Solution
When $x = \frac{2\pi}{3}$, we have

$$\cot\frac{2\pi}{3} - 2\csc\frac{2\pi}{3} = \frac{-\sqrt{3}}{3} - 2\left(\frac{2\sqrt{3}}{3}\right) = \frac{-5\sqrt{3}}{3}$$

so the point of tangency is $P\left(\frac{2\pi}{3}, \frac{-5\sqrt{3}}{3}\right)$. To find the slope of the tangent line at P, we first compute the derivative $\frac{dy}{dx}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\cot x - 2\csc x)\\ &= \frac{d}{dx}\cot x - 2\frac{d}{dx}\csc x \qquad \text{Linearity rule}\\ &= -\csc^2 x - 2(-\csc x\cot x)\\ &= 2\csc x\cot x - \csc^2 x\end{aligned}$$

Then the slope of the tangent line is given by

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=2\pi/3} &= 2\csc\frac{2\pi}{3}\cot\frac{2\pi}{3} - \csc^2\frac{2\pi}{3}\\ &= 2\left(\frac{2}{\sqrt{3}}\right)\left(\frac{-1}{\sqrt{3}}\right) - \left(\frac{2}{\sqrt{3}}\right)^2\\ &= \frac{-8}{3}\end{aligned}$$

so the tangent line at $\left(\frac{2\pi}{3}, \frac{-5\sqrt{3}}{3}\right)$ with slope $-\frac{8}{3}$ is

$$y + \frac{5\sqrt{3}}{3} = -\frac{8}{3}\left(x - \frac{2\pi}{3}\right)$$

$$24x + 9y + 15\sqrt{3} - 16\pi = 0$$

■

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The next theorem, which is easy to prove and remember, is one of the most important results in all of differential calculus.

THEOREM 3.8 Derivative rule for the natural exponential
The natural exponential function e^x is differentiable for all x, with derivative

$$\frac{d}{dx}(e^x) = e^x$$

Proof We will proceed informally. Recall the definition of e:

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$$

Let $n = \frac{1}{\Delta x}$, so that $\lim_{\Delta x\to 0}(1 + \Delta x)^{1/\Delta x} = e$. This means that for Δx very small, $e \approx (1 + \Delta x)^{1/\Delta x}$ or $e^{\Delta x} \approx 1 + \Delta x$, so that $e^{\Delta x} - 1 \approx \Delta x$. Thus, $\lim_{\Delta x\to 0}\frac{e^{\Delta x} - 1}{\Delta x} = 1$.
Finally, using the limit in the definition of derivative for e^x, we obtain

$$\begin{aligned}\frac{d}{dx}(e^x) &= \lim_{\Delta x\to 0}\frac{e^{(x+\Delta x)} - e^x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{e^x(e^{\Delta x} - 1)}{\Delta x}\\ &= e^x\lim_{\Delta x\to 0}\frac{e^{\Delta x} - 1}{\Delta x}\\ &= e^x(1) = e^x\end{aligned}$$

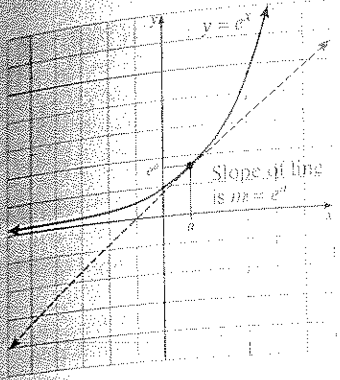

Figure 3.20 The slope of $y = e^x$ at each point (a, e^a) is $m = e^a$.

Note: An easier proof of the derivative formula in Theorem 3.8 is given in Example 6 of Section 3.6 using methods developed in that section.

The fact that $\frac{d}{dx}(e^x) = e^x$ means that the slope of the graph of $y = e^x$ at any point (a, e^a) is $m = e^a$, the y-coordinate of the point, as shown in Figure 3.20. This is one of the features of the exponential function $y = e^x$ that makes it "natural."

EXAMPLE 7 A derivative involving e^x

Differentiate $f(x) = x^2e^x$. For what values of x does $f'(x) = 0$?

Solution

Using the product rule, we find

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2e^x) \\ &= x^2\left[\frac{d}{dx}e^x\right] + \left[\frac{d}{dx}x^2\right]e^x \\ &= x^2e^x + 2xe^x \qquad \text{Exponential and power rules} \end{aligned}$$

To find where $f'(x) = 0$, we solve

$$\begin{aligned} x^2e^x + 2xe^x &= 0 \\ x(x+2)e^x &= 0 \\ x(x+2) &= 0 \qquad \text{Since } e^x \neq 0 \text{ for all } x \\ x &= 0, -2 \end{aligned}$$

■

EXAMPLE 8 A second derivative involving e^x

For $f(x) = e^x \sin x$, find $f'(x)$ and $f''(x)$.

Solution

Apply the product rule twice:

$$\begin{aligned} f'(x) &= e^x(\sin x)' + (e^x)' \sin x \\ &= e^x(\cos x) + e^x(\sin x) \\ &= e^x(\cos x + \sin x) \end{aligned}$$

$$\begin{aligned} f''(x) &= e^x(\cos x + \sin x)' + (e^x)'(\cos x + \sin x) \\ &= e^x(-\sin x + \cos x) + e^x(\cos x + \sin x) \\ &= 2e^x \cos x \end{aligned}$$

■

THEOREM 3.9 Derivative rule for the natural logarithmic function

The natural logarithmic function $\ln x$ is differentiable for all $x > 0$, with derivative

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Proof According to the definition of the derivative, we have

$$\begin{aligned} \frac{d}{dx}(\ln x) &= \lim_{\Delta x \to 0} \frac{\ln(x + \Delta x) - \ln x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(\frac{x + \Delta x}{x}\right) \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right) \qquad \text{Let } h = \frac{x}{\Delta x}, \text{ so } \Delta x = \frac{x}{h}. \end{aligned}$$

$$= \lim_{h\to+\infty} \frac{h}{x} \ln\left(1+\frac{1}{h}\right) \qquad h \to +\infty \text{ as } \Delta x \to 0$$

$$= \frac{1}{x}\left[\lim_{h\to+\infty} h \ln\left(1+\frac{1}{h}\right)\right]$$

$$= \frac{1}{x}\left[\lim_{h\to+\infty} \ln\left(1+\frac{1}{h}\right)^h\right] \qquad \text{Power rule for logarithms}$$

$$= \frac{1}{x}\ln\left[\lim_{h\to+\infty} \left(1+\frac{1}{h}\right)^h\right] \qquad \text{Since } \ln x \text{ is continuous, use the composition limit rule.}$$

$$= \frac{1}{x}\ln e \qquad \text{Definition of } e$$

$$= \frac{1}{x} \qquad \ln e = 1$$

EXAMPLE 9 Derivative of a quotient involving a natural logarithm

Differentiate $f(x) = \dfrac{\ln x}{\sin x}$.

Solution We use the quotient rule.

$$f'(x) = \frac{(\sin x)\frac{d}{dx}(\ln x) - (\ln x)\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{(\sin x)\left(\frac{1}{x}\right) - (\ln x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin x - x\ln x \cos x}{x\sin^2 x}$$

TECHNOLOGY NOTE

When you are using calculators or computer programs such as *Mathematica*, *Derive*, or *Maple*, the form of the derivative may vary. For example, you might obtain

$$\frac{d}{dx}(\tan x) = \tan^2 x + 1$$

instead of $\sec^2 x$ as found in this section;

$$\frac{d}{dx}(\cot x) = -\cot^2 x - 1$$

instead of $-\csc^2 x$;

$$\frac{d}{dx}(\sec x) = \frac{\sin x}{\cos^2 x}$$

instead of $\sec x \tan x$; and

$$\frac{d}{dx}(\csc x) = -\frac{\cos x}{\sin^2 x}$$

instead of $-\csc x \cot x$. A solution to Example 9, using a TI-92, is shown in Figure 3.21.

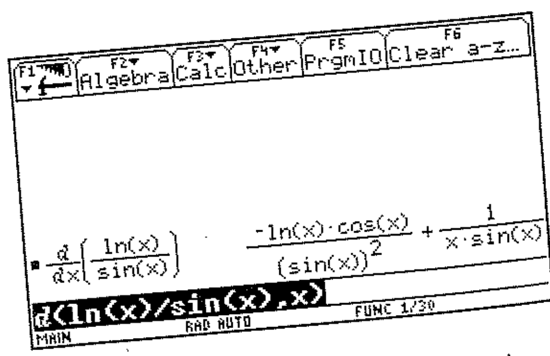

Figure 3.21 Sample output for Example 9. Notice that the form of the answer differs from that shown in the text.

Although the form of this calculator solution is different from the form shown in the example, you should notice that th are equivalent by using algebra (and recalling some of the fundamental identities from trigonometry).

A good test for your calculator is to try to find the derivative of $|x|$ at $x = 0$, which we know does not exist. Ma calculators will give the incorrect answer of 0.