations II

1. Assignment: Sec 12.2, pgs 711-718

2. Objectives:
   a. Understand how the Crank-Nicolson method is derived and how it is used.
   b. Be able to apply the Crank-Nicolson method to a PDE by hand or by computer.

3. Suggested Problems: Sec 12.2: 2a.

Lesson #37 – Hyperbolic Partial Differential Equations

1. Assignment: Sec 12.3, pgs 718-726

2. Objectives:
   a. Apply a finite-difference method to find a numerical solution to the wave equation, an example of a hyperbolic partial-differential equation.
   b. Be able to discuss the expected truncation error versus the actual error obtained when solving a problem using a finite-difference method.

3. Suggested Problems: Sec 12.3: 1

Lesson #38 - Project 6 In-Progress-Review

This is an optional IPR.

Lesson #39 – Finite-Element Method

1. Assignment: Sec 12.4, pgs 726-741

2. Objectives: Understand and apply the finite-element method when the plane region is divided into triangle elements.

3. Suggested Problems: Sec 12.4: 1

Lesson #40 – Course Review

Review Lessons 1 through 39.