# Research Outline

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My research focuses on questions from combinatorial commutative algebra. This uses combinatorial methods and constructions to describe algebraic objects. Primarily, I am interested in the newly discovered correspondence between multigraded modules from commutative algebra and matroids from combinatorics.

## 1. Upper Bounds of Betti Numbers for multigraded modules

Let $R = \Bbbk[x_1, \ldots, x_m]$ be the polynomial ring in $m$ variables over a field $\Bbbk$ with the standard $\mathbb{Z}^m$-grading and $L$ a $\mathbb{Z}^m$-graded (multigraded) $R$-module. Our main result of [2] gives sharp upper bounds for the total Betti number of a multigraded module $L$ in each homological degree in terms of the rank and the first two Betti numbers of $L$. More specifically,

**Theorem 1.1.** [2] *For $i \geq 2$, we have*

$$\beta_i(L) \leq \binom{\beta_1(L)}{\beta_0(L) - \operatorname{rank} L + i - 1}\binom{\beta_0(L) - \operatorname{rank} L + i - 3}{i-2}.$$

These bounds are the maximal ranks of the free modules from a free resolution defined by Tchernev in [6] and they are sharp. If the generators of $L$ and their multidegrees are chosen sufficiently generic, so that the Buchsbaum-Rim-Taylor complex constructed by Charalambous and Tchernev in [3] is the minimal free resolution, then $L$ will attain these bounds in each homological degree. We also use Theorem 1.1 to provide upper bounds for the total multigraded Bass numbers of a multigraded module in terms of the first two total multigraded Bass numbers, by utilizing the Alexander duality functors defined by Miller [4].

## 2. Combinatorial Description of a free resolution

Diana Taylor constructs in [5] a free resolution for every monomial ideal using the topological chain complex of a simplicial complex associated to the monomial ideal. Both the maps and the free modules are described using these combinatorial and topological tools. Tchernev in [6] generalizes Taylor's construction and descibes a free resolution for all multigraded modules. However, this description uses the linear algebra of a presentation of $L$ and not the topology of a combinatorial object associated to $L$. In building a free resolution of a multigraded module, it is usually enough to describe the structure of an underlying vector space complex. In the case of Tchernev's construction a matroid $\mathbf{M}$ on a set $S$ represented by a map $\phi$ obtained from a free presentation of $L$ is used to define an underlying vector space complex called the T-complex with vector spaces Tchernev calls T-spaces, denoted $T_A(\phi)$, where $A$ runs through the so called T-flats of $\mathbf{M}$. The structure of these T-spaces is complicated and depends on the linear algebra of the presentation of $L$. Ultimately, the goal is to define both the maps and vector spaces using a combinatorial or topological model as achieved for monomial ideals by Taylor. This

would effectively remove the dependence on the base field and allow us to define a resolution over $\mathbb{Z}$ instead of $\Bbbk$. Our first step in this direction is to associate a combinatorial object to a T-space. We start by analyzing properties of the T-space.

**Theorem 2.1.** [1] *If* $\mathbf{M}$ *is a representable matroid on* $S$ *with representation* $\phi$ *then* $\dim_{\Bbbk} T_S(\phi) = \beta(\mathbf{M})$*, where* $\beta(\mathbf{M})$ *is Crapo's beta invariant.*

It is well known that for Crapo's beta invariant on has $\beta(\mathbf{M}) = \beta(\mathbf{M}^*)$, where $\mathbf{M}^*$ denotes the matroid dual of $\mathbf{M}$.

This result quantifies the dimensions of the combinatorial objects we want to associate to the (linear algebraically defined) T-spaces. The invariant $\beta(\mathbf{M})$ has been well studied and shown to be the top dimensional integral homology of the reduced broken circuit complex by Ziegler in [7]. Moreover, a canonical basis for this homology is also described in [7]. The reduced broken circuit complex is a simplicial complex defined using only combinatorial properties of the matroid and the homology is from the associated topological chain complex. Since the homology of these reduced broken circuit complexes does not depend on the field, it is a candidate to replace the T-space in an integral resolution. In fact, the next theorem shows this is the correct combinatorial object.

**Theorem 2.2.** [1] *Let* $\mathbf{M}$ *be an ordered representable matroid on a set* $S$ *with rank* $r$ *with representation* $\phi$ *and dual* $\mathbf{M}^*$*. Then there is a canonical isomorphism*

$$\widetilde{H}_{r-2}(\overline{BC}(\mathbf{M}^*);\Bbbk) \xrightarrow{\cong} T_S(\phi).$$

In this way the free modules in the free resolution by Tchernev in [6] can be described completely combinatorially and a canonical basis is provided.

Now, we analyze the maps in the T-complex in order to describe them using combinatorial or topological properties of the matroid as well. Let $m$ be the largest element with respect to the linear ordering on $S$. The matroid $\mathbf{M}\backslash m$ is defined on $S\backslash\{m\}$ by taking the independent sets to be all independent sets of $\mathbf{M}$ that are contained in $S\backslash\{m\}$. The matroid $\mathbf{M}/m$ is defined on $S\backslash\{m\}$ by taking independent sets to be all independent sets of $\mathbf{M}$ that contain $\{m\}$ and then remove $\{m\}$. When $m$ is the largest element in the ordering on $S$ the simplicial complex $\overline{BC}(\mathbf{M}\backslash m)$ is a subcomplex of $\overline{BC}(\mathbf{M})$ consisting of all faces $\overline{BC}(\mathbf{M})$ that do not contain $m$. This induces a canonical exact sequence of reduced chain complexes

$$0 \to \widetilde{C}(\overline{BC}(\mathbf{M}\backslash m)) \to \widetilde{C}(\overline{BC}(\mathbf{M})) \to \widetilde{C}(\overline{BC}(\mathbf{M}/m))[-1] \to 0,$$

which in turn induces a long exact sequence in reduced homology. This sequence commutes with one component of the differentials in the T-complex:

**Theorem 2.3.** [1] *Let* $\mathbf{M}$ *be an ordered representable matroid on a set* $S$ *of rank* $r$ *with representation* $\phi$ *and dual* $\mathbf{M}^*$*. If* $m$ *is the largest element in the ordering on* $S$ *and is not a loop then the following diagram commutes,*

$$\begin{array}{ccccc}
\widetilde{H}_{r-2}(\overline{BC}(M^*\backslash m);\Bbbk) & \longrightarrow & \widetilde{H}_{r-2}(\overline{BC}(M^*);\Bbbk) & \longrightarrow & \widetilde{H}_{r-3}(\overline{BC}(M^*/m);\Bbbk) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
T_{S/m}(\phi) & \xrightarrow{\pi} & T_S(\phi) & \xrightarrow{\psi} & T_{S\backslash m}(\phi)
\end{array}$$

*where the top row is the canonical long exact sequence in homology, the maps in the bottom row* $\pi = \pi^*_{S/m,S}$ *and* $\psi = (-1)^{|S\backslash m|}\phi^{S,S\backslash m}$ *are as defined in* [6]*, and the vertical maps are the isomorphisms defined in Theorem 2.2.*

This provides evidence that the reduced integral homology of the reduced broken circuit complexes is a good choice to replace the T-spaces in a resolution over $\mathbb{Z}$.

## 3. Future research

Here is a sample of projects I am currently working on and will continue to work on in the future.

(1) The current project is to describe the maps of the T-complex completely combinatorially so that they can be defined over $\mathbb{Z}$ not just $\Bbbk$. There are many different topologically defined candidates for such maps. The most natural is to iterate the construction above. For each linear ordering, we get one component of the differential. So we will investigate how to put these component maps together. However, care must be taken because changing the ordering on $S$ changes the reduced homology of the reduced broken circuit complex. We may need to fix a basis and consider other topological maps between these simplicial complexes. By completing this picture, we will have constructed an integral resolution for a representation of $\mathbf{M}$ over $\mathbb{Z}$.

(2) The multiplicity spaces defined by Tchernev in [6] are constructed using the lattice of T-flats of a representable matroid $\mathbf{M}$. The multiplicity spaces are defined as polynomials associated to the maximal chains in this lattice. Creating a matrix from these polynomials describes the multiplicity spaces completely. This matrix is then a representation of another matroid, let us call it the multiplicity matroid. If we change the representation of the original matroid $\mathbf{M}$, it will certainly change the entries of the matrix. However, is the multiplicity matroid independent of the choice of representation? This would be particularly interesting when the representable matroid $\mathbf{M}$ is the uniform matroid. In this case, the multiplicity space is the full symmetric power of a certain vector space. This multiplicity matroid would then describe properties of the symmetric power independent of a choice of basis.

## References


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