A combinatorial description of the free modules in a free resolution of a multigraded module

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A COMBINATORIAL DESCRIPTION OF THE FREE MODULES IN A FREE RESOLUTION OF A MULTIGRADED MODULE

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Abstract. This thesis analyzes the structure of the T-resolution, which is a construction of a free resolution for any multigraded R-module, where $R = \Bbbk[x_1, \ldots, x_m]$. It was orginally described by the linear algebraic structure of a presentation matrix. Here, we will describe much of the resolution from a matroid associated to the finite free presentation without reference to the linear algebraic structure. Encoded in this matroid is the rank of the free modules in the resolution. We use this number to find a suitable simplicial complex whose top reduced homology is canonically isomorphic to the underlying vector space of a free module in the resolution. We completely describe the free modules of the T-resolution combinatorially from a matroid defined from the presentation matrix. We also use the T-resolution to give sharp upper bounds for the Betti numbers of a finitely generated multigraded resolution in terms of the first two Betti numbers and the rank of the module. We use the sharp bounds to establish bounds for the multigraded Bass numbers of a finitely generated multigraded module using Alexander duality.

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Introduction

This thesis is divided into three parts. The first section is background information to aid in our understanding of the problem and the solution. The content of the last two sections are papers that are to be submitted for publication. At this time "Upper Bounds for Betti Numbers of Multigraded Modules" has been accepted for publication by the *Journal of Algebra*. The last chapter shall be submitted soon.

The first section is the background for the results of the last two sections. We begin by introducing multigraded modules and free multigraded resolutions. The Taylor resolution for monomial ideals is described and shown to be supported by the chain complex of a certain simplicial complex. In this way, the Taylor resolution is described completely combinatorially. The main problem is to describe the construction called the T-resolution completely combinatorially as well. The T-resolution is described in section 1.3. Before we can review this, however, we must review properties of matroids in section 1.2, because the T-resolution uses a matroid in its construction. Section 1.4 gives us a convenient way to view matroids as simplicial complexes. We also describe the broken circuit complex and the reduced broken circuit complex. The latter is the simplicial complex whose top dimensional reduced homology will be shown to be canonically isomorphic to the vector spaces of the T-complex, called the T-spaces. We review two methods for identifying the canonical basis for the reduced homology with a set called a β-system. These will be used interchangeable in Section 3 to ultimately create the canonical isomorphism. The last subsection will describe properties of the lattice of flats from Section 1.2, the Möbius function on this lattice, and also a process called Möbius inversion. It also reviews the keystone to our arguments, Crapo's beta invariant. This number will be shown to be both the dimension of the T-spaces and the dimension of the reduced homology of the reduced broken circuit complex.

In "Upper Bound for Betti Number of Multigraded Modules," we give an upper bound for a finitely generated multigraded module in terms of its first two Betti numbers and the rank of the module. More specifically,

Theorem 1. *For $i \geq 2$, we have*

$$\beta_i(L) \leq \binom{\beta_1(L)}{\beta_0(L) - \operatorname{rank} L + i - 1}\binom{\beta_0(L) - \operatorname{rank} L + i - 3}{i-2}.$$

We also give an example of a multigraded module that will achieve these bounds in each homological degree. Therefore, these are sharp upper bounds. We go further to use these bounds to obtain upper bounds for the multigraded Bass numbers of a finitely generated multigraded module.

In Section 3, we describe the T-spaces of the T-complex of a multigraded module L completely in terms of a matroid defined by a presentation matrix. We show

Theorem 2.

$$\dim T_S(\phi) = \beta(\mathbf{M}^*),$$

where $\beta(\mathbf{M}^*)$ is Crapo's beta invariant of the matroid $\mathbf{M}^*$.

This number is also the top dimensional non-zero reduced homology of the reduced broken circuit complex from Section 1.4. We use this to show that

Theorem 3.

$$\widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*); \Bbbk) \xrightarrow[\textit{canonical}]{\cong} T_S(\phi)$$

Using the same construction for each T-flat, we will show that all the vector spaces in the T-complex are isomorphic to reduced homology of a simplicial complex. Since the basic cycles of the reduced broken circuit complex are defined with integer coefficients, this gives a description for the free modules of a resolution of an R-module where $\Bbbk$ is not necessarily a field. The last main result of this thesis describes one component map of the T-complex as the induced connecting map δ from the reduced chain complexes of the reduced broken circuit complex. This is given in the following commutative diagram

Theorem 4.

$$\begin{array}{ccccccccc}
0 \to & \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*\backslash a); \Bbbk) & \to & \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*); \Bbbk) & \xrightarrow{\delta} & \widetilde{H}_{l-2}(\overline{BC}(\mathbf{M}^*/a); \Bbbk) & \to 0 \\
 & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & \\
0 \to & T_{S\backslash a}(\phi/a) & \longrightarrow & T_S(\phi) & \longrightarrow & T_{S\backslash a}(\phi\backslash a) & \to 0,
\end{array}$$

where l is the level of S and $a = \max(S)$.

We hope to describe all maps in the T-resolution completely combinatorially from the matroid presentation. This would allow the entire T-complex construction to extend to the case when $\Bbbk$ is no longer a field.

1. Background

1.1. **Multigraded Modules and Multigraded Resolutions.** We introduce the reader to the definitions of multigraded modules and multigraded resolutions. We define the Taylor resolution for monomial ideals and give its combinatorial description. This creates a template for a combinatorial description of the free resolution an arbitrary multigraded module.

Let $R = \Bbbk[x_1, \dots, x_m]$ be the polynomial ring in m variables over a field $\Bbbk$. The polynomial ring has the decomposition into vector spaces

$$R = \bigoplus_{\boldsymbol{a} \in \mathbb{N}^m} \Bbbk \cdot x^{\boldsymbol{a}},$$

where we write $x^{\boldsymbol{a}}$ for the monomial $x_1^{a_1} \cdots x_m^{a_m} \in R$.

A *multigraded module* over R is a module L with a decomposition

$$L = \bigoplus_{\boldsymbol{b} \in \mathbb{Z}^m} L_{\boldsymbol{b}} \text{ as abelian groups so that } \Bbbk \cdot x^{\boldsymbol{a}} \cdot L_{\boldsymbol{b}} \subseteq L_{\boldsymbol{a}+\boldsymbol{b}}, \text{ for all } \boldsymbol{a}, \boldsymbol{b}$$

where for $\boldsymbol{a} = (a_1, \dots, a_m), \boldsymbol{b} = (b_1, \dots b_m) \in \mathbb{Z}^m$, we write

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, \dots, a_m + b_m)$$

and

$$\operatorname{lcm}(\boldsymbol{a}, \boldsymbol{b}) = \big(\max(a_1, b_1), \dots, \max(a_m, b_m) \big).$$

Thus R is a mulitgraded module over itself.

The element $c \cdot x^{\boldsymbol{a}}$ is a *multihomogeneous element* of R. A *multihomogeneous ideal* of R is an ideal generated by multihomogeneous elements. The homogenenous ideal $\mathfrak{m} = (x_1, \dots x_m)$ is the unique maximal multihomogeneous ideal of R.

If z is a multihomogeneous element inside a multigraded R-module, we write $\deg z$ for its multidegree. In particular, $\deg(x^{\boldsymbol{a}}) = \boldsymbol{a}$. More generally, if $A = \{z_1, \dots, z_k\}$ is a collection of multihomogeneous elements in a multigraded R-module, we set

$$\deg A = \operatorname{lcm}(\deg z_1, \dots, \deg z_k).$$

For example, the $\deg \mathfrak{m} = \mathbf{1} = (1, \dots, 1)$.

The *shift* of a multigraded R-module L, $L(\boldsymbol{a})$, is the multigraded R-module so that $L(\boldsymbol{a})_{\boldsymbol{b}} = L_{\boldsymbol{a}+\boldsymbol{b}}$. Notice that if $L = R$ then $R(\boldsymbol{a})$ is the free R-module generated by the element $x^{-\boldsymbol{a}}$ since $R(\boldsymbol{a})_{-\boldsymbol{a}} = R_{\boldsymbol{a}+(-\boldsymbol{a})} = R_0$. A finitely generated *free multigraded module* F has the form $F \cong \bigoplus_{i=1}^{k} R(-\boldsymbol{d}_i)$, where each summand of F is generated in multidegree $\boldsymbol{d}_i$.

If L and N are multigraded R-modules, then a *multigraded R-module homomorphism* is an R-module homomorphism $\psi : L \to N$ so that $\psi(L_{\boldsymbol{a}}) \subseteq N_{\boldsymbol{a}}$.

A *free presentation* of an R-module L is an exact sequence

$$R^s \to R^t \to L \to 0.$$

A *complex* of R-modules is a sequence of R-modules F_i and R-module homomorphisms $F_i \longrightarrow F_{i-1}$ such that the compositions $F_{i+1} \longrightarrow F_i \longrightarrow F_{i-1}$ are all zero. If $R = \Bbbk$ then we have a *vector space complex.*

A *resolution* $\mathbb{F}$ of an R-module L is a complex of R-modules

$$\mathbb{F} : \cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to 0$$

with the extra properties that

(1) $\operatorname{Im} \varphi_l = \ker \varphi_{l-1}$; i.e., the complex is exact for $l \geq 1$.

(2) $\operatorname{Coker}(\varphi_1) \cong L$.

If, in addition, all F_i are free R-modules then we call $\mathbb{F}$ a *free resolution of L.*

We will often augment the resolution $\mathbb{F}$ and say it is a free resolution if

$$\mathbb{F} : \cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to L \to 0$$

is an exact complex.

A resolution $\mathbb{F}$ is a *multigraded free resolution* of a multigraded module L if the F_i are multigraded free modules and the maps are multigraded R-module homomorphisms.

It is well known that every multigraded R-module L has a *minimal free resolution*,

$$\mathbb{F} : \cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to L \to 0$$

i.e., $\mathbb{F}$ is a multigraded free resolution and the image of $\varphi_l : F_l \to F_{l-1}$ is contained in $\mathfrak{m}F_{l-1}$ for all l.

Theorem 1.1. [6, Theorem 6.3.15] *Any free resolution of a multigraded R-module L contains a minimal free resolution as a direct summand.*

Corollary 1.2. *Any two minimal free resolutions are isomorphic.*

The *ith Betti number in multidegree j of L* is defined to be the number

$$\beta_{i,j}(L) = \dim_{\Bbbk}(Tor_i^R(L, \Bbbk))_j$$

and the numbers $\{\beta_{i,j}(L)\}$ are called the *multigraded Betti numbers*. The number of generators of F_i in multidegree j of the minimal free resolution of L are the graded Betti numbers of L. The *ith Betti number of L* is the number

$$\beta_i(L) = \operatorname{rank} F_i = \sum_j \beta_{i,j}(L).$$

If we have any free resolution of L, then we have upper bounds for the Betti numbers since the minimal free resolution embeds in the given free resolution. The detection of this embedding is elusive, so there are many papers developing tools to determine the minimal free resolution.

It is an open problem to find an explicit combinatorial description for the minimal free resolution of an arbitrary multigraded module over R. This problem is well-studied but still open, even for $L = R/I$ where I is a monomial ideal. The Taylor resolution [15] gives a free multigraded resolution for L where I is an arbitrary monomial ideal, however it is not in general minimal. The most concise description can be found in [8] and it is the one we follow here.

Let $I = (m_1, \ldots, m_t)$ be an ideal generated by t monomials. Let F_s be the free module on basis elements e_J, where J is a subset of length s of $\{1, \ldots, t\}$. Set $m_J = \operatorname{lcm}\{m_j \mid j \in J\}$. For each pair J, K such that J has s elements and K has $s-1$ elements, define

$$c_{J,K} = \begin{cases} 0 & \text{if } K \not\subset J \\ (-1)^{k-1} m_J/m_K & \text{if } J = K \cup \{j_k\} \text{ for some } k \end{cases}$$

where $J = \{j_1, \ldots, j_s\}$ and $j_1 < \ldots < j_s$. Define $d_s : F_s \to F_{s-1}$ to be

$$d_s(e_J) = \sum_K c_{J,K} e_K.$$

We see that all the free modules and all the maps are described completely from the monomial ideal I, or equivalently, from the free presentation

$$R^t \xrightarrow{\begin{pmatrix} m_1 & \ldots & m_t \end{pmatrix}} R \longrightarrow R/I \to 0.$$

From the description and the fact this is a free resolution of R/I we have that $\beta_s(R/I) \leq \binom{t}{s}$ for all s. Chapter 3 is devoted to giving an upper bound for the Betti numbers of any multigraded module.

Example 1.3. *Let $I = (x^2y, xz, z^3)$. Then the Taylor resolution $\mathbb{T}$ of R/I is*

$$0 \to R(-x^2yz^3) \xrightarrow{\begin{pmatrix} z^2 \\ -1 \\ xy \end{pmatrix}} \begin{matrix} R(-x^2yz) \\ \oplus \\ R(-x^2yz^3) \\ \oplus \\ R(-xz^3) \end{matrix} \xrightarrow{\begin{pmatrix} z & z^3 & 0 \\ -xy & 0 & z^2 \\ 0 & -x^2y & -x \end{pmatrix}} \begin{matrix} R(-x^2y) \\ \oplus \\ R(-xz) \\ \oplus \\ R(-z^3) \end{matrix}$$

$$\xrightarrow{\begin{pmatrix} x^2y & xz & z^3 \end{pmatrix}} R \to R/I \to 0$$

We see this resolution is not the minimal free resolution because the image of the the generator in multidegree x^2yz^3 in the second component is the generator with multidegree x^2yz^3 which is not contained in the maximal ideal of $R(-x^2yz^3)$. This can be read from the maps by noticing that the entry corresponding to this map is a nonzero element of the field $\Bbbk$.

Consider the field $\Bbbk$ as the image of the map $R \to \Bbbk$ where we send each x_i to $1 \in \Bbbk$; i.e., $\Bbbk \cong R/(x_1 - 1, \ldots, x_m - 1)$. By tensoring the resolution $\mathbb{T}$ with $\Bbbk$ we

obtain the underlying vector space complex

$$\mathbb{T} \otimes_R \Bbbk : 0 \to \Bbbk \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} \Bbbk \to 0.$$

For an arbitrary monomial ideal I the underlying vector space complex for the Taylor resolution is

$$0 \to \Bbbk \longrightarrow \Bbbk^{\binom{t}{t-1}} \to \ldots \to \Bbbk^{\binom{t}{1}} \longrightarrow \Bbbk,$$

which may be identified as the reduced chain complex of the t-simplex with coefficient in $\Bbbk$. We save the description of the chain complex until Section 1.4 where we will need simplicial complex results.

Alternatively, if we know the underlying vector space complex U and the monomial ideal I, then through a process called *homogenization*, one constructs the free resolution $U \otimes_{\Bbbk} R$ by defining the generator of the free module recursively as follows, $F_1 = R(-m_1) \oplus \ldots \oplus R(-m_t)$ and given F_i define $e_q \in F_{i+1}$ to have multidegree

$$deg(e_q) = lcm\{deg(e_p) \mid e_p \in F_i \text{ and } (p,q) - \text{entry of the matrix} \neq 0\}.$$

So, for monomial ideals, it is enough to construct the underlying complex of vector spaces over $\Bbbk$ and then homogenize.

This thesis makes strides in reinterpreting the construction of the T-resolution described in Section 1.3 in terms of the combinatorics and topology of simplicial complexes associated to a matroid. As an immediate benefit we obtain an explicit basis for the T-spaces that underly the T-resolution, thus providing an approach in the spirit of Taylor's. In addition, our description is the first step in realizing the goal of defining the resolution where $\Bbbk$ is any ring, not just a field.

1.2. **Matroid Theory.** We review basic results from matroid theory and give some examples. In the next subsection, we will use many of these concepts in the construction of the T-Resolution. Additional results are given that will be needed for understanding the last section of this thesis.

A *matroid* $\mathbf{M}$ is a pair $(S, \mathbf{I})$ where S is a finite set and $\mathbf{I}$ is a nonempty collection of subsets of S satisfying the following:

(1) If $X \subseteq Y$ and $Y \in \mathbf{I}$ then $X \in \mathbf{I}$.

(2) If $X, Y \in \mathbf{I}$ and $|X| = |Y|+1$ then there exists an $x \in X \backslash Y$ so that $Y \cup \{x\} \in \mathbf{I}$.

If $X \in \mathbf{I}$ then it is called an *independent* set. If $X \notin \mathbf{I}$ then it is called a *dependent* set. We call a maximal independent set a *basis* of $\mathbf{M}$. A *circuit* is a minimal dependent set. A circuit A so that $|A| = 1$ is called a *loop*.

A matroid $\mathbf{M}$ is called *connected* if for every pair of distinct element x and y in S there is a circuit of $\mathbf{M}$ containing x and y. Otherwise it is *disconnected*. If a matroid $\mathbf{M}$ has a loop, then it is disconnected. Indeed, since a loop $\{x\}$ is a circuit, then any circuit containing $y \neq x$ will not contain x by minimality.

Proposition 1.4. [17, Corollary of Theorem 1.5.1] *Every basis of* $\mathbf{M}$ *has the same cardinality.*

The cardinality of a basis of $\mathbf{M}$ is called the *rank* of $\mathbf{M}$. More generally, the rank of $X \subseteq S$ is the number $r_X^{\mathbf{M}} = \max\{|Y| \mid Y \in \mathbf{I}, Y \subseteq X\}$. Notice that $r_X^{\mathbf{M}} = |X|$ precisely when $X \in \mathbf{I}$ and for any circuit A we have that $r_A^{\mathbf{M}} = |A| - 1$. The level of a subset A is defined to be the number $l_A = |A| - r_A^{\mathbf{M}} - 1$. Notice that the circuits are all of level 0. In [16], Tchernev defines a *T-flat* of $\mathbf{M}$ as a union of circuits of $\mathbf{M}$. We will denote the collection of T-flats of level k by $\mathcal{T}_k(\mathbf{M})$, and the collection of all T-flats by $\mathcal{T}(\mathbf{M})$.

Example 1.5. *Given a finite set of vectors in a vector space, one creates a matroid with them by letting the independent sets be those subsets of the vectors that are linearly independent. In this way, the rank of the subset X is the dimension of the subspace spanned by the vectors of X.*

Example 1.6. *Another way to construct a matroid from a finite set S is from a set map $\phi : S \longrightarrow W$ where W is a $\Bbbk$ vector space. Then we extend this to a map of vector spaces $\phi : U_S \longrightarrow W$ where U_S has basis $\{e_i | i \in S\}$. Continuing with the same notation, we let U_I be the vector space spanned by $\{e_i \mid i \in I \subseteq S\}$, V_I the vector*

space spanned by $\{\phi(e_i) \mid i \in I\}$ *and* $V = V_S$*. Then we form a matroid* $\mathbf{M}$ *on* S *where the independent sets are precisely those subsets* $X \subseteq S$ *such that* $\dim_{\Bbbk} V_X = |X|$*. For instance, let* ϕ *be the map sending*

$$1 \mapsto (1,0)$$

$$2 \mapsto (1,1)$$

$$3 \mapsto (1,2)$$

$$4 \mapsto (1,0)$$

and $S = \{1,2,3,4\}$*. Then*

$$\mathbf{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{12\}, \{13\}, \{23\}, \{24\}, \{34\}\}$$

and the circuits $\mathcal{T}_0(\mathbf{M}) = \{\{123\}, \{234\}, \{14\}\}$*. Also, the rank* $r_{\{14\}} = 1$ *and* $r_{\{24\}} = 2$*. We may also represent the map* ϕ *as a matrix*

$$\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix},$$

where for each $i \in S$*,* e_i *is the* i*th column of the matrix. Thus a matrix defines a matroid on the set of column vectors under linear independence as in the above example.*

When a matroid $\mathbf{M}$ can be formed in this way, we call $\mathbf{M}$ a *representable matroid*. We call $\phi : U_S \to W$ a *representation* of the matroid $\mathbf{M}$.

Example 1.7. *A* uniform matroid*,* $U_{r,n}$ *is the matroid where each* r *element subset of* S *is a basis of* $\mathbf{M}$ *with* $|S| = n$*. The circuits of* $U_{r,n}$ *are all* $r+1$ *element subsets and more generally the set*

$$\mathcal{T}_k(U_{r,n}) = \{X \mid |X| = r + k + 1\}$$

for $0 \leq k \leq n - r - 1$*.*

A subset $Y \subseteq S$ is a *flat* of $\mathbf{M}$ if for every $x \in S \backslash Y$, $r_{Y \cup \{x\}} = r_Y + 1$. A maximal flat is called a *hyperplane* of $\mathbf{M}$. Clearly, if H is a hyperplane of $\mathbf{M}$, then $r_H = r_{\mathbf{M}} - 1$.

The *closure* of A, $\overline{A}$, is the unique smallest proper flat that contains A and has the same rank as A. Clearly, $\overline{F} = F$ where F is a flat of $\mathbf{M}$.

Example 1.8. *When* $\mathbf{M}$ *is a representable matroid, then the vectors in* $\overline{A}$ *span the same subspace as the one spanned by the vectors of* A.

Example 1.9. *The hyperplanes of the uniform matroid* $U_{r,n}$ *are all* $r-1$ *subsets of* S. *The flats of* $U_{r,n}$ *are all subsets of* S *whose cardinality is less than or equal to* $r-1$ *and*

$$\overline{A} = \begin{cases} A & \text{if } |A| \leq r-1 \\ S & \text{otherwise.} \end{cases}$$

Proposition 1.10. [17, Theorem 1.8.3] *A flat is the intersection of hyperplanes.*

Remark 1.11. *Ordering the flats of a matroid by inclusion gives a ranked lattice called the lattice of flats and is stratified by the rank of the flat. Clearly, this is the intersection lattice of the hyperplanes of* $\mathbf{M}$ *from Proposition 1.10.*

The *dual matroid* $\mathbf{M}^*$ of $\mathbf{M}$ has vertex set S and basis of $\mathbf{M}^*$ are all subsets $S - B$, where B is a basis of $\mathbf{M}$. The circuits of $\mathbf{M}^*$ are $C \in S$ so that $S - C$ is a hyperplane of $\mathbf{M}$. Consequently, a set $A \subseteq S$ is a T-flat of a matroid $\mathbf{M}$ if and only if $S \backslash A$ is a proper flat of the dual matroid $\mathbf{M}^*$ by definitions and Proposition 1.10.

In the dual matroid, the rank is the number $r_A^{\mathbf{M}^*} = |A| - r_S^{\mathbf{M}} + r_{S \backslash A}^{\mathbf{M}}$. In particular, the rank $r_S^{\mathbf{M}^*} = |S| - r_S^{\mathbf{M}}$, which is obvious from the definition of $\mathbf{M}^*$.

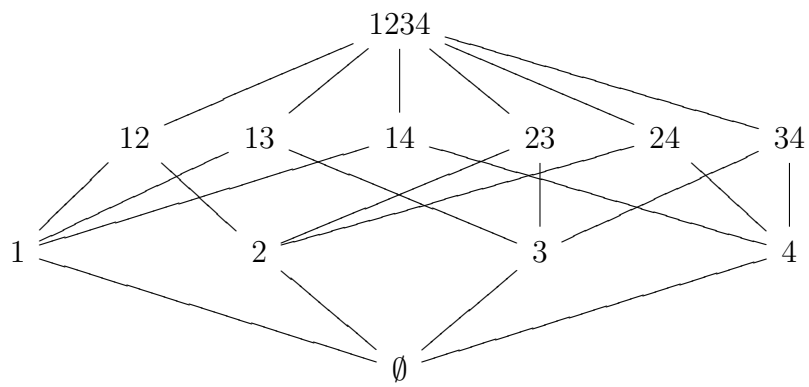
Example 1.12. $U_{r,n}^* = U_{n-r,n}$

Consider the lattice of flats of the dual matroid ordered by inclusion. If we take complements of entries in the lattice, this creates a lattice of T-flats in the matroid ordered by reverse inclusion and ranked by

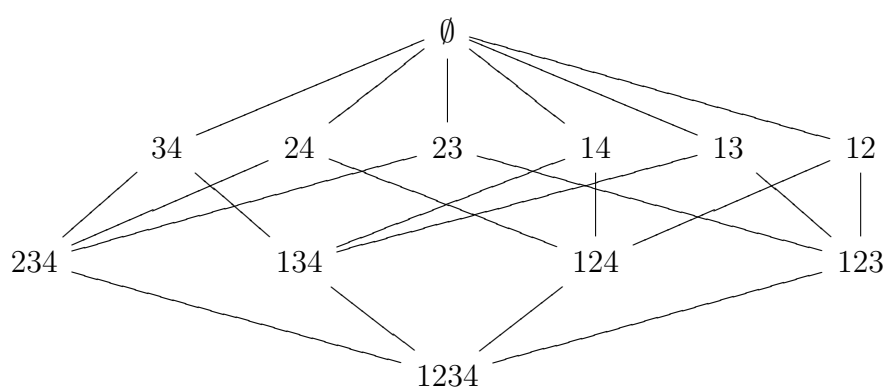
$$r_{S \setminus A}^{\mathbf{M}^*} = |S| - r_S^{\mathbf{M}} - |A| + r_A^{\mathbf{M}} = l_S - l_A,$$

for a T-flat A. Therefore, the lattice of T-flats of a matroid $\mathbf{M}$ ordered by reverse inclusion is isomorphic to the lattice of flats of $\mathbf{M}^*$ ordered by inclusion. We will often use these interchangeably. We need to consider both, because Tchernev [16] uses the lattice of T-flats in the construction of the T-resolution, but the lattice of flats is used by Ziegler [19] to define the labels that define a β-system.

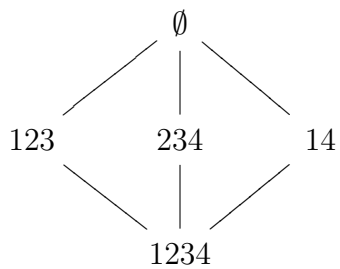
Example 1.13. *The lattice of flats of $U_{3,4}$ ordered by inclusion.*



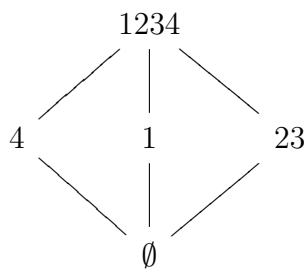
The lattice of T-flats of $U^*_{3,4} = U_{1,4}$ ordered by reverse inclusion.



Example 1.14. *Let* $\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$ *and* $\mathbf{M}$ *the matroid defined by* ϕ *as in Example 1.6. Then the lattice of T-flats of* $\mathbf{M}$ *is*



and the lattice of T-flats of $\mathbf{M}^*$ *is*



We end this subsection by recalling two fundamental matroid operations and how the representation transforms under these operations. The *restriction* of $\mathbf{M}$ to A, $\mathbf{M}|A$ is the matroid on A with $\mathbf{I}|A = \{X \subseteq A \mid X \in \mathbf{I}\}$. We write $\mathbf{M}\backslash a$ for $\mathbf{M}|(S\backslash a)$. The *contraction* of $\mathbf{M}$ to A, $\mathbf{M}.A$ is the matroid on A where $\mathbf{I}.A = \{X \subseteq A \mid \exists Y \subseteq S\backslash A, Y \neq \emptyset$ so that $X \cup Y \in \mathbf{I}\}$. We write $\mathbf{M}/a$ for the contraction $\mathbf{M}.(S\backslash a)$. We will examine these operations further in the Section 1.4.

Definition 1.15. [16, Section 1.3] *Let* ϕ *be a representation of* $\mathbf{M}$*. For a subset* $Y \subseteq S$ *let* $\phi|Y$ *be the restriction of* ϕ *to the subspace* U_Y*. Then*

$$\phi|Y : U_Y \longrightarrow W$$

is a representation of the matroid $\mathbf{M}|Y$*, and we obviously have for any subset* A *of* Y *that* $V_A(\phi) = V_A(\phi|Y)$*. We will write* $\phi\backslash a$ *for* $\phi|(S\backslash a)$*.*

Also, let $\phi.Y = \pi_Y^\phi \circ (\phi|Y)$*, where* $\pi_Y^\phi : W \longrightarrow \overline{W} = W/V_{S\smallsetminus Y}(\phi)$ *is the canonical projection (when* ϕ *and/or* Y *are clear from the context, we will write* π_Y *or simply* π *instead of* π_Y^ϕ*). Then*

$$\phi.Y : U_Y \longrightarrow \overline{W}$$

is a representation of the matroid $\mathbf{M}.Y$. *We will write* ϕ/a *for* $\phi.(S\backslash a)$.

1.3. **T-resolution.** We will describe the T-resolution from [16].

Let

$$E \xrightarrow{\Phi} G \longrightarrow L \longrightarrow 0$$

be a minimal finite free multigraded presentation of L and S a multihomogeneous basis of E. We consider the field $\Bbbk$ as an R-module under the quotient map where we send each variable to 1. In this way, tensoring the free presentation with $\Bbbk$ gives the map

$$E \otimes_R \Bbbk \xrightarrow{\Phi \otimes id} G \otimes_R \Bbbk = W$$

Let $S = S \otimes 1$ be a basis of $E \otimes_R \Bbbk$. Then the set map

$$\phi : S \xrightarrow{\Phi \otimes id} W$$

defines a matroid $\mathbf{M}$ on S as in Example 1.6.

Let I be a T-flat of level l. We write

$$C_{\mathbf{M}}(I) = \{I^{(0)} \subsetneq \cdots \subsetneq I^{(l)} = I \mid \text{ where } I^{(i)} \in \mathcal{T}_i \text{ for each } i \geq 0\}$$

for the collection of all maximal chains of T-flats contained in the T-flat I from the lattice of T-flats described in section 1.2. Let $\mathbb{I}$ be a chain $I^{(0)} \subsetneq \cdots \subsetneq I^{(l)}$ in $C_{\mathbf{M}}(I)$. When $l = 0$, then $\mathbb{I}$ is the only element of $C_{\mathbf{M}}(I)$, and we set $V(\mathbb{I}) = S_0 W = \Bbbk$. When $l \geq 1$, define the vector subspace $V(\mathbb{I}) \subseteq S_l W$ of the lth symmetric power of W as

$$V(\mathbb{I}) \quad = \quad \left(V_{I^{(1)} \smallsetminus I^{(0)}} \cap V_{I^{(0)}}\right) \cdot \ldots \cdot \left(V_{I^{(l)} \smallsetminus I^{(l-1)}} \cap V_{I^{(l-1)}}\right),$$

where the product of these vector spaces is taken in the symmetric algebra of W.

When $l \geq 0$, then the multiplicity space is

$$S_I(\phi) = \begin{cases} 0 & \text{if } I \text{ is not a T-flat} \\ \sum_{\mathbb{I} \in C_{\mathbf{M}}(I)} V(\mathbb{I}) & \text{if } I \text{ is a T-flat} \end{cases}$$

[16, Definition 2.2.3]. When there can be no confusion, we simply write S_I.

Let I be a T-flat of level $l \geq 1$ and let $J \subset I$ be a T-flat of level $l-1$. Multiplication in the symmetric algebra of W induces an injective map

$$\nu \colon (V_J \cap V_{I \smallsetminus J}) \otimes S_J \longrightarrow S_I,$$

which is called the *multiplication map* [16, Definition 2.2.6].

Let $\{a\}$ be independent in $\mathbf{M}$, let $A \subseteq S \backslash a$ be a T-flat of $\mathbf{M}/a$ of level l, let B be a T-flat containing $A \cup \{a\}$, let $\overline{W} = W/V_a$, and let $\pi \colon W \longrightarrow \overline{W}$ be the canonical projection map. Then the canonical surjection of symmetric powers $\pi_l \colon S_l W \longrightarrow S_l \overline{W}$ induces by restriction a surjection of multiplicity spaces $\pi^{\phi}_{S \backslash a, A} = \pi_{S \backslash a, A} \colon S_B(\phi) \longrightarrow S_A(\phi.S \backslash a)$ [16, Theorem 5.4.2].

Theorem 1.16. [16, Theorem 5.4.4] *Let $a \in S$ with $\{a\}$ independent in $\mathbf{M}$, let $A \subseteq S \backslash a$ be a T-flat of $\mathbf{M}.S \backslash a$, and let B be a T-flat containing $A \cup \{a\}$. If $\{a\}$ is a T-part of B then*

$$0 \longrightarrow S_A(\phi) \otimes V_a(\phi) \xrightarrow{\nu} S_B(\phi) \xrightarrow{\pi_{S \backslash a, A}} S_A(\phi.S \backslash a) \longrightarrow 0$$

is an exact sequence of vector spaces.

When $l \geq 0$, the *T-space* of I is

$$T_I(\phi) = S_I(\phi)^* \otimes \bigwedge^{|I|} U_I \otimes \bigwedge^{r_I} V_I(\phi)^*.$$

When $A = \{a\}$, we set

$$T_A(\phi) = T_A = U_a.$$

When there can be no confusion, we simply write T_I.

Since $\bigwedge^{|I|} U_I$ and $\bigwedge^{r_I} V_I^*$ are isomorphic to $\Bbbk$ as they are the top dimensional exterior product,

$$\dim_{\Bbbk} S_I(\phi) = \dim_{\Bbbk} T_I(\phi) \text{ for all T-flats } I.$$

We will use the notation $\mathring{\wedge} X$ to mean the top dimensional nonzero exterior power of X.

Example 1.17. [16, Example 2.2.10] *When* $\mathbf{M} = U_{r,n}$ *and* I *is a T-flat of level* l*, then*

$$S_I(\phi) = S_l V_I,$$

the lth *symmetric power of* V_I *and*

$$T_I(\phi) = S_l V^* \otimes \mathring{\wedge} U_I \otimes \mathring{\wedge} V_I(\phi)^*.$$

The T-spaces are the vector spaces in the T-complex. Now, we describe the maps $\phi_l^{I,J} : T_I(\phi) \to T_J(\phi)$. These maps are a component of the differential in the T-complex.

Definition 1.18. [16, Definition 2.3.1] *Let* $J \subseteq I$ *be subsets of* S*.*

(a) The space K_{IJ} *is given by the formula*

$$K_{IJ} = V_I / V_J.$$

(b) Since $V_I = V_J + V_{I \setminus J}$ *there is a natural commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_J \cap V_{I\setminus J} & \xrightarrow{\subseteq} & V_{I\setminus J} & \longrightarrow & K_{IJ} & \longrightarrow & 0 \\
 & & \downarrow{\subseteq} & & \downarrow{\subseteq} & & \downarrow{=} & & \\
0 & \longrightarrow & V_J & \xrightarrow{\subseteq} & V_I & \longrightarrow & K_{IJ} & \longrightarrow & 0
\end{array}$$

with exact rows; we use it to identify canonically K_{IJ} *and* $V_{I\setminus J}/(V_J \cap V_{I\setminus J})$*.*

(c) There is a natural diagonal isomorphism

$$\boldsymbol{\delta} = \boldsymbol{\delta}_{\mathbf{IJ}} : \quad \mathring{\wedge} U_I \quad \longrightarrow \quad \mathring{\wedge} U_{I\setminus J} \otimes \mathring{\wedge} U_J$$

and a canonical isomorphism

$$\mathbf{a} = \mathbf{a}_{\mathbf{IJ}}^{\mathbf{M}} : \quad \mathring{\wedge} V_I^* \quad \longrightarrow \quad \mathring{\wedge} K_{IJ}^* \otimes \mathring{\wedge} V_J^*$$

induced by the bottom row of the commutative diagram from Part (b) above.

(d) Let I *be a T-flat of level* $l \geq 1$*, and let* J *be a T-flat of level* $l-1$ *inside* I*. In particular, the top row of the commutative diagram from Part (b) induces a canonical homomorphism*

$$\mathbf{b} = \mathbf{b}_{\mathbf{IJ}}^{\mathbf{M}} : \quad (V_J \cap V_{I\setminus J})^* \otimes \mathring{\wedge} K_{IJ}^* \quad \longrightarrow \quad \mathring{\wedge} V_{I\setminus J}^*$$

which is zero when $V_J \cap V_{I\smallsetminus J} = 0$, and is an isomorphism otherwise.

Definition 1.19. [16, Definition 2.3.2] *Let I be a T-flat of level $l > 0$, and let J be an element of $\mathcal{T}_{l-1}(\mathbf{M})$ contained inside I. The homomorphism*

$$\phi_l^{IJ} \colon T_I(\phi) \longrightarrow T_J(\phi)$$

is defined as the composition

$$T_I(\phi) \xrightarrow{\Delta\otimes\boldsymbol{\delta}\otimes\mathbf{a}} \mathcal{Q}_{IJ}(\phi) \xrightarrow{\tau} \mathcal{R}_{IJ}(\phi) \xrightarrow{\mathbf{b}\otimes\wedge\phi\otimes 1} \mathcal{S}_{IJ}(\phi) \xrightarrow{\mu\otimes 1} T_J(\phi)$$

where

$$\mathcal{Q}_{IJ} = (V_J \cap V_{I\smallsetminus J})^* \otimes S_J^* \otimes \big(\mathring{\wedge}\, U_{I\smallsetminus J} \otimes \mathring{\wedge}\, U_J\big) \otimes \big(\mathring{\wedge}\, K_{IJ}^* \otimes \mathring{\wedge}\, V_J^*\big),$$

$$\mathcal{R}_{IJ} = \big[(V_J \cap V_{I\smallsetminus J})^* \otimes \mathring{\wedge}\, K_{IJ}^*\big] \otimes \mathring{\wedge}\, U_{I\smallsetminus J} \otimes \big(S_J^* \otimes \mathring{\wedge}\, U_J \otimes \mathring{\wedge}\, V_J^*\big),$$

$$\mathcal{S}_{IJ} = \mathring{\wedge}\, V_{I\smallsetminus J}^* \otimes \mathring{\wedge}\, V_{I\smallsetminus J} \otimes T_J,$$

where $\mu : X \otimes X^* \longrightarrow \Bbbk$ is the canonical evaluation map, τ is the isomorphism which simply permutes the components of the tensor product as indicated, the map $\Delta \colon S_I^* \longrightarrow (V_J \cap V_{I\smallsetminus J})^* \otimes S_J^*$ is the diagonal map dual to the multiplication map ν, and the map $\wedge\phi \colon \mathring{\wedge}\, U_{I\smallsetminus J} \longrightarrow \mathring{\wedge}\, V_{I\smallsetminus J}$ is zero if $I \smallsetminus J$ is not independent, and is the canonical isomorphism induced by ϕ otherwise.

Definition 1.20. [16, Definition 7.1] *Let A be a T-flat of $\mathbf{M}.Y$, and let $B = S\backslash\overline{(Y\backslash A)}_{\mathbf{M}^*}$, the complements of the closure in $\mathbf{M}^*$.*

(a) By repeatedly contracting one element at a time, it is clear that one always has an equality $r_{B\smallsetminus A}^{\mathbf{M}} + r_A^{\mathbf{M}.Y} = r_B^{\mathbf{M}}$ and an exact sequence

$$0 \longrightarrow V_{B\smallsetminus A}(\phi) \longrightarrow V_B(\phi) \longrightarrow V_A(\phi.Y) \longrightarrow 0.$$

(b) Since $S\smallsetminus Y$ is independent, $r_{B\smallsetminus A}^{\mathbf{M}} = |B\smallsetminus A|$ and we have canonical isomorphisms

$$\mathbf{c}_{Y,A}^{\phi} = \mathbf{c}_Y^{\phi} = \mathbf{c} \colon\ \mathring{\wedge}\, V_A(\phi.Y)^* \otimes \mathring{\wedge}\, V_{B\smallsetminus A}(\phi)^* \ \longrightarrow\ \mathring{\wedge}\, V_B(\phi)^*$$

and

$$\mathbf{d}_{Y,A}^{\phi} = \mathbf{d}_Y^{\phi} = \mathbf{d} \colon\ \Bbbk \ \longrightarrow\ \mathring{\wedge}\, U_{B\smallsetminus A} \otimes \mathring{\wedge}\, V_{B\smallsetminus A}(\phi)^*$$

induced by the canonical projection $\pi \colon W \longrightarrow \overline{W}$ and by the map ϕ, respectively.

Definition 1.21. [16, Definition 7.3] *(a) Let A be a T-flat of $\mathbf{M}.Y$, and let B a T-flat containing A. The canonical inclusion homomorphism is given by*

$$(\pi.Y)_A^{\phi}\colon T_A(\phi.Y) \longrightarrow T_B(\phi)$$

as the composition

$$T_A(\phi.Y) \xrightarrow{1\otimes\mathbf{d}} T_{AB}^{(1)} \xrightarrow{\tau} T_{AB}^{(2)} \xrightarrow{\pi_{Y,A}^{*}\otimes\wedge\otimes\mathbf{c}} T_B(\phi),$$

where

$$\begin{aligned} T_{AB}^{(1)} &= T_A(\phi.Y) \otimes \mathring{\wedge} U_{B\setminus A} \otimes \mathring{\wedge} V_{B\setminus A}(\phi)^*, \\ &= S_A(\phi.Y)^* \otimes \mathring{\wedge} U_A \otimes \mathring{\wedge} V_A(\phi.Y)^* \otimes \mathring{\wedge} U_{B\setminus A} \otimes \mathring{\wedge} V_{B\setminus A}(\phi)^*, \\ T_{AB}^{(2)} &= S_A(\phi.Y)^* \otimes \mathring{\wedge} U_A \otimes \mathring{\wedge} U_{B\setminus A} \otimes \mathring{\wedge} V_A(\phi.Y)^* \otimes \mathring{\wedge} V_{B\setminus A}(\phi)^*, \end{aligned}$$

and the map τ is, as usual, the canonical isomorphism that simply rearranges the components of the tensor product in the indicated order.

(b) For each $l \geq 0$, a canonical injective homomorphism is given by

$$(\pi.Y)_l^{\phi}\colon T_l(\phi.Y) \longrightarrow T_l(\phi)$$

by the requirement that it restricts to the component $T_A(\phi.Y)$ of $T_l(\phi.Y)$ as

$$(\pi.Y)_l^{\phi}\big|_{T_A(\phi.Y)} = (-1)^{l|B\setminus A|}(\pi.Y)_A^{\phi}$$

for every T-flat A in $\mathbf{M}.Y$ of level l.

Definition 1.22. [16, Definition 2.4.5] *The T-complex has the form*

$$0 \to T_S(\phi) \xrightarrow{\phi_{l_S}} \bigoplus_{l_A = l_S - 1} T_A(\phi) \xrightarrow{\phi_{l_S-1}} \cdots \to U_S \xrightarrow{\phi} W \to 0$$

where the $\phi_l\,|_{T_I} = \displaystyle\sum_{\substack{J\in\mathcal{T}_{l-1}(\mathbf{M})\\ J\subseteq I}} (-1)^{|J|}\phi_n^{I,J}$.

Theorem 1.23. [16, Theorem 3.5] *The T-complex is a resolution for the* $\ker\phi$.

Example 1.24. *The T-complex defined by the matroid $U_{1,3}$ is*

$$\mathbb{T} \otimes_R \Bbbk : 0 \to \Bbbk \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} \Bbbk \to 0.$$

Recall this is the underlying vector space complex for the Taylor resolution in Example 1.3.

In order to build a free resolution of L from the T-complex, we must homogenize as in the monomial ideal case. Recall that S was taken to be a multihomogeneous basis of E. For a T-flat $I = \{i_1, \ldots, i_k\} \subset S$

$$\deg I = \operatorname{lcm}(\deg i_1, \ldots \deg i_k).$$

We give the free R-module

$$T_I(\Phi, S) = R \otimes_{\Bbbk} T_I(\phi)$$

a canonical multigrading by the formula

$$\deg(z \otimes v) = \deg z + \deg I$$

for any vector $v \in T_I(\phi)$ and any monomial $z \in R$ [16, Section 4]. The canonical homogenization shows that the construction of a free resolution of a multigraded module is completely determined by the underlying vector space complex just as in Section 1.1 for monomial ideals.

The T-resolution of L from [16, Definition 4.3] has the form (with $\lambda = l_S + 2 = |S| - r_S^{\mathbf{M}} + 1$),

$$T_\bullet(\Phi, S) = 0 \to T_\lambda(\Phi, S) \xrightarrow{\Phi_\lambda} T_{\lambda-1}(\Phi, S) \to \cdots \to T_1(\Phi, S) \xrightarrow{\Phi_1} T_0(\Phi, S) \to 0,$$

and the free R-modules are defined as

$$T_i(\Phi, S) = \bigoplus_{I \in \mathcal{T}_{i-2}(M)} T_I(\Phi, S), \quad 2 \le i \le l_S + 2$$

and

$$T_0(\Phi, S) = G \quad \text{and} \quad T_1(\Phi, S) = E.$$

Example 1.25. *Let*

$$R^t \xrightarrow{\begin{pmatrix} m_1 & \dots & m_t \end{pmatrix}} R \longrightarrow R/I \to 0$$

be a free presentation of R/I and $S = \{m_1, \dots, m_t\}$. Then $U_{1,n}$ is the matroid associated to the presentation matrix and the T-resolution is the Taylor resolution for the monomial ideal generated by S.

Since the T-resolution is a generalization of the Taylor resolution, then the T-complex should have a nice combinatorial description. We see the T-flats of the matroid defined from ϕ are an indexing set for the free modules. We will show that this matroid also describes the rank of the free modules and a simplicial complex to describe the free modules in the resolution. In contrast to Taylor, we will associate one simplicial complex to each free module, not one for the entire resolution. A continuing project is to describe the maps in the T-resolution combinatorially, in order to obtain a complete generalization of the T-resolution when $\Bbbk$ is not a field.

1.4. **Reduced Broken Circuit Complex.** In this subsection, we review how a matroid defines a simplicial complex. We recall the reduced broken circuit complex that we will associate to a T-space in Section 3, the description of a β-system, and show how it is an indexing set for the basic homology cycles of the reduced broken circuit complex.

A *simplicial complex* Δ on a finite set V is a collection of subsets of V so that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. The elements of V are called *vertices*, the subsets of Δ are called *faces*. A face not properly contained in another face is called a *facet*. If $F \in \Delta$ is a face, then $\dim F = |F| - 1$. The $\dim \Delta = \max\{\dim F : F \in \Delta\}$. A simplicial complex is *pure* if all facets are equidimensional.

Example 1.26. *Given a matroid $\mathbf{M}$ on S, one obtains a simplicial complex Δ by letting $V = S$ and the faces are those subsets of S that are independent. A simplicial complex formed in this way is called an* independence complex *of the matroid $\mathbf{M}$. This simplicial complex is pure since each facet is defined from a basis of $\mathbf{M}$ and all bases of $\mathbf{M}$ have the same cardinality by Proposition 1.4. One clearly has* $\dim(\Delta) = r_S^{\mathbf{M}} - 1$.

We will often abuse the language and say that a matroid is a simplicial complex by identifying it in this way.

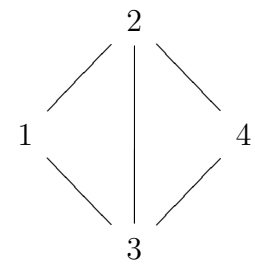
Example 1.27. *The matroid $U_{1,n}$ is the simplicial complex consisting of n points.*

The matroid $U_{2,n} = K_n$, the complete graph on n vertices.

The matroid $U_{r,n}$ is the set of all $(r-1)$-simplices on n vertices.

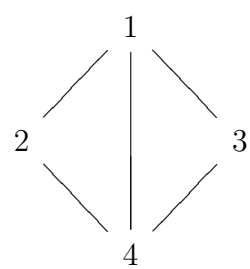
Example 1.28. *Let* $\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$, *then*

$\mathbf{M} =$



and

$\mathbf{M}^* =$



It is clear from the definitions that for the *restriction* of a matroid $\mathbf{M}$ to A, $\mathbf{M}|A$ is the subcomplex

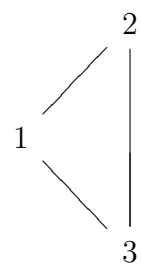
$$\Delta_A = \{F \in \mathbf{M} : F \subseteq A\}.$$

In particular, $\mathbf{M}|(S-a) = \mathbf{M}\backslash a$ is the subcomplex of $\mathbf{M}$ obtained by removing a from all faces of $\mathbf{M}$.

Example 1.29. *If $r \neq n$ then $U_{r,n}\backslash a = U_{r,n-1}$ and if $r = n$ then $U_{n,n}\backslash a = U_{n-1,n-1}$.*

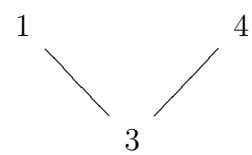
Example 1.30. *Let* $\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$, *then*

$\mathbf{M}\backslash 4 =$



and

$\mathbf{M}\backslash 2 =$



Similarly, the *contraction* of a matroid $\mathbf{M}$ on A, $\mathbf{M}.A$ is the subcomplex

$$\Delta_{\mathbf{M}.A} = \{F \subseteq (S - A) : F \cup B \in \mathbf{M}, \text{ for some } B \subseteq A\}.$$

In particular, $\mathbf{M}.(S - a) = \mathbf{M}/a = \{F \subseteq S - a : F \cup \{a\} \in \mathbf{M}\}$ is the subcomplex $\text{link}(a)$.

Example 1.31. *The matroid* $U_{r,n}/a = U_{r-1,n-1}$.

Example 1.32. *Let* $\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$, *then*

$\mathbf{M}/4 =$

and

$\mathbf{M}/2 =$

Theorem 1.33. *A simplicial complex is the independence complex of a matroid if and only if it is pure and all subcomplexes are pure.* $\square$

An *ordered matroid* is a matroid with a linear ordering on the vertices. Since S is finite, we may assume $S = \{1, \ldots, n\}$ where we assign these numbers so that $1 < 2 < \cdots < n$. In an ordered matroid a *broken circuit* is obtained from a circuit by deleting its smallest element. The family of all subsets of S that contain no broken circuits is called the *broken circuit complex* of $\mathbf{M}$, written $BC(\mathbf{M})$ and the family of all subsets of $S-1$ that contain no broken circuits is called the *reduced broken circuit complex* of $\mathbf{M}$, written $\overline{BC}(\mathbf{M})$.

Example 1.34. *The circuits of $U_{r,n}$ are all $r+1$ element subsets. The broken circuits of $U_{r,n}$ are all r element subsets except those that contain 1. Thus $BC(U_{r,n})$ are all r simplices that contain 1 and $\overline{BC}(U_{r,n}) = U_{r-1,n-1}$.*

Example 1.35. *When* $\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$, *then*

$$BC(\mathbf{M}) = \begin{tikzcd} & 2 \\ 1 \arrow[ur, dash] \arrow[dr, dash] & \\ & 3 \end{tikzcd}$$

and

$$\overline{BC}(\mathbf{M}) = \begin{tikzcd} \bullet 2 \\ \\ \bullet 3 \end{tikzcd}$$

Theorem 1.36. [3, Proposition 3.1 and 3.2] *Let $\mathbf{M}$ be an ordered matroid of rank r. Then*

(1) $\overline{BC}(\mathbf{M}) \subseteq BC(\mathbf{M}) \subseteq \mathbf{M}$.

(2) *$BC(\mathbf{M})$ is a pure (r-1)-dimensional complex of* $\mathbf{M}$.

(3) *$BC(\mathbf{M})$ is a cone over $\overline{BC}(\mathbf{M})$ with apex* $\min(S)$*, whose facets we call nbc-basis.*

(4) *$\overline{BC}(\mathbf{M})$ is a pure (r-2)-dimensional subcomplex of* $\mathbf{M}$.

We will now review some basic topological results about simplicial complexes. Let $F = \{v_0, v_1, \ldots, v_s\}$ be a face of a simplicial complex Δ with vertex set V. Giving an ordering to the vertices of V, we write $F = [v_0, \ldots, v_s]$ when $v_0 < v_1 < \ldots < v_s$ in the given ordering. Let $\widetilde{C}_s(\Delta)$ be the free abelian group generated by the s-dimensional faces of Δ which we canonically identify with the symbols $[v_0, \ldots, v_s]$. The group homomorphism $\partial_s : \widetilde{C}_s(\Delta) \to \widetilde{C}_{s-1}(\Delta)$ is defined on the basis elements as

$$\partial_s[v_0, \ldots, v_s] = \sum_{i=0}^{s} (-1)^i [v_0, \ldots, \hat{v_i}, \ldots v_s].$$

The elements $\rho \in \widetilde{C}_s(\Delta)$ such that $\partial_s(\rho) = 0$ are called *cycles* and form a subgroup $Z_s(\Delta)$. The elements $\sigma \in \widetilde{C}_s(\Delta)$ such that $\sigma = \partial_{s+1}(\tau)$ for some $\tau \in \widetilde{C}_{s+1}(\Delta)$ are called *boundaries* and form a subgroup $B_s(\Delta)$. It is a direct calculation to see that $\partial_s \circ \partial_{s+1} = 0$ for all $s \in \mathbb{Z}$. Thus $B_s(\Delta) \subseteq Z_s(\Delta)$ for all $s \in \mathbb{Z}$. The quotient group $\widetilde{H}_s(\Delta) = Z_s(\Delta)/B_s(\Delta)$ is the s-dimensional *reduced homology group* of Δ. A complex Δ is called *acyclic* if $\widetilde{H}_s(\Delta) = 0$ for all $s \in \mathbb{Z}$.

A *shelling* of a pure simplicial complex of dimension d is a linear ordering of the facets $F_1, F_2, \ldots, F_k$ so that for all $i > 1$ the intersection $F_i \cap (\bigcup_{j<i} F_j)$ is a nonempty union of facets of the boundary ∂F_i. In a shelling, F_i is a *homology facet* if $F_i \cap (\bigcup_{j<i} F_j) = \partial F_i$. This name reflects the fact that these particular faces index a basis for the homology of the shellable simplicial complex:

Theorem 1.37. [1, Theorem 7.7.2] *Let Δ be a shellable simplicial complex of dimension d with homology facets $\{F_1, \ldots, F_k\}$. Then*

(1)

$$\widetilde{H}_i(\Delta) = \begin{cases} \mathbb{Z}^k & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}.$$

(2) *There are cycles* $\sigma_1, \ldots, \sigma_k \in \widetilde{H}_d(\Delta)$ *uniquely determined by*

$$\sigma_i(F_j) = \delta_{ij} \quad \textit{(Kronecker's delta)},$$

where $\sigma(F)$ *is the coefficient of* F *in the formal linear combination* $\sigma \in \widetilde{C}_d(\Delta)$.

(3) $\{\sigma_1, \ldots, \sigma_k\}$ *is a basis of the free group* $\widetilde{H}_d(\Delta)$ *and are called the basic cycles corresponding to the shelling.*

Theorem 1.38. [1, Theorem 7.4.3] *The lexicographic ordering on the facets of the simplicial complexes* $BC(\mathbf{M})$ *and* $\overline{BC}(\mathbf{M})$ *induces a shelling.*

Hence, $\overline{BC}(\mathbf{M})$ is a shellable simplicial complex under any fixed ordering of S and this will give a canonical set of generators for the homology. However, we need to be able to identify the homology facets of $\overline{BC}(\mathbf{M})$. This is accomplished in [19] with a set called a β-system, denoted $\beta\text{nbc}(\mathbf{M})$.

A basis B has *internal activity* if there is an element $b \in B,\ b = \min(c^*(B, b))$ where $c^*(B, b)$ is the unique circuit in the dual matroid $\mathbf{M}^*$ containing b and contained in $(S \backslash B) \cup \{b\}$ ([19], Definition 1.3). If $1 \in B$ then B will have 1 as an internally active element. We say that B has no non-trivial internal activity if the only element of B that is internally active is 1.

Definition 1.39. [19, Definition 1.3] *A* β*-system of* $\mathbf{M}$ *is the set*

$$\beta nbc(\mathbf{M}) = \{B \textit{ is an nbc-basis } \mid B \textit{ has no non-trivial internal activity}\}.$$

Theorem 1.40. [19, Theorem 1.4] *If* $\mathbf{M}$ *is an ordered loopless matroid, then* $B \in \beta nbc(\mathbf{M})$ *if and only if* $B \backslash 1$ *is a homology facet for* $\overline{BC}(\mathbf{M})$.

Theorem 1.41. [19, Section 1] *Let* $B \in \beta nbc(\mathbf{M})$ *and define the set map* $\varphi : B \to S$ *which sends* $\varphi(b) = \min(c^*(B, b))$. *We find that if* $b \neq 1$, *then* $\min(c^*(B, b)) < b$ *because* $b \in c^*(B, b)$ *and* B *has no nontrivial internal activity. Set* $A_i = \{i\} \cup \phi^{-1}(i)$. *Then*

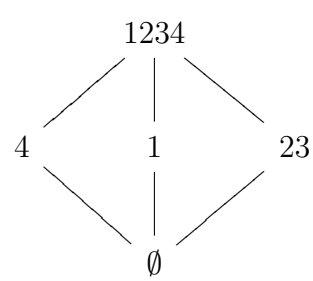
$$\overline{\sigma}_B = \sum_{i_1=0}^{e_1} \sum_{i_2=0}^{e_2} \ldots \sum_{i_k=0}^{e_k} (-1)^{i_1+\ldots+i_k} (A_1 - a_{i_1}^1) \cup (A_2 - a_{i_2}^2) \cup \ldots \cup (A_k - a_{i_k}^k)$$

is the canonical basic cycle in the reduced homology of the reduced broken circuit complex associated to B with respect to the lexicographic ordering, where the union is understood to be the elements of the cycle listed in increasing order and $e_i = |A_i|$.

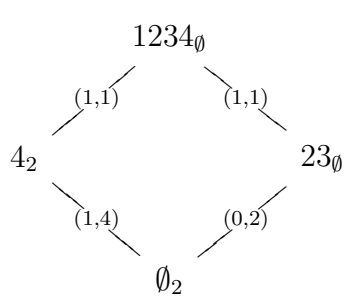
An alternate method of identifying a β-system is given in [19, Section 2], by establishing a certain shelling called an EL-shelling on $\overline{BC}(\mathbf{M})$ as follows. Create the lattice of flats for $\mathbf{M}$. Remove the entries of the lattice that contain the smallest element 1, but keep S as the maximal flat of the remaining lattice. For a flat F, let $A(F)$ be the lexicographically least basis of $\mathbf{M}/(1 \cup F)$ in the fixed ordering. Let G cover F, which we will denote $F - G$ in the revised lattice. The label for the cover $F - G$ is

$$\lambda(F, G) = \begin{cases} (0, \min(G \cap A(F))) & \text{if } G \cap A(F) \neq \emptyset; \\ (1, \min(G \backslash F)) & \text{otherwise.} \end{cases}$$

Example 1.42. *Let* $\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$ *and* $\mathbf{M}$ *be the matroid defined by* ϕ*. Recall the lattice of flats from Example 1.14,*

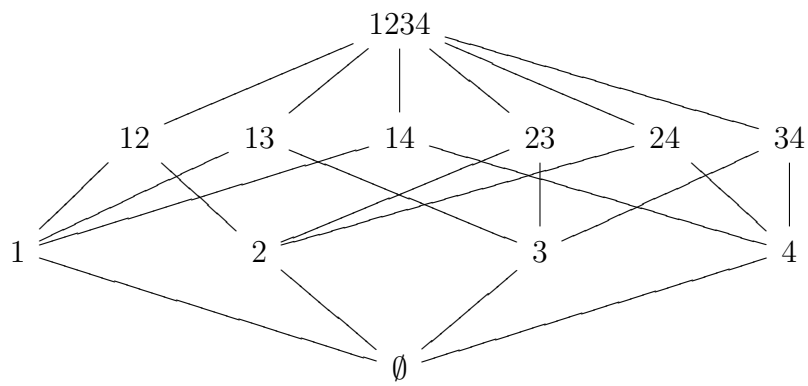


Labeling the edges as prescribed above we get

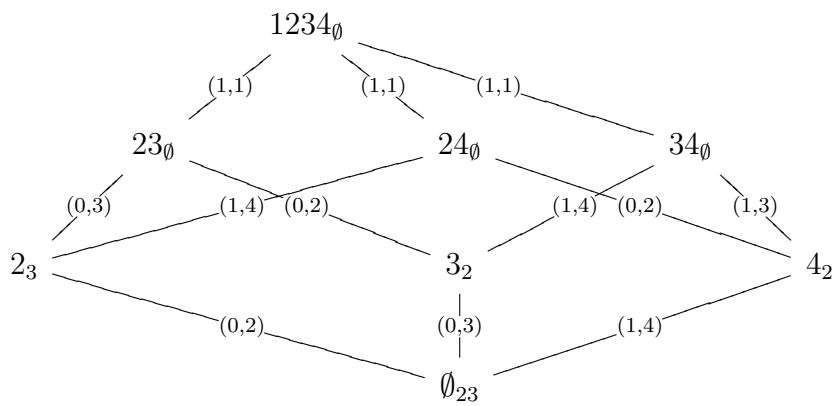


where the subscripts denote $A(F)$.

Example 1.43. *The lattice of flats of $U_{3,4}$ ordered by inclusion.*



and with the labels we have



Theorem 1.44. [19, Theorem 2.4] *Let* $\mathbf{M}$ *be an ordered matroid then*

$$B = \{b_r > b_{r-1} > \ldots > b_2 > 1\} \in \beta nbc(\mathbf{M})$$

if and only if

$$(1, b_r), (1, b_{r-1}), \ldots, (1, b_2), (1, 1)$$

is the decreasing sequence of labels for the EL-shelling.

Example 1.45. *The uniform matroid has*

$$\beta nbc(U_{r,n}) = \{X \subseteq S \mid |X| = r, 1 \in X, \text{ and } 2 \notin X\}.$$

As X is a basis of $\mathbf{M}$, so $|X| = r$. Also, since X contains no broken circuits, then $1 \in X$ from Theorem 1.36. For each $b \in B$, $c^*(B, b) = S \backslash B \cup \{b\}$, so that if $2 \in B$, then 2 is an internally active element and if $2 \notin B$, then all $b \neq 1$ have $\min(c^*(B, b)) = 2$.

1.5. **Beta Invariant.** The *Möbius function* μ on a partially ordered set, poset P, is a function $\mu : P \times P \to \mathbb{Z}$ defined as follows

$$\mu(X, X) = 1,$$

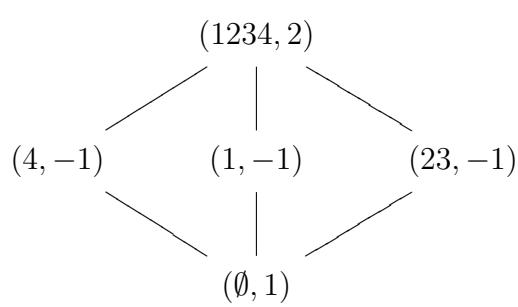
$$\sum_{X \leq Z \leq Y} \mu(X, Z) = 0 \text{ if } X < Y,$$

and

$$\mu(X, Z) = 0 \text{ otherwise.}$$

Example 1.46. *For the lattice of flats of the matroid $U_{r,n}$, one has $\mu(\emptyset, F) = (-1)^{|F|}$ for a flat $F \neq S$ and $\mu(\emptyset, S) = (-1)^{l_S - 1}$.*

Example 1.47. *Let $\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$ and let $\mathbf{M}$ be the matroid defined by ϕ as in Example 1.6. Then the lattice of flats of $\mathbf{M}^*$ as in 1.14 is labeled below by $(F, \mu(\emptyset, F))$, for each flat F.*



Theorem 1.48 (Möbius Inversion)**.** [17, Theorem 15.2.1] *If f is a real valued function on a poset P and*

$$f(X) = \sum_{X \leq Y} g(Y)$$

for some real valued function g on P, then

$$g(X) = \sum_{X \leq Y} \mu(X, Y) f(Y).$$

Remark 1.49. *The Möbius function for the lattice of T-flats ordered by reverse inclusion is precisely the same as the Möbius function for the lattice of flats ordered by inclusion in the dual matroid.*

Crapo's beta invariant is the number

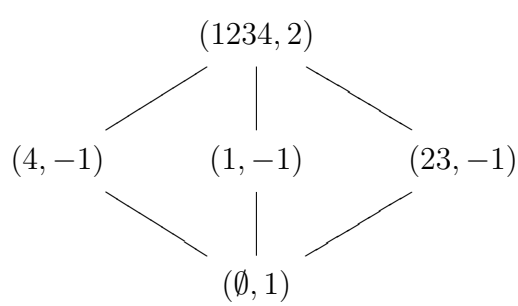
$$\beta(\mathbf{M}) = (-1)^{r_S^{\mathbf{M}}} \sum_{X \in \mathfrak{L}(\mathbf{M})} \mu(\emptyset, X) r_X^{\mathbf{M}},$$

where $\mathfrak{L}(\mathbf{M})$ is the lattice of flats for the matroid $\mathbf{M}$ on S.

Example 1.50. *We have $\beta(U_{r,n}) = \binom{n-2}{r-1}$ from [7] using direct computation. However, this is also clear from Example 1.50 and Lemma 1.53. We see from Example 1.43 that $U_{3,4}$ has one decreasing chain and that*

$$\beta(U_{3,4}) = \binom{4-2}{3-1} = \binom{2}{2} = 1.$$

Example 1.51. *Let $\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$ and let $\mathbf{M}$ be the matroid defined by ϕ then the lattice of flats of $\mathbf{M}^*$ with the Möbius function μ from Example 1.47 is*



Thus, $\beta(\mathbf{M}) = (-1)^2[0 + (-1)(3)(1) + (2)(2)] = 1$ and this coincides with the number of decreasing chains from Example 1.42.

We will show that Crapo's beta invariant of $\mathbf{M}$ is the dimension of the T-spaces $T_S(\phi)$ in the T-complex. First, however, we recall some facts about this number and give support for the reduced broken circuit complex to be the correct simplicial complex to associate to this T-space.

Theorem 1.52. [7] *Crapo's beta invariant satisfies the following properties.*

(1) *The number* $\beta(\mathbf{M}) \geq 0$ *for every matroid* $\mathbf{M}$.

(2) $\beta(\mathbf{M}) = 0$ *precisely when* $\mathbf{M}$ *is disconnected.*

(3) *For any matroid* $\mathbf{M}$ *with dual* $\mathbf{M}^*$, $\beta(\mathbf{M}) = \beta(\mathbf{M}^*)$

(4) *If* a *is not a loop or a coloop, then* $\beta(M) = \beta(M/a) + \beta(M\backslash a)$.

In order to establish the isomorphism between the reduced homology of the broken circuit complex and the dimension of a T-space, we first must equate their dimension. Theorems 1.37, 1.38 and 1.39 together show that for a matroid $\mathbf{M}$ of rank r the only non-zero reduced homology of the reduced broken circuit complex is in degree $r-2$ and of dimension $|\beta\text{nbc}(\mathbf{M})|$.

Lemma 1.53. [19, Section 1] *Let* $\mathbf{M}$ *be an ordered matroid, then* $|\beta nbc(\mathbf{M})| = \beta(\mathbf{M})$.

2. Upper Bounds for Betti Numbers of Multigraded Modules

This section gives a sharp upper bound for the Betti numbers of a finitely generated multigraded R-module. Much work has been done on establishing lower bounds for Betti numbers of L, initially motivated by the Buchsbaum-Eisenbud-Horrocks conjecture for finite modules over regular local rings. This conjecture was shown to hold for R/I when I is a monomial ideal by Evans and Griffith [9] and generally for all multigraded modules by Charalambous [4] and Santoni [14].

On the other hand, little is known about upper bounds for the Betti numbers. The main result of this section gives sharp upper bounds for the total Betti number of L in each homological degree in terms of the rank and the first two Betti numbers of the module L.

Theorem 1. *For $i \geq 2$, we have*

$$\beta_i(L) \leq \binom{\beta_1(L)}{\beta_0(L) - \operatorname{rank} L + i - 1}\binom{\beta_0(L) - \operatorname{rank} L + i - 3}{i-2}.$$

These bounds are precisely the ranks of the free modules in the multigraded Buchsbaum-Rim complex from [5] (called there the Buchsbaum-Rim-Taylor complex). However, there seems to be no appropriate map that achieves comparison between the minimal free resolution of L and the Buchsbaum-Rim complex. Instead, we obtain our bound using the combinatorial structure of the (not necessarily minimal) free resolution defined by Tchernev [16]. We show that our bounds are sharp by giving a class of examples that attains them simultaneously in each homological degree. To achieve this, we make the generators of L and their multidegrees sufficiently generic, so that the Buchsbaum-Rim-Taylor complex is the minimal free resolution. In Section 2.2, as a corollary to our main theorem, we give bounds for the total multigraded Bass numbers of multigraded modules in terms of the first two total multigraded Bass numbers, by using the Alexander duality functors defined by Miller [11].

2.1. Betti Numbers of Multigraded Modules.

We shall need the following lemma to prove Theorem 1. Then we give an example to show that the bounds are sharp.

Lemma 2.1. *Let* $\mathbf{M}$ *be a matroid on a finite set* S*. If* $|S| = n$ *and* $r_S^{\mathbf{M}} = r$*, then*

$$|\mathcal{T}_k(\mathbf{M})| \leq \binom{n}{r+k+1}.$$

Proof. First, rewrite the rank of the dual matroid as follows

$$r_A^{\mathbf{M}^*} = |A| - r_S^{\mathbf{M}} + r_{S\setminus A}^{\mathbf{M}} = |S| - r_S^{\mathbf{M}} - l_{S\setminus A}^{\mathbf{M}} - 1.$$

In the above notation, we see for a T-flat A of level k, the rank of its complementary flat in the dual matroid is

$$r_{S\setminus A}^{\mathbf{M}^*} = n - r - k - 1.$$

By definition

$$|\mathcal{T}_k(\mathbf{M})| = |\mathcal{F}_{n-r-k-1}(\mathbf{M}^*)|,$$

where $\mathcal{F}_\rho(\mathbf{M})$ denotes the collection of all flats with rank ρ in the matroid $\mathbf{M}$. Since a subset has the property that $r_A^{\mathbf{M}} \leq |A|$, we see that the number of flats with rank $n-r-k-1$ must be less than the total number of subsets of S with cardinality $n-r-k-1$. Thus

$$|\mathcal{T}_k(\mathbf{M})| = |\mathcal{F}_{n-r-k-1}(\mathbf{M}^*)| \leq \binom{n}{n-r-k-1} = \binom{n}{r+k+1},$$

which verifies our bound for the number of T-flats of level k. □

Proof of Theorem 1. Let $E \xrightarrow{\Phi} G \longrightarrow L \longrightarrow 0$ be a minimal finite free multigraded presentation of L so that $\beta_0(L) = \operatorname{rank}(G)$ and $\beta_1(L) = \operatorname{rank}(E)$. When $i \geq 2$ we have

$$\beta_i(L) \leq \operatorname{rank}\big(T_i(\Phi, S)\big) = \operatorname{rank}\left(\bigoplus_{I \in \mathcal{T}_{i-2}(\mathbf{M})} R \otimes T_I(\phi)\right)$$

by definition and

$$\operatorname{rank}\left(\bigoplus_{I \in \mathcal{T}_{i-2}(\mathbf{M})} R \otimes T_I(\phi)\right) \leq |\mathcal{T}_{i-2}(\mathbf{M})| \max_{I \in \mathcal{T}_{i-2}(\mathbf{M})} (\dim_{\Bbbk} T_I(\phi)).$$

It follows from the definition [16, Definition 2.2.3] that for each $I \in \mathcal{T}_k(\mathbf{M})$ one has

$$\dim T_I(\phi) \leq \dim S_k(V) = \binom{r+k-1}{k},$$

where $S_k(V)$ denotes the kth symmetric power of $V = Im\phi$ and $r = \dim_{\Bbbk}(V)$. Therefore, by Lemma 2.1, we get

$$\beta_i(L) \leq \binom{|S|}{r_S^{\mathbf{M}} + i - 1}\binom{r_S^{\mathbf{M}} + i - 3}{i - 2}.$$

Since

$$|S| = \operatorname{rank}(E) = \beta_1(L)$$

and

$$r_S^{\mathbf{M}} = \dim_{\Bbbk} V = \operatorname{rank}(\phi) = \operatorname{rank}(\Phi) = \operatorname{rank} G - \operatorname{rank} L = \beta_0(L) - \operatorname{rank} L,$$

the bounds on Betti numbers of multigraded modules have been established. □

Next, we give an example of a finite multigraded module that achieves these bounds simultaneously in each homological degree.

Example 2.2. *Let L be a finite multigraded module with a minimal presentation matrix of uniform rank. In the above set up, this is equivalent to saying that if the rank of the matrix ϕ is r then the image of every r-element subset of S has dimension r. As such, the free R-modules described in* [16, Definition 4.2] *become precisely those of the Buchsbaum-Rim-Taylor complex defined in* [5, Definition 4.3]. *In addition, choose the multidegrees of the generators of E to be generic. In that case, no two T-flats have the same multidegree. Therefore, the Buchsbaum-Rim-Taylor complex is the minimal free multigraded resolution of L. Thus for $i \geq 2$,*

$$\beta_i(L) = \binom{|S|}{r + i - 1}\binom{r + i - 3}{i - 2};$$

our bounds are simultaneously achieved.

2.2. **Multigraded Bass numbers of multigraded modules.** We write $\mu_i(\mathfrak{p}, L)$ for the ith Bass number of an R-module L at a prime $\mathfrak{p}$. Since we are over a polynomial ring and L is multigraded, there are only a finite number of mulitgraded primes $\mathfrak{p}$ generated by $\{x_{j_1}, \ldots, x_{j_l}\}$ for $\{j_1, \ldots, j_l\} \subset \{1, \ldots, m\}$. Thus it makes sense to call

the well-defined integer

$$^*\mu_i(L) = \sum_{\mathfrak{p}:\text{multigraded prime ideal}} \mu_i(\mathfrak{p}, L),$$

the total multigraded i^{th} Bass number of L.

Theorem 2.3. *Let $R = \Bbbk[x_1, \ldots, x_m]$ be a polynomial ring over a field $\Bbbk$ with the standard grading, L a finite multigraded R-module, then the bounds for $i \geq 2$ of the total multigraded Bass numbers of L are*

$$^*\mu_i(L) \leq \binom{^*\mu_1(L)}{^*\mu_0(L) + i - 1}\binom{^*\mu_0(L) + i - 3}{i - 2}.$$

Proof. For any $\boldsymbol{c}$, $^*\mu_i(L(-\boldsymbol{c})) = {}^*\mu_i(L)$. Since L is finitely generated and has a finite multigraded injective resolution, there is a $\boldsymbol{c}$ so that $L(-\boldsymbol{c})$ only has non-zero degrees greater than or equal to $\mathbf{1}$ and so that the nonzero Bass numbers at each multigraded prime occur in only positive degrees. Thus it suffices to establish our bound when L satisfies these two conditions.

Let $\boldsymbol{a}$ be the componentwise maximum of the degrees of the 0th and 1st Betti numbers of L. Thus L is a positively $\boldsymbol{a}$-determined module as defined in [11, Definition 2.1] and the minimal free resolution will be positively $\boldsymbol{a}$-determined [11, Proposition 2.5].

We write $L^{\boldsymbol{a}}$ for the Alexander dual of an R-module L with respect to degree $\boldsymbol{a}$ as defined in [11]. We write $B_{\boldsymbol{a}}L$ of an R-module for the quotient of L by the submodule $\bigoplus_{\boldsymbol{b} \not\leq \boldsymbol{a}} L_{\boldsymbol{b}}$ as in [11]. In the case when L is generated in degrees greater than $\mathbf{1}$, we have that $B_{\boldsymbol{a}}L^{\boldsymbol{a}} = L^{\boldsymbol{a}}$, since $L^{\boldsymbol{a}}$ is bounded inside the interval $[\mathbf{0}, \boldsymbol{a} - \mathbf{1}]$. Thus we will use Miller's results for $B_{\boldsymbol{a}}L^{\boldsymbol{a}}$ and $L^{\boldsymbol{a}}$ interchangeably throughout the remainder of this section.

Let

$$0 \longrightarrow L \longrightarrow I \xrightarrow{\Lambda} J$$

be a minimal multigraded copresentation of L. Miller [11, Theorem 4.5] shows that the matrix Λ is also a minimal presentation matrix for a free resolution of $L^{\boldsymbol{a}}$ after some appropriate degree shifts. In establishing bounds for the Betti numbers in the

previous section, we obtain the coefficient matrix of the vector space complex, which is independent of the degrees of the original free modules. Thus the coefficient matrix of Λ defines a representable matroid for the multigraded module $L^{\boldsymbol{a}}$. One easily sees that the bounds established in the Main Theorem depend only on the presentation matrix. Since $\operatorname{rank} L^{\boldsymbol{a}} = 0$ we have for $i \geq 2$

$$\beta_i(L^{\boldsymbol{a}}) \leq \binom{\beta_1(L^{\boldsymbol{a}})}{\beta_0(L^{\boldsymbol{a}}) + i - 1}\binom{\beta_0(L^{\boldsymbol{a}}) + i - 3}{i - 2}.$$

Further, Miller [11, Theorem 5.3] establishes the relation

$$\beta_{i,\boldsymbol{b}}(L^{\boldsymbol{a}}) = \mu_{i,(\boldsymbol{a}\setminus\boldsymbol{b})-supp(\boldsymbol{b})}(\mathfrak{m}^{supp(\boldsymbol{b})}, L)$$

where $0 \leq \boldsymbol{b} \leq \boldsymbol{a} \cdot supp(\boldsymbol{b})$. By summing over all possible degrees and since the nonzero Bass numbers at each multigraded prime occur in positive degrees, we see that ${}^*\mu_0(L) = \beta_0(L^{\boldsymbol{a}}) =$ number of rows of Λ, ${}^*\mu_1(L) = \beta_1(L^{\boldsymbol{a}}) =$ number of columns of Λ, and

$${}^*\mu_i(L) = \beta_i(L^{\boldsymbol{a}}).$$

Therefore, the bounds for the total multigraded Bass numbers have been established. □

The following corollary generalizes this bound to all Bass numbers.

Corollary 2.4. *Let $R = \Bbbk[x_1, \ldots, x_m]$ be a polynomial ring over a field $\Bbbk$ with the standard $\mathbb{Z}^m$ grading, L a finite multigraded R-module and $\mathfrak{p}$ a prime ideal of R. Let $d = \dim R_{\mathfrak{p}}/\mathfrak{p}^* R_{\mathfrak{p}}$, where $\mathfrak{p}^*$ denotes the largest multigraded prime ideal of R contained in $\mathfrak{p}$. Then, for $i \geq 2 + d$, the bounds for Bass numbers of L are*

$$\mu_i(\mathfrak{p}, L) \leq \binom{{}^*\mu_1(L)}{{}^*\mu_0(L) + i - d - 1}\binom{{}^*\mu_0(L) + i - d - 3}{i - d - 2}.$$

Proof. Goto and Watanabe showed in [10] that

$$\mu_i(\mathfrak{p}, L) = \mu_{i-d}(\mathfrak{p}^*, L).$$

Clearly, $\mu_{i-d}(\mathfrak{p}^*, L) \leq {}^*\mu_{i-d}(L)$. Thus Theorem 2.3 establishes our bounds. □

3. The Combinatorial Description

Section 1.3 described how the T-resolution of a multigraded R-module L is determined from the T-complex by homogenization. Thus to understand the structure of the T-resolution, it is enough to understand the structure of the T-complex. In this section, we study the vector spaces of the T-complex called the T-spaces. In the first subsection, we show that the $\dim_{\Bbbk} T_A(\phi) = \beta((\mathbf{M}|A)^*)$, Crapo's beta invariant on the restricted matroid $(\mathbf{M}|A)^*$. The second subsection is devoted to understanding the structure of the reduced broken circuit complex under restriction and contraction away from the biggest element a with respect to a fixed ordering. There is a natural decomposition of the complex which gives the short exact sequence

$$0 \to \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*\backslash a);\Bbbk) \xrightarrow{\epsilon} \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*);\Bbbk) \xrightarrow{\delta} \widetilde{H}_{l-2}(\overline{BC}(\mathbf{M}^*/a);\Bbbk) \to 0,$$

because the only non-zero reduced homology occurs in the top dimension by Theorem 1.37. If $r = \operatorname{rank}(\mathbf{M}^*)$ then the top dimension for the reduced broken circuit complex by Theorem 1.36 is $r - 2 = |S| - r_S^{\mathbf{M}} - 2 = l_S - 1$.

The final subsection will show that the set $\beta\mathrm{nbc}(\mathbf{M}^*)$ is a canonical indexing set for both S_S and T_S. In addition, we construct a canonical isomorphism

$$\widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*);\Bbbk) \xrightarrow{\theta} T_S(\phi).$$

This allows us to describe an explicit basis for each T-space $T_A(\phi)$. Moreover, this isomorphism gives each vertical map in a commutative diagram

$$\begin{array}{ccccccc}
0 \to \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*\backslash a);\Bbbk) & \to & \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*);\Bbbk) & \xrightarrow{\delta} & \widetilde{H}_{l-2}(\overline{BC}(\mathbf{M}^*/a);\Bbbk) \to 0 \\
\downarrow{\scriptstyle \theta/a} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta\backslash a} \\
0 \to T_{S\backslash a}(\phi/a) & \longrightarrow & T_S(\phi) & \longrightarrow & T_{S\backslash a}(\phi\backslash a) \to 0,
\end{array}$$

where the top sequence is induced from the reduced chain complex and the sequence on the bottom row is from [16, Proof of Theorems 3.4, 3.5, and 5.45].

3.1. **Dimension of a T-space.**

We will find the dimension of the T-space to be Crapo's beta invariant discussed in Section 1.5.

Theorem 3.1. *The dimension of a T-space of a T-flat A is given by*

$$(-1)^{l_A} \dim(T_A) = \sum_{B \subseteq A} \mu(A, B)(l_B + 1)$$

where $\mu(A, B)$ is the Möbius function for the partially ordered set of connected T-flats for the matroid $\mathbf{M}$ under reverse inclusion.

Proof. The T-complex is a resolution for $\ker \phi$. By [16, Theorem 5.4.1], restricting the resolution of the $\ker \phi$ to A the result will be a resolution of $\ker(\phi|A)$. This new resolution will end at $T_A(\phi)$ with all remaining T-spaces of the form $T_B(\phi)$ where $B \subseteq A$, and

$$0 = |A| - r_A^{\mathbf{M}} - \sum_{B \subseteq A} (-1)^{l_B} \dim(T_B(\phi)).$$

We can simply rewrite this as

$$l_A + 1 = \sum_{B \subseteq A} (-1)^{l_B} \dim(T_B(\phi)).$$

Next, let us define

$$f(A) = l_A + 1$$

and

$$g(B) = (-1)^{l_B} \dim T_B(\phi),$$

so that with the poset defined with reverse inclusion the above equation can be written

$$f(A) = \sum_{B \subseteq A} g(B).$$

Using Möbius inversion from Theorem 1.48 we obtain

$$g(A) = \sum_{B \subseteq A} \mu(A, B) f(B).$$

Thus our equation is verified. □

Theorem 2. *Given a representable matroid $\mathbf{M}$ on a finite set S with representation ϕ,*

$$\dim T_S(\phi) = \beta(\mathbf{M}^*).$$

Proof. $\beta(\mathbf{M}^*) = (-1)^{r_S^{\mathbf{M}^*}} \sum_{x \in \mathfrak{L}(\mathbf{M}^*)} \mu(\emptyset, x) r_x^{\mathbf{M}^*}$, where $\mathfrak{L}(\mathbf{M}^*)$ is the lattice of flats of the dual matroid. This yields

$$\begin{aligned}
&\beta(\mathbf{M}^*) \\
&= (-1)^{l_S+1} \textstyle\sum_{x \in \mathfrak{L}(\mathbf{M}^*)} \mu(\emptyset, x) \left[(l_S + 1) - (l_{S \setminus x} + 1)\right] \\
&= (-1)^{l_S+1}(l_S + 1) \left[\textstyle\sum_{x \in \mathfrak{L}(\mathbf{M}^*)} \mu(\emptyset, x)\right] + (-1)^{l_S} \left[\textstyle\sum_{x \in \mathfrak{L}(\mathbf{M}^*)} \mu(\emptyset, x)(l_{S \setminus x} + 1)\right] \\
&= (-1)^{l_S} \textstyle\sum_{x \in \mathfrak{L}(\mathbf{M}^*)} \mu(\emptyset, x)(l_{S \setminus x} + 1) \\
&= (-1)^{l_S} \textstyle\sum_{B \in \mathcal{T}(\mathbf{M})} \mu(S, B)(l_B + 1) = \dim T_S(\phi).
\end{aligned}$$

□

Example 3.2. *If* $\mathbf{M} = U_{r,n}$*, then* $\dim T_S(\phi) = \binom{n-2}{r-1} = \beta(U_{r,n}) = \beta(U_{r,n}^*)$ *where the first equality follows by* [16, Example 2.2.10]*, the second is Example 1.50, and the third is Theorem 1.52.*

Corollary 3.3. *Given a representable matroid* $\mathbf{M}$ *on a finite set* S *with representation* ϕ *and a T-flat* A *of* $\mathbf{M}$*, then* $\dim T_A(\phi) = \beta((\mathbf{M}|A)^*)$.

Proof. This corollary is immediate from Theorem 2 since $T_A(\phi) = T_A(\phi|A)$ [16, Theorem 5.4.1] where $\phi|A$ is the submatrix of ϕ whose columns are indexed by A. Thus replacing the $\mathbf{M}$ by $\mathbf{M}|A$ we obtain the result for all T-flats A. □

3.2. **Broken Circuit Complex.** Since Crapo's beta invariant satisfies the decomposition property from Theorem 1.36(4), the decreasing chains of the shelling on the lattice must also satisfy this relation. The next proposition shows exactly how this transfers.

Proposition 3.4. *Consider a linear ordering on* S *of a connected matroid* $\mathbf{M}$ *and let* $a = \max(S)$*. If* a *is a loop, then the decreasing chains of* $\mathbf{M}$ *are precisely the decreasing chains of* $\mathbf{M} \backslash a$*. Otherwise, the decreasing chains of* $\mathbf{M} \backslash a$ *are those decreasing chains of* $\mathbf{M}$ *that do not pass through* $\overline{\{a\}}$*. Also, the decreasing chains of* $\mathbf{M}/a$ *are precisely those decreasing chains of* $\mathbf{M}$ *that pass through* $\overline{\{a\}}$ *with* a *removed.*

Proof. Consider the decreasing chains of $\mathbf{M}/a$. The lattice of flats for $\mathbf{M}/a$ is precisely the interval $[\overline{\{a\}}, S]$ of the lattice of flats of $\mathbf{M}$ with each flat F replaced by $F\backslash a$. Consider the cover in a decreasing chain $F - G$ that has label $(1, y)$ and $F - G \in [\overline{\{a\}}, S]$. Then $F\backslash a - G\backslash a$ is a cover in $\mathbf{M}/a$. In this way, $A(F\backslash a) =$ lexicographically least basis element of $(\mathbf{M}/a)/(F\backslash a \cup \{1\}) = \mathbf{M}/(F \cup \{1\})$. Thus $G\backslash a \cap A(F\backslash a) = G\backslash a \cap A(F) = \emptyset$ since $G \cap A(F) = \emptyset$. Since $(G\backslash a)\backslash(F\backslash a) = G\backslash F$, $y = \min\{(G\backslash a)\backslash(F\backslash a)\}$. Thus decreasing chains keep their labels under contraction, and it follows that the decreasing chains of $\mathbf{M}/a$ are those of $\mathbf{M}$ passing through $\overline{\{a\}}$.

The lattice of flats for $\mathbf{M}\backslash a$ is obtained from the lattice of flats of $\mathbf{M}$ by removing the element a from each flat containing a and collapsing redundant edges. We establish the correspondence of decreasing chains of $\mathbf{M}\backslash a$ in the remaining two cases.

Now, consider the decreasing chains of $\mathbf{M}$. Suppose $F - G$ is a cover in the chains of $\mathbf{M}$ so that $a \notin G$. Then $A(F\backslash a)$ is the lexicographic least basis of $(\mathbf{M}\backslash a)/(F\backslash a \cup \{1\}) = (\mathbf{M}/(F \cup \{1\}))\backslash a$ and so $A(F\backslash a) = A(F)\backslash a$ and $G \cap A(F\backslash a) = G \cap A(F)\backslash a = \emptyset$. Clearly, $\min(G\backslash F)$ remains the same label.

Suppose $F - G$ is a cover in $\mathbf{M}$ so that $a \in G$ but $a \notin F$. If $G = F \cup \{a\}$ then the decreasing chain has label $(1, a)$, but since that is the biggest possible label, this implies that $F = \emptyset$. In this case, the edge $F - G$ would collapse under restriction and so could not be the maximal chain in $\mathbf{M}\backslash a$ since $\mathbf{M}$ is connected. So, there exists $z \in G, z \neq a$, so that $z \notin F$. Since $G \cap A(F) = \emptyset$ by assumption, then $a \notin A(F)$. Thus, $G\backslash a \cap A(F\backslash a) = G\backslash a \cap A(F)\backslash a$ as above and so $G\backslash a \cap A(F\backslash a) = \emptyset$. Also, $\min(G\backslash F) = \min((G\backslash a)\backslash F)$ since $a = \max(S)$. Therefore, each decreasing chain is a decreasing chain under contraction if it passes through $\overline{\{a\}}$ or restriction otherwise. $\square$

The basic cycles of the reduced homology of the reduced broken circuit complex decompose as in $\beta(\mathbf{M})$ in Theorem 1.52 (4) and $\beta\mathrm{nbc}(\mathbf{M}^*)$ in Proposition 3.4. When $\mathbf{M}$ is a connected matroid and a is the largest element in the ordering on S the simplicial complex $\overline{BC}(\mathbf{M}\backslash a)$ is a subcomplex of $\overline{BC}(\mathbf{M})$ consisting of all faces $\overline{BC}(\mathbf{M})$ that do not contain a by Proposition 3.4. This induces a canonical exact sequence of

reduced chain complexes

$$0 \to \widetilde{C}(\overline{BC}(\mathbf{M}\backslash a)) \to \widetilde{C}(\overline{BC}(\mathbf{M})) \to \widetilde{C}(\overline{BC}(\mathbf{M}/a))[-1] \to 0,$$

which induces the following long exact sequence in reduced homology:

$$0 \to \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}\backslash a);\Bbbk) \xrightarrow{\epsilon} \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M});\Bbbk) \xrightarrow{\delta} \widetilde{H}_{l-2}(\overline{BC}(\mathbf{M}/a);\Bbbk) \to 0$$

where l is the level of $\mathbf{M}$.

Lemma 3.5. *For $\overline{\sigma}_B \in \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}\backslash a);\Bbbk)$,*

$$\epsilon(\overline{\sigma}_B) = \overline{\sigma}_B \in \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M});\Bbbk)$$

and the map δ evaluated on a cycle $\overline{\sigma}_B \in \widetilde{H}_{l-2}(\overline{BC}(\mathbf{M});\Bbbk)$ yields

$$\delta(\overline{\sigma}_B) = \begin{cases} \overline{\sigma}_{B\backslash a} & \text{if } a \in B \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First we show that the map $\varphi : B \to S$ where $B \in \beta\mathrm{nbc}(\mathbf{M})$ is preserved under restriction if $a \notin B$ and under contraction if $a \in B$.

If $a \notin B$, then $c^*_{\mathbf{M}\backslash a}(B,b) = c^*_{\mathbf{M}}(B,b)\backslash a$. Also, we know that $a \neq \min c^*(B,b)$ since $\min c^*(B,b) < b < a$. Therefore, the map φ is preserved under restriction.

For the contraction, we use a similar approach. Since $a \in B$ and $a \neq b$ and $c^*(B,b) \subseteq (S\backslash B) \cup \{b\}$ so that $a \notin c^*(B,b)$ and thus $c^*(B,b)$ is a circuit in $\mathbf{M}\backslash a$. Therefore, the map φ is also preserved under contraction.

The map ϵ is induced from the inclusion. Moreover, since we have that $B \in \beta\mathrm{nbc}(\mathbf{M}\backslash a)$ is also in $\beta\mathrm{nbc}(\mathbf{M})$ and φ is preserved under restriction on the maximal element, we see the subsets A_i will not change so that the basis element given by B in $\widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}\backslash a);\Bbbk)$ is the same as the element given by B in $\widetilde{H}_{l-1}(\overline{BC}(\mathbf{M});\Bbbk)$.

The map δ is induced from the projection onto $\overline{BC}(\mathbf{M}/a) * a$ then removing a. Moreover, we have that

$$
\begin{aligned}
&\delta(\overline{\sigma}_B) \\
&= \sum_{i_1=0}^{e_1} \cdots \sum_{i_t=0}^{e_t-1} \cdots \sum_{i_k=0}^{e_k} (-1)^{i_1+\ldots+i_k} (A_1 - a_{i_1}^1) \cup \ldots ((A_t \backslash a) - a_{i_t}^t) \ldots \cup (A_k - a_{i_k}^k) \\
&= \overline{\sigma}_{B\backslash a}
\end{aligned}
$$

because φ preserves the A_i under contraction and removing e_t does not change the coefficient. □

3.3. **Describing the T-spaces.** The problem of finding a suitable combinatorial model for the vector spaces $T_A(\phi)$ has now been changed to finding a combinatorial model for Crapo's beta invariant that is completely determined by the matroid $\mathbf{M}$. The remainder of this section will show that this combinatorial object is the reduced broken circuit complex.

Definition 3.6. *Each $B \in \beta nbc(\mathbf{M}^*)$ determines canonically an element x_B of the multiplicity space $S_S(\phi)$ defined as follows.*

Taking the complements of the flats in $\mathbf{M}^*$ of the decreasing chain corresponding to B and excluding $\emptyset$ yields a chain of T-flats in $\mathbf{M}$, $J_0 \subset J_1 \subset \ldots \subset J_{l_S}$. Set

$$x_B = \phi(u_1) \cdot \phi(u_2) \cdot \ldots \cdot \phi(u_{l_S}),$$

where each $u_i = e_p + \sum_{j<p} x_j e_j$ is the unique element of $U_{J_i \backslash J_{i-1}}$, where p is the largest element of $J_i \backslash J_{i-1}$ in the fixed ordering so that $\phi(u_i) \in V_{J_i \backslash J_{i-1}} \cap V_{J_{i-1}}$ [16, Remark 5.2.3].

Theorem 3.7. *The collection $\{x_B \mid B \in \beta nbc(\mathbf{M}^*)\}$ is a basis for the multiplicity space $S_S(\phi)$.*

Proof. If $\mathbf{M}$ is disconnected, then $S_S(\phi) = 0$ by [16, Theorem 5.3.5], $\beta(\mathbf{M}^*) = 0$ by Theorem 1.52 and $\beta\text{nbc}(\mathbf{M}^*) = \emptyset$. Therefore, we may assume that $\mathbf{M}$ is a connected matroid. We will use induction on $|S|$. Suppose $|S| = 1$, then $S = \{a\}$ is dependent or independent. If S is independent, then $S_S(\phi) = 0$ by [16, Definition 2.2.3]. Also, S is a loop in $\mathbf{M}^*$, so $r_{\mathbf{M}^*} = 0$ and $\beta\text{nbc}(\mathbf{M}^*) = \emptyset$. Therefore, there is nothing to show.

If S is dependent then S is independent in $\mathbf{M}^*$, so $r_{\mathbf{M}^*} = 1$ and has no circuits and therefore no broken circuits. Therefore, $\beta\text{nbc}(\mathbf{M}^*) = \{\{a\}\}$. Thus $x_{\{a\}}$ is the product of zero factors, hence $x_{\{a\}} = 1 \in \Bbbk = S_S(\phi)$ by Theorem 3.6 and [16, Definition 2.2.3 (c)].

Now, suppose $n = |S| > 1$. Let $a = \max(S)$. By our inductive hypothesis and Proposition 3.4, $\{x_B \mid B \in \beta\text{nbc}(\mathbf{M}^*\backslash a) = \beta\text{nbc}((\mathbf{M}/a)^*)\}$ is a basis for $S_{S\backslash a}(\phi/a)$ and $\{x_B \mid B \in \beta\text{nbc}(\mathbf{M}^*/a) = \beta\text{nbc}((\mathbf{M}\backslash a)^*)\}$ is a basis for $S_{S\backslash a}(\phi\backslash a)$.

We have the short exact sequence of vector spaces from [16, Theorem 5.4.5]

$$0 \to S_{S\backslash a}(\phi) \otimes V_a(\phi) \xrightarrow{\nu} S_S(\phi) \xrightarrow{\pi_{S/a}} S_{S\backslash a}(\phi.S\backslash a) \to 0$$

When $B \in \beta\text{nbc}(\mathbf{M}^*)$ and $a \notin B$ we have $\pi_{S/a}(x_B) = x_B$ where $B \in \beta\text{nbc}(\mathbf{M}^*\backslash a)$. Let x_B be a basis element in $S_{S\backslash a}(\phi)$ where $B \in \beta\text{nbc}(\mathbf{M}^*/a)$ and $\phi(e_a)$ the linear form in $V_a(\phi)$. Then

$$\nu(x_B \otimes \phi(e_a)) = \phi(e_a) \cdot x_B = x_{B\cup a} \in S_S(\phi),$$

because $e_a = u_{l_S}$ in $U_{S\backslash(S\backslash a)} = U_a$. Thus, $\{x_B \mid B \in \beta\text{nbc}(\mathbf{M}^*)\}$ gives a spanning set of $S_S(\phi)$. However, there are $\beta(\mathbf{M}^*)$ elements in this spanning set and $\dim_{\Bbbk} S_S(\phi) = \dim_{\Bbbk} T_S(\phi) = \beta(\mathbf{M}) = \beta(\mathbf{M}^*)$ by Theorem 2 and Theorem 1.52, so $\{x_B \mid B \in \beta\text{nbc}(\mathbf{M}^*)\}$ is a basis of $S_S(\phi)$. □

Example 3.8. *For the uniform matroid $U_{r,n}$, we have*

$$\beta nbc(U^*_{r,n}) = \{X \subseteq S \mid |X| = n - r, 1 \in X \text{ but } 2 \notin X\}$$

from Example 1.50. Therefore, from [16, Example 2.2.10] *and Theorem 3.7 if ϕ is a representation of $U_{r,n}$, then the set*

$$\{\phi(e_B) \mid |B| = n - r, 1 \in B, \text{ but } 2 \notin B\}$$

is a basis for $S_S(\phi) = S_{n-r-1}V$, the $(n - r - 1)$-th symmetric power of $V = \operatorname{Im}\phi$.

Definition 3.9. [16, Proof of Theorem 6.1] *Using the fixed linear ordering on S, we identify each subset of S with the increasing sequence of its elements. For a sequence*

$K = (a_1, \ldots, a_q)$, we set

$$e_K = e_{a_1} \wedge \cdots \wedge e_{a_q} \in \wedge^q U \quad \textit{and} \quad v_K = \phi(e_{a_1}) \wedge \cdots \wedge \phi(e_{a_q}) \in \wedge^q V.$$

If K is a subset of S and the elements $\phi(e_{a_1}), \ldots, \phi(e_{a_q})$ form a basis of V_K, we write e_K^* and v_K^* for the unique elements of $\wedge^q U_K^*$ and $\wedge^q V_K^*$ respectively, such that $e_K^*(e_K) = 1$ and $v_K^*(v_K) = 1$. In particular, if B is a basis of $\mathbf{M}^*$ then $v_{S\setminus B}$ is a basis of $\wedge^{r_S} V_S$ and $v_{S\setminus B}^*$ is the dual basis of $\wedge^{r_S} V_S^*$.

Theorem 3. *Let $\mathbf{M}$ be the ordered matroid on S obtained from ϕ of level l. Then there is a canonical isomorphism*

$$\widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*); \Bbbk) \xrightarrow[\cong]{\theta} T_S(\phi)$$

which sends

$$\overline{\sigma}_B \mapsto x_B^* \otimes e_S \otimes v_{S\setminus B}^*,$$

where $\{x_B^* \mid B \in \beta nbc(\mathbf{M}^*)\}$ is the basis of $S_S^*(\phi)$ dual to the basis $\{x_B \mid B \in \beta nbc(\mathbf{M}^*)\}$ of $S_S(\phi)$.

Proof. If $\mathbf{M}$ is disconnected, then $T_S(\phi) = 0$ and $\overline{BC}(\mathbf{M}^*) = \emptyset$ which gives that $\widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*)) = 0$ unless $l = 0$, in which case $\mathbf{M}$ is connected. Therefore, we may assume $\mathbf{M}$ is a connected matroid. We showed in Theorem 3.7 that $\{x_B \mid B \in \beta\mathrm{nbc}(\mathbf{M}^*)\}$ is a canonical basis of $S_S(\phi)$, so $\{x_B^* \otimes e_S \otimes v_{S\setminus B}^* \mid B \in \beta\mathrm{nbc}(\mathbf{M}^*)\}$ is a canonical basis for $T_S(\phi)$. Thus, both sets of basis elements are indexed by $\beta\mathrm{nbc}(\mathbf{M}^*)$, hence θ is an isomorphism. □

Corollary 3.10. *Let $\mathbf{M}$ be the ordered matroid obtained from ϕ. Then for each T-flat A of level l_A there is a canonical isomorphism*

$$\widetilde{H}_{l_A-1}(\overline{BC}((\mathbf{M}|A)^*); \Bbbk) \xrightarrow[\cong]{\theta|A} T_A(\phi).$$

Proof. This follows by considering the matroid $\mathbf{M}|A$ as in proof of Corollary 3.3. □

Lemma 3.11. *Let* $\mathbf{M}$ *be a connected matroid on* S *with representation* ϕ*. If* S *and* $S\backslash a$ *are T-flats of* $\mathbf{M}$ *and* $l = l_S$*, then the sequence of* $\Bbbk$ *vector spaces*

$$0 \to T_{S\backslash a}(\phi/a) \xrightarrow{(-1)^l(\pi.S\backslash a)^{\phi}_{S\backslash a}} T_S(\phi) \xrightarrow{(-1)^{|S\backslash a|}\phi_l^{S,S\backslash a}} T_{S\backslash a}(\phi) \to 0$$

is exact.

Proof. Although it was not stated explicitly, this was shown in [16]. The injective map is one component of the injective complexes of Theorem 7.7. The vector space $T_{S\backslash a}(\phi)$ being the cokernal of $(-1)^l(\pi.S\backslash a)^{\phi}_{S\backslash a}$, denoted by C_l, is shown within the proof of Theorem 3.4, 3.5 and 5.45 in Section 9. The author shows that the map $\gamma_l^S : C_l \to T_{S\backslash a}(\phi)$ induced from $(-1)^{|S\backslash a|}\phi_n^{S,S\backslash a} : T_S(\phi) \to T_{S\backslash a}(\phi)$ is an isomorphism. □

Theorem 3.12. *Let* $\mathbf{M}$ *be an ordered matroid on* S *with representation* ϕ *and* $l = l_S$*. If* $a = \max(S)$ *and* $S\backslash a$ *is a T-flat of* $\mathbf{M}$*, then the following diagram commutes,*

$$\begin{array}{ccccccc}
0 \to \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*\backslash a);\Bbbk) & \xrightarrow{\tilde{\epsilon}} & \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*);\Bbbk) & \xrightarrow{\delta} & \widetilde{H}_{l-2}(\overline{BC}(\mathbf{M}^*/a);\Bbbk) \to 0 \\
\downarrow{\scriptstyle \theta/a} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta\backslash a} \\
0 \to T_{S\backslash a}(\phi/a) \longrightarrow & & T_S(\phi) \longrightarrow & & T_{S\backslash a}(\phi\backslash a) \to 0,
\end{array}$$

where the maps $\tilde{\epsilon} = (-1)^l\epsilon$ *and* δ *are from Lemma 3.5 and the bottom row is from Lemma 3.11.*

Proof. Notice that the top row of the diagram is not the sequence induced from the reduced chains of the reduced broken circuit complex, even so this sequence is exact. In order to show that our diagram is commutative, it is enough to show that the dual diagram

$$\begin{array}{ccccc}
0 \to T_{S\backslash a}(\phi\backslash a)^* \to & & T_S(\phi)^* \to & & T_{S\backslash a}(\phi/a)^* \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \widetilde{H}_{l-2}(\overline{BC}(\mathbf{M}^*/a);\Bbbk)^* & \xrightarrow{\delta^*} & \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*);\Bbbk)^* & \xrightarrow{\tilde{\epsilon}^*} & \widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*\backslash a);\Bbbk)^* \to 0
\end{array}$$

is commutative. We will do this by investigating the images of the basis elements indexed by $\beta\mathrm{nbc}(\mathbf{M}^*)$ under each map.

Let $\{\overline{\sigma}^*_B | B \in \beta\text{nbc}(\mathbf{M}^*)\}$ in $\widetilde{H}_{l-1}(\overline{BC}(\mathbf{M}^*); \Bbbk)^*$ be the basis dual to $\{\overline{\sigma}_B | B \in \beta\text{nbc}(\mathbf{M}^*)\}$. By taking duals, Lemma 3.5 shows that

$$\delta^*(\overline{\sigma}^*_B) = \overline{\sigma}^*_{B \cup a}$$

and

$$(-1)^l \epsilon^*(\overline{\sigma}^*_B) = \begin{cases} (-1)^l \overline{\sigma}^*_B & \text{if } a \notin B \\ 0 & \text{otherwise.} \end{cases}$$

Also, Theorem 3 gives

$$\theta^*(x_B \otimes e^*_S \otimes v_{S \setminus B}) = \overline{\sigma}^*_B,$$

$$(\theta \backslash a)^*(x_B \otimes e^*_{S \setminus a} \otimes v_{(S \setminus a) \setminus B}) = \overline{\sigma}^*_B,$$

and

$$(\theta / a)^*(x_B \otimes e^*_{S \setminus a} \otimes \overline{v}_{(S \setminus a) \setminus B}) = \overline{\sigma}^*_B.$$

Therefore, it remains to show that

$$(-1)^{|S|-1} (\phi_l^{S, S \setminus a})^*(x_B \otimes e^*_{S \setminus a} \otimes v_{(S \setminus a) \setminus B}) = x_{B \cup a} \otimes e^*_S \otimes v_{S \setminus (B \cup a)}$$

and

$$(-1)^l (\pi^{\phi}_{S \setminus a})^*_{S \setminus a}(x_B \otimes e^*_S \otimes v_{S \setminus B}) = \begin{cases} (-1)^l x_B \otimes e^*_{S \setminus a} \otimes \overline{v}_{(S \setminus a) \setminus B} & \text{if } a \notin B \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{M}$ is connected, $V_S = V_{S \setminus a}$ and thus in Definition 1.18 we have that $K_{S, S \setminus a} = 0$. Therefore, the map $\phi_l^{S, S \setminus a}$ from Definition 1.19 can be rewritten as the composition

$$\begin{array}{c}
T_S = S^*_S \otimes \mathring{\wedge} U_S \mathring{\wedge} V^*_S \\
\Big\downarrow {\scriptstyle \Delta \otimes \boldsymbol{\delta} \otimes 1} \\
V^*_a \otimes S^*_{S \setminus a} \otimes U_a \otimes \mathring{\wedge} U_{S \setminus a} \otimes \mathring{\wedge} V^*_{S \setminus a} \\
\Big\downarrow {\scriptstyle \tau} \\
V^*_a \otimes U_a \otimes S^*_{S \setminus a} \otimes \mathring{\wedge} U_{S \setminus a} \otimes \mathring{\wedge} V^*_{S \setminus a} \\
\Big\downarrow {\scriptstyle 1 \otimes \phi \otimes 1} \\
V^*_a \otimes V_a \otimes S^*_{S \setminus a} \otimes \mathring{\wedge} U_{S \setminus a} \otimes \mathring{\wedge} V^*_{S \setminus a} \\
\Big\downarrow {\scriptstyle \mu \otimes 1} \\
S^*_{S \setminus a} \otimes \mathring{\wedge} U_{S \setminus a} \otimes \mathring{\wedge} V^*_{S \setminus a} = T_{S \setminus a}
\end{array}$$

The next diagram gives the composition of the dual maps in the left sequence and the image of $x_B \otimes e^*_{S\backslash a} \otimes v_{S\backslash B}$ in the right sequence.

$$\begin{array}{ccc}
S_{S\backslash a} \otimes \mathring{\wedge} U^*_{S\backslash a} \mathring{\wedge} V_{S\backslash a} & & x_B \otimes e^*_{S\backslash a} \otimes v_{S\backslash B} \\
\Big\downarrow{\scriptstyle \mu^*\otimes 1} & & \Big\downarrow \\
V_a \otimes V_a^* \otimes S_{S\backslash a} \otimes \mathring{\wedge} U^*_{S\backslash a} \otimes \mathring{\wedge} V_{S\backslash a} & & v_a \otimes v_a^* \otimes x_B \otimes e^*_{S\backslash a} \otimes v_{S\backslash B} \\
\Big\downarrow{\scriptstyle 1\otimes\phi^*\otimes 1} & & \Big\downarrow \\
V_a \otimes U_a^* \otimes S_{S\backslash a} \otimes \mathring{\wedge} U^*_{S\backslash a} \otimes \mathring{\wedge} V_{S\backslash a} & & v_a \otimes e_a^* \otimes x_B \otimes e^*_{S\backslash a} \otimes v_{S\backslash B} \\
\Big\downarrow{\scriptstyle \tau^*} & & \Big\downarrow \\
V_a \otimes S_{S\backslash a} \otimes U_a^* \otimes \mathring{\wedge} U^*_{S\backslash a} \otimes \mathring{\wedge} V_{S\backslash a} & & v_a \otimes x_B \otimes e_a^* \otimes e^*_{S\backslash a} \otimes v_{S\backslash B} \\
\Big\downarrow{\scriptstyle \nu\otimes\wedge\otimes 1} & & \Big\downarrow \\
S_S \otimes \mathring{\wedge} U_S^* \otimes \mathring{\wedge} V_S & & (-1)^{|S\backslash a|} x_{B\cup a} \otimes e_S^* \otimes v_{S\backslash B}
\end{array}$$

Therefore,

$$(-1)^{|S|-1}(\phi_l^{S,S\backslash a})^*(x_B \otimes e^*_{S\backslash a} \otimes v_{(S\backslash a)\backslash B}) = x_{B\cup a} \otimes e_S^* \otimes v_{S\backslash(B\cup a)},$$

so that the left square commutes.

The map

$$(\pi.S\backslash a)^{\phi}_{S\backslash a} : T_{S\backslash a}(\phi/a) \to T_S(\phi)$$

in Definition 1.21 is given by the composition of the sequence of maps

$$\begin{array}{c}
T_{S\backslash a}(\phi/a) = S_{S\backslash a}(\phi/a)^* \otimes \mathring{\wedge} U_{S\backslash a} \mathring{\wedge} V_{S\backslash a}(\phi/a)^* \\
\Big\downarrow{\scriptstyle 1\otimes \boldsymbol{d}} \\
S_{S\backslash a}(\phi/a)^* \otimes \mathring{\wedge} U_{S\backslash a} \otimes \mathring{\wedge} V_{S\backslash a}(\phi/a)^* \otimes U_a \otimes V_a^* \\
\Big\downarrow{\scriptstyle \tau} \\
S_{S\backslash a}(\phi/a)^* \otimes \mathring{\wedge} U_{S\backslash a} \otimes U_a \otimes \mathring{\wedge} V_{S\backslash a}(\phi/a)^* \otimes V_a^* \\
\Big\downarrow{\scriptstyle \pi^*\otimes\wedge\otimes \boldsymbol{c}} \\
S_S(\phi)^* \otimes \mathring{\wedge} U_S \otimes \mathring{\wedge} V_S(\phi)^* = T_S(\phi).
\end{array}$$

Similarly, we obtain image of a basis element $x_B \otimes e_S^* \otimes v_{S\setminus B}$ by composition of the sequence of dual maps as follows.

$$\begin{array}{c}
x_B \otimes e_S^* \otimes v_{S\setminus B} \\
\Big\downarrow {\scriptstyle \pi\otimes\boldsymbol{\delta}\otimes\mathbf{c}^*} \\
\overline{x_B} \otimes e_{S\setminus a}^* \otimes e_a^* \otimes \overline{v}_{(S\setminus B)\setminus a} \otimes v_a \\
\Big\downarrow {\scriptstyle \tau^*} \\
\overline{x_B} \otimes e_{S\setminus a}^* \otimes \overline{v}_{(S\setminus B)\setminus a} \otimes e_a^* \otimes v_a \\
\Big\downarrow {\scriptstyle 1\otimes\boldsymbol{d}^*} \\
\overline{x_B} \otimes e_{S\setminus a}^* \otimes \overline{v}_{(S\setminus a)\setminus B}
\end{array}$$

Therefore,

$$(-1)^l(\pi_{S\setminus a}^{\phi})_{S\setminus a}^*(x_B \otimes e_S^* \otimes v_{S\setminus B}) = \begin{cases} (-1)^l x_B \otimes e_{S\setminus a}^* \otimes v_{(S\setminus a)\setminus B} & \text{if } a \notin B \\ 0 & \text{otherwise} \end{cases}$$

and the diagram commutes. □

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