PROBABILITY AND STATISTICS

FOR ENGINEERING AND THE SCIENCES

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SIXTH EDITION

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Probability

Introduction

The term **probability** refers to the study of randomness and uncertainty. In any situation in which one of a number of possible outcomes may occur, the theory of probability provides methods for quantifying the chances, or likelihoods, associated with the various outcomes. The language of probability is constantly used in an informal manner in both written and spoken contexts. Examples include such statements as "It is likely that the Dow-Jones average will increase by the end of the year," "There is a 50–50 chance that the incumbent will seek reelection," "There will probably be at least one section of that course offered next year," "The odds favor a quick settlement of the strike," and "It is expected that at least 20,000 concert tickets will be sold." In this chapter, we introduce some elementary probability concepts, indicate how probabilities can be interpreted, and show how the rules of probability can be applied to compute the probabilities of many interesting events. The methodology of probability will then permit us to express in precise language such informal statements as those given above.

The study of probability as a branch of mathematics goes back over 300 years, where it had its genesis in connection with questions involving games of chance. Many books are devoted exclusively to probability, but our objective here is to cover only that part of the subject that has the most direct bearing on problems of statistical inference.

2.1 Sample Spaces and Events

An **experiment** is any action or process whose outcome is subject to uncertainty. Although the word *experiment* generally suggests a planned or carefully controlled laboratory testing situation, we use it here in a much wider sense. Thus experiments that may be of interest include tossing a coin once or several times, selecting a card or cards from a deck, weighing a loaf of bread, ascertaining the commuting time from home to work on a particular morning, obtaining blood types from a group of individuals, or measuring the compressive strengths of different steel beams.

The Sample Space of an Experiment

DEFINITION

> The **sample space** of an experiment, denoted by $\mathcal{S}$, is the set of all possible outcomes of that experiment.

Example 2.1 The simplest experiment to which probability applies is one with two possible outcomes. One such experiment consists of examining a single fuse to see whether it is defective. The sample space for this experiment can be abbreviated as $\mathcal{S} = \{N, D\}$, where N represents not defective, D represents defective, and the braces are used to enclose the elements of a set. Another such experiment would involve tossing a thumbtack and noting whether it landed point up or point down, with sample space $\mathcal{S} = \{U, D\}$, and yet another would consist of observing the gender of the next child born at the local hospital, with $\mathcal{S} = \{M, F\}$. ■

Example 2.2 If we examine three fuses in sequence and note the result of each examination, then an outcome for the entire experiment is any sequence of N's and D's of length 3, so

$$\mathcal{S} = \{NNN, NND, NDN, NDD, DNN, DND, DDN, DDD\}$$

If we had tossed a thumbtack three times, the sample space would be obtained by replacing N by U in $\mathcal{S}$ above, with a similar notational change yielding the sample space for the experiment in which the genders of three newborn children are observed. ■

Example 2.3 Two gas stations are located at a certain intersection. Each one has six gas pumps. Consider the experiment in which the number of pumps in use at a particular time of day is determined for each of the stations. An experimental outcome specifies how many pumps are in use at the first station and how many are in use at the second one. One possible outcome is (2, 2), another is (4, 1), and yet another is (1, 4). The 49 outcomes in $\mathcal{S}$ are displayed in the accompanying table. The sample space for the experiment in which a six-sided die is thrown twice results from deleting the 0 row and 0 column from the table, giving 36 outcomes.

		Second Station						
		0	**1**	**2**	**3**	**4**	**5**	**6**
	0	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)
	1	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
First Station	**3**	(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 0)	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

■

Example 2.4 If a new type-D flashlight battery has a voltage that is outside certain limits, that battery is characterized as a failure (F); if the battery has a voltage within the prescribed limits, it is a success (S). Suppose an experiment consists of testing each battery as it comes off an assembly line until we first observe a success. Although it may not be very likely, a possible outcome of this experiment is that the first 10 (or 100 or 1000 or . . .) are F's and the next one is an S. That is, for any positive integer n, we may have to examine n batteries before seeing the first S. The sample space is $\mathcal{S} = \{S, FS, FFS, FFFS, \ldots\}$, which contains an infinite number of possible outcomes. The same abbreviated form of the sample space is appropriate for an experiment in which, starting at a specified time, the gender of each newborn infant is recorded until the birth of a male is observed. ■

Events

In our study of probability, we will be interested not only in the individual outcomes of $\mathcal{S}$ but also in any collection of outcomes from $\mathcal{S}$.

DEFINITION

> An **event** is any collection (subset) of outcomes contained in the sample space $\mathcal{S}$. An event is said to be **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

When an experiment is performed, a particular event A is said to occur if the resulting experimental outcome is contained in A. In general, exactly one simple event will occur, but many compound events will occur simultaneously.

Example 2.5 Consider an experiment in which each of three vehicles taking a particular freeway exit turns left (L) or right (R) at the end of the exit ramp. The eight possible outcomes that comprise the sample space are LLL, RLL, LRL, LLR, LRR, RLR, RRL, and RRR. Thus there are eight simple events, among which are $E_1 = \{LLL\}$ and $E_5 = \{LRR\}$. Some compound events include

$A = \{RLL, LRL, LLR\}$ = the event that exactly one of the three vehicles turns right

$B = \{LLL, RLL, LRL, LLR\}$ = the event that at most one of the vehicles turns right

$C = \{LLL, RRR\}$ = the event that all three vehicles turn in the same direction

Suppose that when the experiment is performed, the outcome is LLL. Then the simple event E_1 has occurred and so also have the events B and C (but not A). ■

Example 2.6 (Example 2.3 continued)

When the number of pumps in use at each of two six-pump gas stations is observed, there are 49 possible outcomes, so there are 49 simple events: $E_1 = \{(0, 0)\}$, $E_2 = \{(0, 1)\}, \ldots, E_{49} = \{(6, 6)\}$. Examples of compound events are

$A = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$ = the event that the number of pumps in use is the same for both stations

$B = \{(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)\}$ = the event that the total number of pumps in use is four

$C = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ = the event that at most one pump is in use at each station ■

Example 2.7 (Example 2.4 continued)

The sample space for the battery examination experiment contains an infinite number of outcomes, so there are an infinite number of simple events. Compound events include

$A = \{S, FS, FFS\}$ = the event that at most three batteries are examined

$E = \{FS, FFFS, FFFFFS, \ldots\}$ = the event that an even number of batteries are examined ■

Some Relations from Set Theory

An event is nothing but a set, so relationships and results from elementary set theory can be used to study events. The following operations will be used to construct new events from given events.

DEFINITION

1. The **union** of two events A and B, denoted by $A \cup B$ and read "A *or* B," is the event consisting of all outcomes that are *either in A or in B or in both events* (so that the union includes outcomes for which both A and B occur as well as outcomes for which exactly one occurs)—that is, all outcomes in at least one of the events.
2. The **intersection** of two events A and B, denoted by $A \cap B$ and read "A *and* B," is the event consisting of all outcomes that are in *both A and B.*
3. The **complement** of an event A, denoted by A', is the set of all outcomes in $\mathcal{S}$ that are not contained in A.

Example 2.8 (Example 2.3 continued)

For the experiment in which the number of pumps in use at a single six-pump gas station is observed, let $A = \{0, 1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{1, 3, 5\}$. Then

$$A \cup B = \{0, 1, 2, 3, 4, 5, 6\} = \mathcal{S}, \quad A \cup C = \{0, 1, 2, 3, 4, 5\},$$

$$A \cap B = \{3, 4\}, \quad A \cap C = \{1, 3\}, \quad A' = \{5, 6\}, \quad \{A \cup C\}' = \{6\}$$ ■

Example 2.9 (Example 2.4 continued)

In the battery experiment, define A, B, and C by

$$A = \{S, FS, FFS\}$$
$$B = \{S, FFS, FFFFS\}$$

and

$$C = \{FS, FFFS, FFFFFS, \ldots\}$$

Then

$$A \cup B = \{S, FS, FFS, FFFFS\}$$
$$A \cap B = \{S, FFS\}$$
$$A' = \{FFFS, FFFFS, FFFFFS, \ldots\}$$

and

$$C' = \{S, FFS, FFFFS, \ldots\} = \{\text{an odd number of batteries are examined}\}$$

Sometimes A and B have no outcomes in common, so that the intersection of A and B contains no outcomes.

DEFINITION

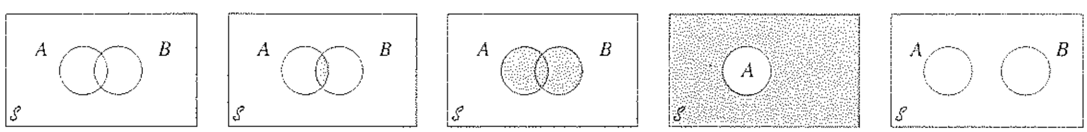
When A and B have no outcomes in common, they are said to be **mutually exclusive** or **disjoint** events.

Example 2.10

A small city has three automobile dealerships: a GM dealer selling Chevrolets, Pontiacs, and Buicks; a Ford dealer selling Fords and Mercurys; and a Chrysler dealer selling Plymouths and Chryslers. If an experiment consists of observing the brand of the next car sold, then the events $A = \{\text{Chevrolet, Pontiac, Buick}\}$ and $B = \{\text{Ford, Mercury}\}$ are mutually exclusive because the next car sold cannot be both a GM product and a Ford product.

The operations of union and intersection can be extended to more than two events. For any three events A, B, and C, the event $A \cup B \cup C$ is the set of outcomes contained in at least one of the three events, whereas $A \cap B \cap C$ is the set of outcomes contained in all three events. Given events $A_1, A_2, A_3, \ldots$, these events are said to be mutually exclusive (or pairwise disjoint) if no two events have any outcomes in common.

A pictorial representation of events and manipulations with events is obtained by using Venn diagrams. To construct a Venn diagram, draw a rectangle whose interior will represent the sample space $\mathcal{S}$. Then any event A is represented as the interior of a closed curve (often a circle) contained in $\mathcal{S}$. Figure 2.1 shows examples of Venn diagrams.



(a) Venn diagram of events A and B (b) Shaded region is A ∩ B (c) Shaded region is A ∪ B (d) Shaded region is A′ (e) Mutually exclusive events

Figure 2.1 Venn diagrams

Exercises | Section 2.1 (1–10)

1. Four universities—1, 2, 3, and 4—are participating in a holiday basketball tournament. In the first round, 1 will play 2 and 3 will play 4. Then the two winners will play for the championship, and the two losers will also play. One possible outcome can be denoted by 1324 (1 beats 2 and 3 beats 4 in first-round games, and then 1 beats 3 and 2 beats 4).

a. List all outcomes in $\mathcal{S}$.

b. Let A denote the event that 1 wins the tournament. List outcomes in A.

c. Let B denote the event that 2 gets into the championship game. List outcomes in B.

d. What are the outcomes in $A \cup B$ and in $A \cap B$? What are the outcomes in A'?

2. Suppose that vehicles taking a particular freeway exit can turn right (R), turn left (L), or go straight (S). Consider observing the direction for each of three successive vehicles.

a. List all outcomes in the event A that all three vehicles go in the same direction.

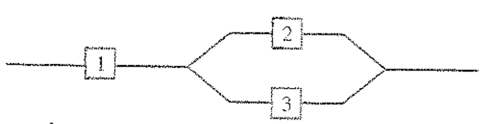
b. List all outcomes in the event B that all three vehicles take different directions.

c. List all outcomes in the event C that exactly two of the three vehicles turn right.

d. List all outcomes in the event D that exactly two vehicles go in the same direction.

e. List outcomes in D', $C \cup D$, and $C \cap D$.

3. Three components are connected to form a system as shown in the accompanying diagram. Because the components in the 2–3 subsystem are connected in parallel, that subsystem will function if at least one of the two individual components functions. For the entire system to function, component 1 must function and so must the 2–3 subsystem.



The experiment consists of determining the condition of each component [S (success) for a functioning component and F (failure) for a nonfunctioning component].

a. What outcomes are contained in the event A that exactly two out of the three components function?

b. What outcomes are contained in the event B that at least two of the components function?

c. What outcomes are contained in the event C that the system functions?

d. List outcomes in C', $A \cup C$, $A \cap C$, $B \cup C$, and $B \cap C$.

4. Each of a sample of four home mortgages is classified as fixed rate (F) or variable rate (V).

a. What are the 16 outcomes in $\mathcal{S}$?

b. Which outcomes are in the event that exactly three of the selected mortgages are fixed rate?

c. Which outcomes are in the event that all four mortgages are of the same type?

d. Which outcomes are in the event that at most one of the four is a variable-rate mortgage?

e. What is the union of the events in parts (c) and (d), and what is the intersection of these two events?

f. What are the union and intersection of the two events in parts (b) and (c)?

5. A family consisting of three persons—A, B, and C—belongs to a medical clinic that always has a doctor at each of stations 1, 2, and 3. During a certain week, each member of the family visits the clinic once and is assigned at random to a station. The experiment consists of recording the station number for each member. One outcome is (1, 2, 1) for A to station 1, B to station 2, and C to station 1.

a. List the 27 outcomes in the sample space.

b. List all outcomes in the event that all three members go to the same station.

c. List all outcomes in the event that all members go to different stations.

d. List all outcomes in the event that no one goes to station 2.

6. A college library has five copies of a certain text on reserve. Two copies (1 and 2) are first printings, and the other three (3, 4, and 5) are second printings. A student examines these books in random order, stopping only when a second printing has been selected. One possible outcome is 5, and another is 213.

a. List the outcomes in $\mathcal{S}$.

b. Let A denote the event that exactly one book must be examined. What outcomes are in A?

c. Let B be the event that book 5 is the one selected. What outcomes are in B?

d. Let C be the event that book 1 is not examined. What outcomes are in C?

7. An academic department has just completed voting by secret ballot for a department head. The ballot box contains four slips with votes for candidate A and three slips with votes for candidate B. Suppose these slips are removed from the box one by one.
   a. List all possible outcomes.
   b. Suppose a running tally is kept as slips are removed. For what outcomes does A remain ahead of B throughout the tally?

8. An engineering construction firm is currently working on power plants at three different sites. Let A_i denote the event that the plant at site i is completed by the contract date. Use the operations of union, intersection, and complementation to describe each of the following events in terms of A_1, A_2, and A_3, draw a Venn diagram, and shade the region corresponding to each one.
   a. At least one plant is completed by the contract date.
   b. All plants are completed by the contract date.
   c. Only the plant at site 1 is completed by the contract date.
   d. Exactly one plant is completed by the contract date.
   e. Either the plant at site 1 or both of the other two plants are completed by the contract date.

9. Use Venn diagrams to verify the following two relationships for any events A and B (these are called De Morgan's laws):
   a. $(A \cup B)' = A' \cap B'$
   b. $(A \cap B)' = A' \cup B'$

10. **a.** In Example 2.10, identify three events that are mutually exclusive.
   b. Suppose there is no outcome common to all three of the events A, B, and C. Are these three events necessarily mutually exclusive? If your answer is yes, explain why; if your answer is no, give a counterexample using the experiment of Example 2.10.

2.2 Axioms, Interpretations, and Properties of Probability

Given an experiment and a sample space $\mathcal{S}$, *the objective of probability is to assign to each event A a number P(A), called the probability of the event A, which will give a precise measure of the chance that A will occur.* To ensure that the probability assignments will be consistent with our intuitive notions of probability, all assignments should satisfy the following axioms (basic properties) of probability.

AXIOM 1 For any event A, $P(A) \geq 0$.

AXIOM 2 $P(\mathcal{S}) = 1$.

AXIOM 3 **a.** If $A_1, A_2, \ldots, A_k$ is a finite collection of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = \sum_{i=1}^{k} P(A_i)$$

b. If $A_1, A_2, A_3, \ldots$ is an infinite collection of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$$

Axiom 1 reflects the intuitive notion that the chance of A occurring should be nonnegative. The sample space is by definition the event that must occur when the experiment is performed ($\mathcal{S}$ contains all possible outcomes), so Axiom 2 says that the maximum possible probability of 1 is assigned to $\mathcal{S}$. The third axiom formalizes the idea that if we wish the probability that at least one of a number of events will occur and no two of the events can occur simultaneously, then the chance of at least one occurring is the sum of the chances of the individual events.

Example 2.11 In the experiment in which a single coin is tossed, the sample space is $\mathcal{S} = \{H, T\}$. The axioms specify $P(\mathcal{S}) = 1$, so to complete the probability assignment, it remains only to determine $P(H)$ and $P(T)$. Since H and T are disjoint events and $H \cup T = \mathcal{S}$, Axiom 3 implies that

$$1 = P(\mathcal{S}) = P(H) + P(T)$$

This implies that $P(T) = 1 - P(H)$. The only freedom allowed by the axioms in this experiment is the probability assigned to H. One possible assignment of probabilities is $P(H) = .5$, $P(T) = .5$, whereas another possible assignment is $P(H) = .75$, $P(T) = .25$. In fact, letting p represent any fixed number between 0 and 1, $P(H) = p$ and $P(T) = 1 - p$ is an assignment consistent with the axioms. ■

Example 2.12 Consider the experiment in Example 2.4, in which batteries coming off an assembly line are tested one by one until one having a voltage within prescribed limits is found. The simple events are $E_1 = \{S\}$, $E_2 = \{FS\}$, $E_3 = \{FFS\}$, $E_4 = \{FFFS\}$, ... Suppose the probability of any particular battery being satisfactory is .99. Then it can be shown that $P(E_1) = .99$, $P(E_2) = (.01)(.99)$, $P(E_3) = (.01)^2(.99)$, . . . is an assignment of probabilities to the simple events that satisfies the axioms. In particular, because the E_i's are disjoint and $\mathcal{S} = E_1 \cup E_2 \cup E_3 \cup \ldots$, it must be the case that

$$\begin{aligned} 1 = P(\mathcal{S}) &= P(E_1) + P(E_2) + P(E_3) + \cdots \\ &= .99[1 + .01 + (.01)^2 + (.01)^3 + \cdots] \end{aligned}$$

The validity of this equality is a consequence of a mathematical result concerning the sum of a geometric series.

However, another legitimate (according to the axioms) probability assignment of the same "geometric" type is obtained by replacing .99 by any other number p between 0 and 1 (and .01 by $1 - p$). ■

Interpreting Probability

Examples 2.11 and 2.12 show that the axioms do not completely determine an assignment of probabilities to events. The axioms serve only to rule out assignments inconsistent with our intuitive notions of probability. In the coin-tossing experiment of Example 2.11, two particular assignments were suggested. The appropriate or correct assignment depends on the manner in which the experiment is carried out and also on one's interpretation of probability. The interpretation that is most frequently used and most easily understood is based on the notion of relative frequencies.

Consider an experiment that can be repeatedly performed in an identical and independent fashion, and let A be an event consisting of a fixed set of outcomes of the experiment. Simple examples of such repeatable experiments include the coin-tossing and die-tossing experiments previously discussed. If the experiment is performed n times, on some of the replications the event A will occur (the outcome will be in the set A), and on others, A will not occur. Let $n(A)$ denote the number of replications on which A does occur. Then the ratio $n(A)/n$ is called the *relative frequency* of occurrence of the event A in the sequence of n replications. Empirical evidence, based on the results of many of these sequences of repeatable experiments, indicates that as n grows large, the relative frequency $n(A)/n$ stabilizes, as pictured in Figure 2.2. That is, as n gets arbitrarily large, the relative frequency approaches a limiting value we refer to as the *limiting relative frequency* of the event A. The objective interpretation of probability identifies this limiting relative frequency with $P(A)$.

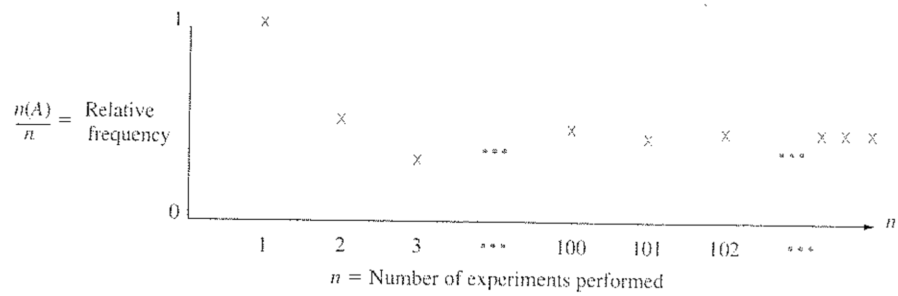


Figure 2.2 Stabilization of relative frequency

If probabilities are assigned to events in accordance with their limiting relative frequencies, then we can interpret a statement such as "The probability of that coin landing with the head facing up when it is tossed is .5" to mean that in a large number of such tosses, a head will appear on approximately half the tosses and a tail on the other half.

This relative frequency interpretation of probability is said to be objective because it rests on a property of the experiment rather than on any particular individual concerned with the experiment. For example, two different observers of a sequence of coin tosses should both use the same probability assignments since the observers have nothing to do with limiting relative frequency. In practice, this interpretation is not as objective as it might seem, since the limiting relative frequency of an event will not be known. Thus we will have to assign probabilities based on our beliefs about the limiting relative frequency of events under study. Fortunately, there are many experiments for which there will be a consensus with respect to probability assignments. When we speak of a fair coin, we shall mean $P(H) = P(T) = .5$, and a fair die is one for which limiting relative frequencies of the six outcomes are all $\frac{1}{6}$, suggesting probability assignments $P(\{1\}) = \cdots = P(\{6\}) = \frac{1}{6}$.

Because the objective interpretation of probability is based on the notion of limiting frequency, its applicability is limited to experimental situations that are repeatable. Yet the language of probability is often used in connection with situations that are inherently unrepeatable. Examples include: "The chances are good for a peace agreement"; "It is likely that our company will be awarded the contract"; and "Because their best quarterback is injured, I expect them to score no more than 10 points against us." In such situations we would like, as before, to assign numerical probabilities to various outcomes and events (e.g., the probability is .9 that we will get the contract). We must therefore adopt an alternative interpretation of these probabilities. Because different observers may have different prior information and opinions concerning such experimental situations, probability assignments may now differ from individual to individual. Interpretations in such situations are thus referred to as *subjective*. The book by Robert Winkler listed in the chapter references gives a very readable survey of several subjective interpretations.

Properties of Probability

PROPOSITION For any event A, $P(A) = 1 - P(A')$.

Proof

In Axiom 3a, let $k = 2$, $A_1 = A$, and $A_2 = A'$. Since by definition of A', $A \cup A' = \mathcal{S}$ while A and A' are disjoint, $1 = P(\mathcal{S}) = P(A \cup A') = P(A) + P(A')$, from which the desired result follows. ■

This proposition is surprisingly useful because there are many situations in which $P(A')$ is more easily obtained by direct methods than is $P(A)$.

Example 2.13 Consider a system of five identical components connected in series, as illustrated in Figure 2.3.

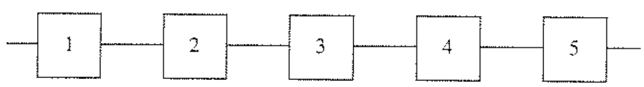


Figure 2.3 A system of five components connected in series

Denote a component that fails by F and one that doesn't fail by S (for success). Let A be the event that the *system* fails. For A to occur, at least one of the individual components must fail. Outcomes in A include *SSFSS* (1, 2, 4, and 5 all work, but 3 does not), *FFSSS*, and so on. There are in fact 31 different outcomes in A. However, A', the event that the system works, consists of the single outcome *SSSSS*. We will see in Section 2.5 that if 90% of all these components do not fail and different components fail independently of one another, then $P(A') = P(SSSSS) = .9^5 = .59$. Thus $P(A) = 1 - .59 = .41$; so among a large number of such systems, roughly 41% will fail. ■

In general, the foregoing proposition is useful when the event of interest can be expressed as "at least . . . ," since then the complement "less than . . ." may be easier to work with (in some problems, "more than . . ." is easier to deal with than "at most . . ."). When you are having difficulty calculating $P(A)$ directly, think of determining $P(A')$.

PROPOSITION If A and B are mutually exclusive, then $P(A \cap B) = 0$.

Proof

Because $A \cap B$ contains no outcomes, $(A \cap B)' = \mathcal{S}$. Thus we have that $1 = P[(A \cap B)'] = 1 - P(A \cap B)$, which implies $P(A \cap B) = 1 - 1 = 0$. ■

When events A and B are mutually exclusive, Axiom 3 gives $P(A \cup B) = P(A) + P(B)$. When A and B are not mutually exclusive, the probability of the union is obtained from the following result.

PROPOSITION

For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Notice that the proposition is valid even if A and B are mutually exclusive, since then $P(A \cap B) = 0$. The key idea is that, in adding $P(A)$ and $P(B)$, the probability of the intersection $A \cap B$ is actually counted twice, so $P(A \cap B)$ must be subtracted out.

Proof

Note first that $A \cup B = A \cup (B \cap A')$, as illustrated in Figure 2.4. Since A and $(B \cap A')$ are mutually exclusive, $P(A \cup B) = P(A) + P(B \cap A')$. But $B = (B \cap A) \cup (B \cap A')$ (the union of that part of B in A and that part of B not in A), with $(B \cap A)$ and $(B \cap A')$ mutually exclusive, so that $P(B) = P(B \cap A) + P(B \cap A')$. Combining these results gives

$$\begin{aligned} P(A \cup B) = P(A) + P(B \cap A') &= P(A) + [P(B) - P(A \cap B)] \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

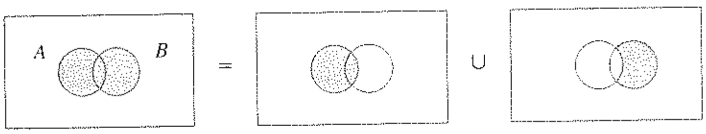


Figure 2.4 Representing $A \cup B$ as a union of disjoint events ■

Example 2.14 In a certain residential suburb, 60% of all households subscribe to the metropolitan newspaper published in a nearby city, 80% subscribe to the local paper, and 50% of all households subscribe to both papers. If a household is selected at random, what is the probability that it subscribes to (1) at least one of the two newspapers and (2) exactly one of the two newspapers?

With A = {subscribes to the metropolitan paper} and B = {subscribes to the local paper}, the given information implies that $P(A) = .6$, $P(B) = .8$, and $P(A \cap B) = .5$. The previous proposition then applies to give

$$\begin{aligned} &P(\text{subscribes to at least one of the two newspapers}) \\ &\qquad = P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .8 - .5 = .9 \end{aligned}$$

The event that a household subscribes only to the local paper can be written as $A' \cap B$ [(not metropolitan) and local]. Now Figure 2.4 implies that

$$.9 = P(A \cup B) = P(A) + P(A' \cap B) = .6 + P(A' \cap B)$$

from which $P(A' \cap B) = .3$. Similarly, $P(A \cap B') = P(A \cup B) - P(B) = .1$. This is all illustrated in Figure 2.5, from which we see that

$$P(\text{exactly one}) = P(A \cap B') + P(A' \cap B) = .1 + .3 = .4$$

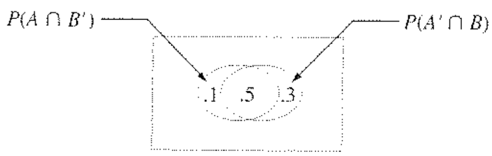


Figure 2.5 Probabilities for Example 2.14 ■

The probability of a union of more than two events can be computed analogously. For three events A, B, and C, the result is

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

This can be seen by examining a Venn diagram of $A \cup B \cup C$, which is shown in Figure 2.6. When $P(A)$, $P(B)$, and $P(C)$ are added, certain intersections are counted twice, so they must be subtracted out, but this results in $P(A \cap B \cap C)$ being subtracted once too often.

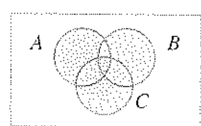


Figure 2.6 $A \cup B \cup C$

Determining Probabilities Systematically

When the number of possible outcomes (simple events) is large, there will be many compound events. A simple way to determine probabilities for these events that avoids violating the axioms and derived properties is to first determine probabilities $P(E_i)$ for all simple events. These should satisfy $P(E_i) \ge 0$ and $\sum_{\text{all } i} P(E_i) = 1$. Then the probability of any compound event A is computed by adding together the $P(E_i)$'s for all E_i's in A:

$$P(A) = \sum_{\text{all } E_i\text{'s in } A} P(E_i)$$

Example 2.15 Denote the six elementary events $\{1\}, \ldots, \{6\}$ associated with tossing a six-sided die once by $E_1, \ldots, E_6$. If the die is constructed so that any of the three even outcomes is twice as likely to occur as any of the three odd outcomes, then an appropriate assignment of probabilities to elementary events is $P(E_1) = P(E_3) = P(E_5) = \frac{1}{9}$, $P(E_2) = P(E_4) = P(E_6) = \frac{2}{9}$. Then for the event $A = \{\text{outcome is even}\} = E_2 \cup E_4 \cup E_6$, $P(A) = P(E_2) + P(E_4) + P(E_6) = \frac{6}{9} = \frac{2}{3}$; for $B = \{\text{outcome} \le 3\} = E_1 \cup E_2 \cup E_3$, $P(B) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$. ■

Equally Likely Outcomes

In many experiments consisting of N outcomes, it is reasonable to assign equal probabilities to all N simple events. These include such obvious examples as tossing a fair coin or fair die once or twice (or any fixed number of times), or selecting one or several cards from a well-shuffled deck of 52. With $p = P(E_i)$ for every i,

$$1 = \sum_{i=1}^{N} P(E_i) = \sum_{i=1}^{N} p = p \cdot N \qquad \text{so } p = \frac{1}{N}$$

That is, if there are N possible outcomes, then the probability assigned to each is $1/N$.

Now consider an event A, with $N(A)$ denoting the number of outcomes contained in A. Then

$$P(A) = \sum_{E_i \text{ in } A} P(E_i) = \sum_{E_i \text{ in } A} \frac{1}{N} = \frac{N(A)}{N}$$

Once we have counted the number N of outcomes in the sample space, to compute the probability of any event we must count the number of outcomes contained in that event and take the ratio of the two numbers. Thus when outcomes are equally likely, computing probabilities reduces to counting.

Example 2.16 When two dice are rolled separately, there are $N = 36$ outcomes (delete the first row and column from the table in Example 2.3). If both the dice are fair, all 36 outcomes are equally likely, so $P(E_i) = \frac{1}{36}$. Then the event $A = \{\text{sum of two numbers} = 7\}$ consists of the six outcomes (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), and (6, 1), so

$$P(A) = \frac{N(A)}{N} = \frac{6}{36} = \frac{1}{6}$$
■

Exercises | Section 2.2 (11–28)

11. A mutual fund company offers its customers several different funds: a money-market fund, three different bond funds (short, intermediate, and long-term), two stock funds (moderate and high-risk), and a balanced fund. Among customers who own shares in just one fund, the percentages of customers in the different funds are as follows:

Money-market	20%	High-risk stock	18%
Short bond	15%	Moderate-risk stock	25%
Intermediate bond	10%	Balanced	7%
Long bond	5%		

A customer who owns shares in just one fund is randomly selected.

a. What is the probability that the selected individual owns shares in the balanced fund?
b. What is the probability that the individual owns shares in a bond fund?
c. What is the probability that the selected individual does not own shares in a stock fund?

12. Consider randomly selecting a student at a certain university, and let A denote the event that the selected individual has a Visa credit card and B be the analogous event for a MasterCard. Suppose that $P(A) = .5$, $P(B) = .4$, and $P(A \cap B) = .25$.
a. Compute the probability that the selected individual has at least one of the two types of cards (i.e., the probability of the event $A \cup B$).
b. What is the probability that the selected individual has neither type of card?
c. Describe, in terms of A and B, the event that the selected student has a Visa card but not a MasterCard, and then calculate the probability of this event.

13. A computer consulting firm presently has bids out on three projects. Let $A_i = \{\text{awarded project } i\}$, for $i = 1, 2, 3$, and suppose that $P(A_1) = .22$, $P(A_2) = .25$, $P(A_3) = .28$, $P(A_1 \cap A_2) = .11$, $P(A_1 \cap A_3) = .05$, $P(A_2 \cap A_3) = .07$, $P(A_1 \cap A_2 \cap A_3) = .01$. Express in words each of the following events, and compute the probability of each event:
a. $A_1 \cup A_2$
b. $A_1' \cap A_2'$ [*Hint:* $(A_1 \cup A_2)' = A_1' \cap A_2'$]
c. $A_1 \cup A_2 \cup A_3$
d. $A_1' \cap A_2' \cap A_3'$
e. $A_1' \cap A_2' \cap A_3$
f. $(A_1' \cap A_2') \cup A_3$

14. A utility company offers a lifeline rate to any household whose electricity usage falls below 240 kWh during a particular month. Let A denote the event that a randomly selected household in a certain community does not exceed the lifeline usage during January, and let B be the analogous event for the month of July (A and B refer to the same household). Suppose $P(A) = .8$, $P(B) = .7$, and $P(A \cup B) = .9$. Compute the following:
a. $P(A \cap B)$.
b. The probability that the lifeline usage amount is exceeded in exactly one of the two months. Describe this event in terms of A and B.

15. Consider the type of clothes dryer (gas or electric) purchased by each of five different customers at a certain store.
a. If the probability that at most one of these purchases an electric dryer is .428, what is the probability that at least two purchase an electric dryer?
b. If $P(\text{all five purchase gas}) = .116$ and $P(\text{all five purchase electric}) = .005$, what is the probability that at least one of each type is purchased?

16. An individual is presented with three different glasses of cola, labeled C, D, and P. He is asked to taste all three and then list them in order of preference. Suppose the same cola has actually been put into all three glasses.
a. What are the simple events in this ranking experiment, and what probability would you assign to each one?
b. What is the probability that C is ranked first?
c. What is the probability that C is ranked first and D is ranked last?

17. Let A denote the event that the next request for assistance from a statistical software consultant relates to the SPSS package, and let B be the event that the next request is for help with SAS. Suppose that $P(A) = .30$ and $P(B) = .50$.
a. Why is it not the case that $P(A) + P(B) = 1$?
b. Calculate $P(A')$.
c. Calculate $P(A \cup B)$.
d. Calculate $P(A' \cap B')$.

18. A box contains four 40-W bulbs, five 60-W bulbs, and six 75-W bulbs. If bulbs are selected one by one in random order, what is the probability that at least two bulbs must be selected to obtain one that is rated 75 W?

19. Human visual inspection of solder joints on printed circuit boards can be very subjective. Part of the problem stems from the numerous types of solder defects (e.g., pad nonwetting, knee visibility, voids) and even the degree to which a joint possesses one or more of these defects. Consequently, even highly trained inspectors can disagree on the disposition of a particular joint. In one batch of 10,000 joints, inspector A found 724 that were judged defective, inspector B found 751 such joints, and 1159 of the joints were judged defective by at least one of the inspectors. Suppose that one of the 10,000 joints is randomly selected.
a. What is the probability that the selected joint was judged to be defective by neither of the two inspectors?
b. What is the probability that the selected joint was judged to be defective by inspector B but not by inspector A?

20. A certain factory operates three different shifts. Over the last year, 200 accidents have occurred at the factory. Some of these can be attributed at least in part to unsafe working conditions, whereas the others are unrelated to working conditions. The accompanying table gives the percentage of accidents falling in each type of accident–shift category.

		Unsafe Conditions	Unrelated to Conditions
Shift	*Day*	10%	35%
	Swing	8%	20%
	Night	5%	22%

Suppose one of the 200 accident reports is randomly selected from a file of reports, and the shift and type of accident are determined.

a. What are the simple events?

b. What is the probability that the selected accident was attributed to unsafe conditions?

c. What is the probability that the selected accident did not occur on the day shift?

21. An insurance company offers four different deductible levels—none, low, medium, and high—for its homeowner's policyholders and three different levels—low, medium, and high—for its automobile policyholders. The accompanying table gives proportions for the various categories of policyholders who have both types of insurance. For example, the proportion of individuals with both low homeowner's deductible and low auto deductible is .06 (6% of all such individuals).

	Homeowner's			
Auto	N	L	M	H
L	.04	.06	.05	.03
M	.07	.10	.20	.10
H	.02	.03	.15	.15

Suppose an individual having both types of policies is randomly selected.

a. What is the probability that the individual has a medium auto deductible and a high homeowner's deductible?

b. What is the probability that the individual has a low auto deductible? A low homeowner's deductible?

c. What is the probability that the individual is in the same category for both auto and homeowner's deductibles?

d. Based on your answer in part (c), what is the probability that the two categories are different?

e. What is the probability that the individual has at least one low deductible level?

f. Using the answer in part (e), what is the probability that neither deductible level is low?

22. The route used by a certain motorist in commuting to work contains two intersections with traffic signals. The probability that he must stop at the first signal is .4, the analogous probability for the second signal is .5, and the probability that he must stop at at least one of the two signals is .6. What is the probability that he must stop

a. At both signals?

b. At the first signal but not at the second one?

c. At exactly one signal?

23. The computers of six faculty members in a certain department are to be replaced. Two of the faculty members have selected laptop machines and the other four have chosen desktop machines. Suppose that only two of the setups can be done on a particular day, and the two computers to be set up are randomly selected from the six (implying 15 equally likely outcomes; if the computers are numbered 1, 2, . . . , 6, then one outcome consists of computers 1 and 2, another consists of computers 1 and 3, and so on).

a. What is the probability that both selected setups are for laptop computers?

b. What is the probability that both selected setups are desktop machines?

c. What is the probability that at least one selected setup is for a desktop computer?

d. What is the probability that at least one computer of each type is chosen for setup?

24. Use the axioms to show that if one event A is contained in another event B (i.e., A is a subset of B), then $P(A) \le P(B)$. [*Hint:* For such A and B, A and $B \cap A'$ are disjoint and $B = A \cup (B \cap A')$, as can be seen from a Venn diagram.] For general A and B, what does this imply about the relationship among $P(A \cap B)$, $P(A)$, and $P(A \cup B)$?

25. The three major options on a certain type of new car are an automatic transmission (A), a sunroof (B), and a stereo with compact disc player (C). If 70% of all purchasers request A, 80% request B, 75% request C, 85% request A or B, 90% request A or C, 95% request B or C, and 98% request A or B or C, compute the probabilities of the following events. [*Hint:* "A or B" is the event that at least one of the two options is requested; try drawing a Venn diagram and labeling all regions.]

a. The next purchaser will request at least one of the three options.

b. The next purchaser will select none of the three options.

c. The next purchaser will request only an automatic transmission and not either of the other two options.

d. The next purchaser will select exactly one of these three options.

26. A certain system can experience three different types of defects. Let A_i ($i = 1, 2, 3$) denote the event that the system has a defect of type i. Suppose that

$$P(A_1) = .12 \quad P(A_2) = .07 \quad P(A_3) = .05$$
$$P(A_1 \cup A_2) = .13 \quad P(A_1 \cup A_3) = .14$$
$$P(A_2 \cup A_3) = .10 \quad P(A_1 \cap A_2 \cap A_3) = .01$$

a. What is the probability that the system does not have a type 1 defect?

b. What is the probability that the system has both type 1 and type 2 defects?

c. What is the probability that the system has both type 1 and type 2 defects but not a type 3 defect?

d. What is the probability that the system has at most two of these defects?

27. An academic department with five faculty members—Anderson, Box, Cox, Cramer, and Fisher—must select two of its members to serve on a personnel review committee. Because the work will be time-consuming, no one is anxious to serve, so it is decided that the representative will be selected by putting five slips of paper in a box, mixing them, and selecting two.

a. What is the probability that both Anderson and Box will be selected? (*Hint:* List the equally likely outcomes.)

b. What is the probability that at least one of the two members whose name begins with C is selected?

c. If the five faculty members have taught for 3, 6, 7, 10, and 14 years, respectively, at the university, what is the probability that the two chosen representatives have at least 15 years' teaching experience at the university?

28. In Exercise 5, suppose that any incoming individual is equally likely to be assigned to any of the three stations irrespective of where other individuals have been assigned. What is the probability that

a. All three family members are assigned to the same station?

b. At most two family members are assigned to the same station?

c. Every family member is assigned to a different station?

2.3 Counting Techniques

When the various outcomes of an experiment are equally likely (the same probability is assigned to each simple event), the task of computing probabilities reduces to counting. In particular, if N is the number of outcomes in a sample space and $N(A)$ is the number of outcomes contained in an event A, then

$$P(A) = \frac{N(A)}{N} \tag{2.1}$$

If a list of the outcomes is available or easy to construct and N is small, then the numerator and denominator of Equation (2.1) can be obtained without the benefit of any general counting principles.

There are, however, many experiments for which the effort involved in constructing such a list is prohibitive because N is quite large. By exploiting some general counting rules, it is possible to compute probabilities of the form (2.1) without a listing of outcomes. These rules are also useful in many problems involving outcomes that are not equally likely. Several of the rules developed here will be used in studying probability distributions in the next chapter.

The Product Rule for Ordered Pairs

Our first counting rule applies to any situation in which a set (event) consists of ordered pairs of objects and we wish to count the number of such pairs. By an ordered pair, we mean that, if O_1 and O_2 are objects, then the pair (O_1, O_2) is different from the pair (O_2, O_1). For example, if an individual selects one airline for a trip from Los Angeles to Chicago and (after transacting business in Chicago) a second one for continuing on to New York, one possibility is (American, United), another is (United, American), and still another is (United, United).

PROPOSITION

> If the first element or object of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second element of the pair can be selected in n_2 ways, then the number of pairs is n_1n_2.

Example 2.17 A homeowner doing some remodeling requires the services of both a plumbing contractor and an electrical contractor. If there are 12 plumbing contractors and 9 electrical contractors available in the area, in how many ways can the contractors be chosen? If we denote the plumbers by $P_1, \ldots, P_{12}$ and the electricians by $Q_1, \ldots, Q_9$, then we wish the number of pairs of the form (P_i, Q_j). With $n_1 = 12$ and $n_2 = 9$, the product rule yields $N = (12)(9) = 108$ possible ways of choosing the two types of contractors. ■

In Example 2.17, the choice of the second element of the pair did not depend on which first element was chosen or occurred. As long as there is the same number of choices of the second element for each first element, the product rule is valid even when the set of possible second elements depends on the first element.

Example 2.18 A family has just moved to a new city and requires the services of both an obstetrician and a pediatrician. There are two easily accessible medical clinics, each having two obstetricians and three pediatricians. The family will obtain maximum health insurance benefits by joining a clinic and selecting both doctors from that clinic. In how many ways can this be done? Denote the obstetricians by O_1, O_2, O_3, and O_4 and the pediatricians by $P_1, \ldots, P_6$. Then we wish the number of pairs (O_i, P_j) for which O_i and P_j are associated with the same clinic. Because there are four obstetricians, $n_1 = 4$, and for each there are three choices of pediatrician, so $n_2 = 3$. Applying the product rule gives $N = n_1n_2 = 12$ possible choices. ■

Tree Diagrams

In many counting and probability problems, a configuration called a *tree diagram* can be used to represent pictorially all the possibilities. The tree diagram associated with Example 2.18 appears in Figure 2.7. Starting from a point on the left side of the diagram, for each possible first element of a pair a straight-line segment emanates rightward. Each of these lines is referred to as a first-generation branch. Now for any given first-generation branch we construct another line segment emanating from the tip of the branch for each possible choice of a second element of the pair. Each such line segment is a second-generation branch. Because there are four obstetricians, there are four first-generation branches, and three pediatricians for each obstetrician yields three second-generation branches emanating from each first-generation branch.

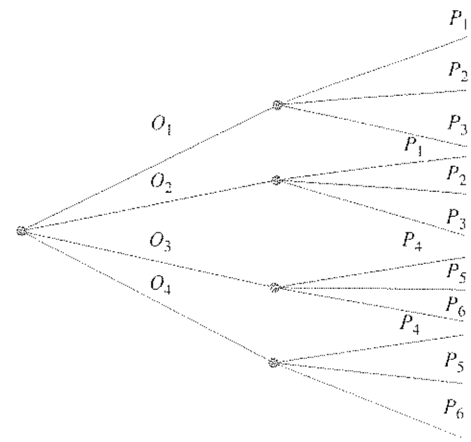


Figure 2.7 Tree diagram for Example 2.18

Generalizing, suppose there are n_1 first-generation branches, and for each first-generation branch there are n_2 second-generation branches. The total number of second-generation branches is then $n_1 n_2$. Since the end of each second-generation branch corresponds to exactly one possible pair (choosing a first element and then a second puts us at the end of exactly one second-generation branch), there are $n_1 n_2$ pairs, verifying the product rule.

The construction of a tree diagram does not depend on having the same number of second-generation branches emanating from each first-generation branch. If the second clinic had four pediatricians, then there would be only three branches emanating from two of the first-generation branches and four emanating from each of the other two first-generation branches. A tree diagram can thus be used to represent pictorially experiments other than those to which the product rule applies.

A More General Product Rule

If a six-sided die is tossed five times in succession rather than just twice, then each possible outcome is an ordered collection of five numbers such as (1, 3, 1, 2, 4) or (6, 5, 2, 2, 2). We will call an ordered collection of k objects a ***k*-tuple** (so a pair is a 2-tuple and a triple is a 3-tuple). Each outcome of the die-tossing experiment is then a 5-tuple.

Product Rule for k-Tuples

Suppose a set consists of ordered collections of k elements (k-tuples) and that there are n_1 possible choices for the first element; for each choice of the first element, there are n_2 possible choices of the second element; . . .; for each possible choice of the first $k - 1$ elements, there are n_k choices of the kth element. Then there are $n_1 n_2 \cdot \cdots \cdot n_k$ possible k-tuples.

This more general rule can also be illustrated by a tree diagram; simply construct a more elaborate diagram by adding third-generation branches emanating from the tip of each second generation, then fourth-generation branches, and so on, until finally kth-generation branches are added.

Example 2.19 (Example 2.17 continued) Suppose the home remodeling job involves first purchasing several kitchen appliances. They will all be purchased from the same dealer, and there are five dealers in the area. With the dealers denoted by $D_1, \ldots, D_5$, there are $N = n_1 n_2 n_3 = (5)(12)(9) = 540$ 3-tuples of the form (D_i, P_j, Q_k), so there are 540 ways to choose first an appliance dealer, then a plumbing contractor, and finally an electrical contractor. ■

Example 2.20 (Example 2.18 continued) If each clinic has both three specialists in internal medicine and two general surgeons, there are $n_1 n_2 n_3 n_4 = (4)(3)(3)(2) = 72$ ways to select one doctor of each type such that all doctors practice at the same clinic. ■

Permutations

So far the successive elements of a k-tuple were selected from entirely different sets (e.g., appliance dealers, then plumbers, and finally electricians). In several tosses of a die, the set from which successive elements are chosen is always $\{1, 2, 3, 4, 5, 6\}$, but the choices are made "with replacement" so that the same element can appear more than once. We now consider a fixed set consisting of n distinct elements and suppose that a k-tuple is formed by selecting successively from this set *without replacement* so that an element can appear in at most one of the k positions.

DEFINITION Any ordered sequence of k objects taken from a set of n distinct objects is called a **permutation** of size k of the objects. The number of permutations of size k that can be constructed from the n objects is denoted by $P_{k,n}$.

The number of permutations of size k is obtained immediately from the general product rule. The first element can be chosen in n ways, for each of these n ways the second element can be chosen in $n - 1$ ways, and so on; finally, for each way of choosing the first $k - 1$ elements, the kth element can be chosen in $n - (k - 1) = n - k + 1$ ways, so

$$P_{k,n} = n(n-1)(n-2)\cdot\cdots\cdot(n-k+2)(n-k+1)$$

Example 2.21 There are ten teaching assistants available for grading papers in a particular course. The first exam consists of four questions, and the professor wishes to select a different assistant to grade each question (only one assistant per question). In how many ways can assistants be chosen to grade the exam? Here n = the number of assistants = 10 and k = the number of questions = 4. The number of different grading assignments is then $P_{4,10} = (10)(9)(8)(7) = 5040$. ■

The use of factorial notation allows $P_{k,n}$ to be expressed more compactly.

DEFINITION

> For any positive integer m, $m!$ is read "m factorial" and is defined by $m! = m(m-1) \cdot \cdots \cdot (2)(1)$. Also, $0! = 1$.

Using factorial notation, $(10)(9)(8)(7) = (10)(9)(8)(7)(6!)/6! = 10!/6!$. More generally,

$$P_{k,n} = n(n-1) \cdot \cdots \cdot (n-k+1)$$
$$= \frac{n(n-1) \cdot \cdots \cdot (n-k+1)(n-k)(n-k-1) \cdot \cdots \cdot (2)(1)}{(n-k)(n-k-1) \cdot \cdots \cdot (2)(1)}$$

which becomes

$$P_{k,n} = \frac{n!}{(n-k)!}$$

For example, $P_{3,9} = 9!/(9-3)! = 9!/6! = 9 \cdot 8 \cdot 7 \cdot 6!/6! = 9 \cdot 8 \cdot 7$. Note also that because $0! = 1$, $P_{n,n} = n!/(n-n)! = n!/0! = n!/1 = n!$, as it should.

Combinations

There are many counting problems in which one is given a set of n distinct objects and wishes to count the number of *unordered* subsets of size k. For example, in bridge it is only the 13 cards in a hand and not the order in which they are dealt that is important; in the formation of a committee, the order in which committee members are listed is frequently unimportant.

DEFINITION

> Given a set of n distinct objects, any unordered subset of size k of the objects is called a **combination.** The number of combinations of size k that can be formed from n distinct objects will be denoted by $\binom{n}{k}$. (This notation is more common in probability than $C_{k,n}$, which would be analogous to notation for permutations.)

The number of combinations of size k from a particular set is smaller than the number of permutations because, when order is disregarded, a number of permutations correspond to the same combination. Consider, for example, the set $\{A, B, C, D, E\}$ consisting of five elements. There are $5!/(5-3)! = 60$ permutations of size 3. There are six permutations of size 3 consisting of the elements A, B, and C since these three can be ordered $3 \cdot 2 \cdot 1 = 3! = 6$ ways: (A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), and (C, B, A). These six permutations are equivalent to the single combination $\{A, B, C\}$. Similarly, for any other combination of size 3, there are $3!$ permutations, each obtained by ordering the three objects. Thus,

$$60 = P_{3,5} = \binom{5}{3} \cdot 3!; \quad \text{so} \binom{5}{3} = \frac{60}{3!} = 10$$

These ten combinations are

$$\{A, B, C\}\ \{A, B, D\}\ \{A, B, E\}\ \{A, C, D\}\ \{A, C, E\}\ \{A, D, E\},\ \{B, C, D\}$$
$$\{B, C, E\}\ \{B, D, E\}\ \{C, D, E\}$$

When there are n distinct objects, any permutation of size k is obtained by ordering the k unordered objects of a combination in one of $k!$ ways, so the number of permutations is the product of $k!$ and the number of combinations. This gives

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

Notice that $\binom{n}{n} = 1$ and $\binom{n}{0} = 1$ since there is only one way to choose a set of (all) n elements or of no elements, and $\binom{n}{1} = n$ since there are n subsets of size 1.

Example 2.22 A bridge hand consists of any 13 cards selected from a 52-card deck without regard to order. There are $\binom{52}{13} = 52!/13!39!$ different bridge hands, which works out to approximately 635 billion. Since there are 13 cards in each suit, the number of hands consisting entirely of clubs and/or spades (no red cards) is $\binom{26}{13} = 26!/13!13! = 10{,}400{,}600$. One of these $\binom{26}{13}$ hands consists entirely of spades, and one consists entirely of clubs, so there are $[\binom{26}{13} - 2]$ hands that consist entirely of clubs and spades with both suits represented in the hand. Suppose a bridge hand is dealt from a well-shuffled deck (i.e., 13 cards are randomly selected from among the 52 possibilities) and let

$$A = \{\text{the hand consists entirely of spades and clubs with both suits represented}\}$$
$$B = \{\text{the hand consists of exactly two suits}\}$$

The $N = \binom{52}{13}$ possible outcomes are equally likely, so

$$P(A) = \frac{N(A)}{N} = \frac{\binom{26}{13} - 2}{\binom{52}{13}} = .0000164$$

Since there are $\binom{4}{2} = 6$ combinations consisting of two suits, of which spades and clubs is one such combination,

$$P(B) = \frac{6\left[\binom{26}{13} - 2\right]}{\binom{52}{13}} = .0000983$$

That is, a hand consisting entirely of cards from exactly two of the four suits will occur roughly once in every 10,000 hands. If you play bridge only once a month, it is likely that you will never be dealt such a hand. ■

Example 2.23 A university warehouse has received a shipment of 25 printers, of which 10 are laser printers and 15 are inkjet models. If 6 of these 25 are selected at random to be checked by a particular technician, what is the probability that exactly 3 of those selected are laser printers (so that the other 3 are inkjets)?

Let D_3 = {exactly 3 of the 6 selected are inkjet printers}. Assuming that any particular set of 6 printers is as likely to be chosen as is any other set of 6, we have equally likely outcomes, so $P(D_3) = N(D_3)/N$, where N is the number of ways of choosing 6 printers from the 25 and $N(D_3)$ is the number of ways of choosing 3 laser printers and 3 inkjet models. Thus $N = \binom{25}{6}$. To obtain $N(D_3)$, think of first choosing 3 of the 15 inkjet models and then 3 of the laser printers. There are $\binom{15}{3}$ ways of choosing the 3 inkjet models, and there are $\binom{10}{3}$ ways of choosing the 3 laser printers; $N(D_3)$ is now the product of these two numbers (visualize a tree diagram—we are really using a product rule argument here), so

$$P(D_3) = \frac{N(D_3)}{N} = \frac{\binom{15}{3}\binom{10}{3}}{\binom{25}{6}} = \frac{\frac{15!}{3!12!} \cdot \frac{10!}{3!7!}}{\frac{25!}{6!19!}} = .3083$$

Let D_4 = {exactly 4 of the 6 printers selected are inkjet models} and define D_5 and D_6 in an analogous manner. Then the probability that at least 3 inkjet printers are selected is

$$P(D_3 \cup D_4 \cup D_5 \cup D_6) = P(D_3) + P(D_4) + P(D_5) + P(D_6)$$

$$= \frac{\binom{15}{3}\binom{10}{3}}{\binom{25}{6}} + \frac{\binom{15}{4}\binom{10}{2}}{\binom{25}{6}} + \frac{\binom{15}{5}\binom{10}{1}}{\binom{25}{6}} + \frac{\binom{15}{6}\binom{10}{0}}{\binom{25}{6}} = .8530$$

Exercises Section 2.3 (29–44)

29. The Student Engineers Council at a certain college has one student representative from each of the five engineering majors (civil, electrical, industrial, materials, and mechanical). In how many ways can

a. Both a council president and a vice president be selected?

b. A president, a vice president, and a secretary be selected?

c. Two members be selected for the Dean's Council?

30. A friend of mine is giving a dinner party. His current wine supply includes 8 bottles of zinfandel, 10 of merlot, and 12 of cabernet (he only drinks red wine), all from different wineries.

a. If he wants to serve 3 bottles of zinfandel and serving order is important, how many ways are there to do this?

b. If 6 bottles of wine are to be randomly selected from the 30 for serving, how many ways are there to do this?

c. If 6 bottles are randomly selected, how many ways are there to obtain two bottles of each variety?

d. If 6 bottles are randomly selected, what is the probability that this results in two bottles of each variety being chosen?

e. If 6 bottles are randomly selected, what is the probability that all of them are the same variety.

31. a. Beethoven wrote 9 symphonies and Mozart wrote 27 piano concertos. If a university radio station announcer wishes to play first a Beethoven symphony and then a Mozart concerto, in how many ways can this be done?

b. The station manager decides that on each successive night (7 days per week), a Beethoven symphony will be played, followed by a Mozart piano concerto, followed by a Schubert string quartet (of which there are 15). For roughly how many years could this policy be continued before exactly the same program would have to be repeated?

32. A chain of stereo stores is offering a special price on a complete set of components (receiver, compact disc player, speakers, cassette deck). A purchaser is offered a choice of manufacturer for each component:

Receiver: Kenwood, Onkyo, Pioneer, Sony, Sherwood
Compact disc player: Onkyo, Pioneer, Sony, Technics
Speakers: Boston, Infinity, Polk
Cassette deck: Onkyo, Sony, Teac, Technics

A switchboard display in the store allows a customer to hook together any selection of components (consisting of one of each type). Use the product rules to answer the following questions:

a. In how many ways can one component of each type be selected?
b. In how many ways can components be selected if both the receiver and the compact disc player are to be Sony?
c. In how many ways can components be selected if none is to be Sony?
d. In how many ways can a selection be made if at least one Sony component is to be included?
e. If someone flips switches on the selection in a completely random fashion, what is the probability that the system selected contains at least one Sony component? Exactly one Sony component?

33. Shortly after being put into service, some buses manufactured by a certain company have developed cracks on the underside of the main frame. Suppose a particular city has 25 of these buses, and cracks have actually appeared in 8 of them.

a. How many ways are there to select a sample of 5 buses from the 25 for a thorough inspection?
b. In how many ways can a sample of 5 buses contain exactly 4 with visible cracks?
c. If a sample of 5 buses is chosen at random, what is the probability that exactly 4 of the 5 will have visible cracks?
d. If buses are selected as in part (c), what is the probability that at least 4 of those selected will have visible cracks?

34. A production facility employs 20 workers on the day shift, 15 workers on the swing shift, and 10 workers on the graveyard shift. A quality control consultant is to select 6 of these workers for in-depth interviews. Suppose the selection is made in such a way that any particular group of 6 workers has the same chance of being selected as does any other group (drawing 6 slips without replacement from among 45).

a. How many selections result in all 6 workers coming from the day shift? What is the probability that all 6 selected workers will be from the day shift?
b. What is the probability that all 6 selected workers will be from the same shift?
c. What is the probability that at least two different shifts will be represented among the selected workers?
d. What is the probability that at least one of the shifts will be unrepresented in the sample of workers?

35. An academic department with five faculty members narrowed its choice for department head to either candidate A or candidate B. Each member then voted on a slip of paper for one of the candidates. Suppose there are actually three votes for A and two for B. If the slips are selected for tallying in random order, what is the probability that A remains ahead of B throughout the vote count (e.g., this event occurs if the selected ordering is $AABAB$, but not for $ABBAA$)?

36. An experimenter is studying the effects of temperature, pressure, and type of catalyst on yield from a certain chemical reaction. Three different temperatures, four different pressures, and five different catalysts are under consideration.

a. If any particular experimental run involves the use of a single temperature, pressure, and catalyst, how many experimental runs are possible?
b. How many experimental runs are there that involve use of the lowest temperature and two lowest pressures?

37. Refer to Exercise 36 and suppose that five different experimental runs are to be made on the first day of experimentation. If the five are randomly selected from among all the possibilities, so that any group of five has the same probability of selection, what is the probability that a different catalyst is used on each run?

38. A box in a certain supply room contains four 40-W lightbulbs, five 60-W bulbs, and six 75-W bulbs. Suppose that three bulbs are randomly selected.

a. What is the probability that exactly two of the selected bulbs are rated 75 W?

b. What is the probability that all three of the selected bulbs have the same rating?
c. What is the probability that one bulb of each type is selected?
d. Suppose now that bulbs are to be selected one by one until a 75-W bulb is found. What is the probability that it is necessary to examine at least six bulbs?

39. Fifteen telephones have just been received at an authorized service center. Five of these telephones are cellular, five are cordless, and the other five are corded phones. Suppose that these components are randomly allocated the numbers 1, 2, . . . , 15 to establish the order in which they will be serviced.
a. What is the probability that all the cordless phones are among the first ten to be serviced?
b. What is the probability that after servicing ten of these phones, phones of only two of the three types remain to be serviced?
c. What is the probability that two phones of each type are among the first six serviced?

40. Three molecules of type A, three of type B, three of type C, and three of type D are to be linked together to form a chain molecule. One such chain molecule is $ABCDABCDABCD$, and another is $BCDDAAABDBCC$.
a. How many such chain molecules are there? (*Hint:* If the three A's were distinguishable from one another—A_1, A_2, A_3—and the B's, C's, and D's were also, how many molecules would there be? How is this number reduced when the subscripts are removed from the A's?)
b. Suppose a chain molecule of the type described is randomly selected. What is the probability that all three molecules of each type end up next to one another (such as in $BBBAAADDDCCC$)?

41. A mathematics professor wishes to schedule an appointment with each of her eight teaching assistants, four men and four women, to discuss her calculus course. Suppose all possible orderings of appointments are equally likely to be selected.
a. What is the probability that at least one female assistant is among the first three with whom the professor meets?
b. What is the probability that after the first five appointments she has met with all female assistants?
c. Suppose the professor has the same eight assistants the following semester and again schedules appointments without regard to the ordering during the first semester. What is the probability that the orderings of appointments are different?

42. Three married couples have purchased theater tickets and are seated in a row consisting of just six seats. If they take their seats in a completely random fashion (random order), what is the probability that Jim and Paula (husband and wife) sit in the two seats on the far left? What is the probability that Jim and Paula end up sitting next to one another? What is the probability that at least one of the wives ends up sitting next to her husband?

43. In five-card poker, a straight consists of five cards with adjacent denominations (e.g., 9 of clubs, 10 of hearts, jack of hearts, queen of spades, and king of clubs). Assuming that aces can be high or low, if you are dealt a five-card hand, what is the probability that it will be a straight with high card 10? What is the probability that it will be a straight? What is the probability that it will be a straight flush (all cards in the same suit)?

44. Show that $\binom{n}{k} = \binom{n}{n-k}$. Give an interpretation involving subsets.

2.4 Conditional Probability

The probabilities assigned to various events depend on what is known about the experimental situation when the assignment is made. Subsequent to the initial assignment, partial information about or relevant to the outcome of the experiment may become available. Such information may cause us to revise some of our probability assignments. For a particular event A, we have used $P(A)$ to represent the probability assigned to A; we now think of $P(A)$ as the original or unconditional probability of the event A.

In this section, we examine how the information "an event B has occurred" affects the probability assigned to A. For example, A might refer to an individual having a particular disease in the presence of certain symptoms. If a blood test is performed on the individual and the result is negative (B = negative blood test), then the probability of having the disease will change (it should decrease, but not usually to zero, since blood tests are not infallible). We will use the notation $P(A \mid B)$ to represent the **conditional probability of A given that the event B has occurred.**

Example 2.24 Complex components are assembled in a plant that uses two different assembly lines, A and A'. Line A uses older equipment than A', so it is somewhat slower and less reliable. Suppose on a given day line A has assembled 8 components, of which 2 have been identified as defective (B) and 6 as nondefective (B'), whereas A' has produced 1 defective and 9 nondefective components. This information is summarized in the accompanying table.

		Condition	
		B	B'
Line	A	2	6
	A'	1	9

Unaware of this information, the sales manager randomly selects 1 of these 18 components for a demonstration. Prior to the demonstration

$$P(\text{line } A \text{ component selected}) = P(A) = \frac{N(A)}{N} = \frac{8}{18} = .44$$

However, if the chosen component turns out to be defective, then the event B has occurred, so the component must have been 1 of the 3 in the B column of the table. Since these 3 components are equally likely among themselves after B has occurred,

$$P(A \mid B) = \frac{2}{3} = \frac{\frac{2}{18}}{\frac{3}{18}} = \frac{P(A \cap B)}{P(B)} \tag{2.2}$$

In Equation (2.2), the conditional probability is expressed as a ratio of unconditional probabilities: The numerator is the probability of the intersection of the two events, whereas the denominator is the probability of the conditioning event B. A Venn diagram illuminates this relationship (Figure 2.8).

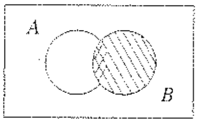


Figure 2.8 Motivating the definition of conditional probability

Given that B has occurred, the relevant sample space is no longer $\mathcal{S}$ but consists of outcomes in B; A has occurred if and only if one of the outcomes in the intersection occurred, so the conditional probability of A given B is proportional to $P(A \cap B)$. The proportionality constant $1/P(B)$ is used to ensure that the probability $P(B \mid B)$ of the new sample space B equals 1.

The Definition of Conditional Probability

Example 2.24 demonstrates that when outcomes are equally likely, computation of conditional probabilities can be based on intuition. When experiments are more complicated, though, intuition may fail us, so we want to have a general definition of conditional probability that will yield intuitive answers in simple problems. The Venn diagram and Equation (2.2) suggest the appropriate definition.

DEFINITION

For any two events A and B with $P(B) > 0$, the **conditional probability of A given that B has occurred** is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{2.3}$$

Example 2.25 Suppose that of all individuals buying a certain digital camera, 60% include an optional memory card in their purchase, 40% include an extra battery, and 30% include both a card and battery. Consider randomly selecting a buyer and let $A = \{\text{memory card purchased}\}$ and $B = \{\text{battery purchased}\}$. Then $P(A) = .60$, $P(B) = .40$, and $P(\text{both purchased}) = P(A \cap B) = .30$. Given that the selected individual purchased an extra battery, the probability that an optional card was also purchased is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{.30}{.40} = .75$$

That is, of all those purchasing an extra battery, 75% purchased an optional memory card. Similarly,

$$P(\text{battery} \mid \text{memory card}) = P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{.30}{.60} = .50$$

Notice that $P(A \mid B) \neq P(A)$ and $P(B \mid A) \neq P(B)$.

Example 2.26 A news magazine publishes three columns entitled "Art" (A), "Books" (B), and "Cinema" (C). Reading habits of a randomly selected reader with respect to these columns are

Read regularly	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
Probability	.14	.23	.37	.08	.09	.13	.05

(See Figure 2.9.)

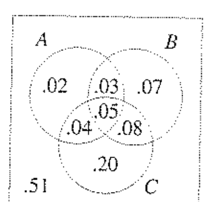


Figure 2.9 Venn diagram for Example 2.26

We thus have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{.08}{.23} = .348$$

$$P(A \mid B \cup C) = \frac{P(A \cap (B \cup C))}{P(B \cup C)} = \frac{.04 + .05 + .03}{.47} = \frac{.12}{.47} = .255$$

$$P(A \mid \text{reads at least one}) = P(A \mid A \cup B \cup C) = \frac{P(A \cap (A \cup B \cup C))}{P(A \cup B \cup C)}$$

$$= \frac{P(A)}{P(A \cup B \cup C)} = \frac{.14}{.49} = .286$$

and

$$P(A \cup B \mid C) = \frac{P((A \cup B) \cap C)}{P(C)} = \frac{.04 + .05 + .08}{.37} = .459$$

The Multiplication Rule for $P(A \cap B)$

The definition of conditional probability yields the following result, obtained by multiplying both sides of Equation (2.3) by $P(B)$.

The Multiplication Rule

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

This rule is important because it is often the case that $P(A \cap B)$ is desired, whereas both $P(B)$ and $P(A \mid B)$ can be specified from the problem description. Consideration of $P(B \mid A)$ gives $P(A \cap B) = P(B \mid A) \cdot P(A)$.

Example 2.27 Four individuals have responded to a request by a blood bank for blood donations. None of them has donated before, so their blood types are unknown. Suppose only type O+ is desired and only one of the four actually has this type. If the potential donors are selected in random order for typing, what is the probability that at least three individuals must be typed to obtain the desired type?

Making the identification $B = \{\text{first type not O+}\}$ and $A = \{\text{second type not O+}\}$, $P(B) = \frac{3}{4}$. Given that the first type is not O+, two of the three individuals left are not O+, so $P(A \mid B) = \frac{2}{3}$. The multiplication rule now gives

$$\begin{aligned} P(\text{at least three individuals are typed}) &= P(A \cap B) \\ &= P(A \mid B) \cdot P(B) \\ &= \frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12} \\ &= .5 \end{aligned}$$

The multiplication rule is most useful when the experiment consists of several stages in succession. The conditioning event B then describes the outcome of the first stage and A the outcome of the second, so that $P(A \mid B)$—conditioning on what occurs first—will often be known. The rule is easily extended to experiments involving more than two stages. For example,

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_3 \mid A_1 \cap A_2) \cdot P(A_1 \cap A_2) \\ &= P(A_3 \mid A_1 \cap A_2) \cdot P(A_2 \mid A_1) \cdot P(A_1) \qquad (2.4) \end{aligned}$$

where A_1 occurs first, followed by A_2, and finally A_3.

Example 2.28 For the blood typing experiment of Example 2.27,

$$\begin{aligned} P(\text{third type is O+}) &= P(\text{third is} \mid \text{first isn't} \cap \text{second isn't}) \\ &\quad \cdot P(\text{second isn't} \mid \text{first isn't}) \cdot P(\text{first isn't}) \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4} = .25 \end{aligned}$$

When the experiment of interest consists of a sequence of several stages, it is convenient to represent these with a tree diagram. Once we have an appropriate tree diagram, probabilities and conditional probabilities can be entered on the various branches; this will make repeated use of the multiplication rule quite straightforward.

Example 2.29 A chain of video stores sells three different brands of VCRs. Of its VCR sales, 50% are brand 1 (the least expensive), 30% are brand 2, and 20% are brand 3. Each manufacturer offers a 1-year warranty on parts and labor. It is known that 25% of brand 1's VCRs require warranty repair work, whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively.

1. What is the probability that a randomly selected purchaser has bought a brand 1 VCR that will need repair while under warranty?
2. What is the probability that a randomly selected purchaser has a VCR that will need repair while under warranty?
3. If a customer returns to the store with a VCR that needs warranty repair work, what is the probability that it is a brand 1 VCR? A brand 2 VCR? A brand 3 VCR?

The first stage of the problem involves a customer selecting one of the three brands of VCR. Let $A_i = \{\text{brand } i \text{ is purchased}\}$, for $i = 1, 2,$ and 3. Then $P(A_1) = .50$, $P(A_2) = .30$, and $P(A_3) = .20$. Once a brand of VCR is selected, the second stage involves observing whether the selected VCR needs warranty repair. With $B = \{\text{needs repair}\}$ and $B' = \{\text{doesn't need repair}\}$, the given information implies that $P(B \mid A_1) = .25$, $P(B \mid A_2) = .20$, and $P(B \mid A_3) = .10$.

The tree diagram representing this experimental situation is shown in Figure 2.10. The initial branches correspond to different brands of VCRs; there are two second-generation branches emanating from the tip of each initial branch, one for "needs repair" and the other for "doesn't need repair." The probability $P(A_i)$ appears on the ith initial branch, whereas the conditional probabilities $P(B \mid A_i)$ and $P(B' \mid A_i)$ appear on the second-generation branches. To the right of each second-generation branch corresponding to the occurrence of B, we display the product of probabilities on the branches leading out to that point. This is simply the multiplication rule in action. The answer to the question posed in 1 is thus $P(A_1 \cap B) = P(B \mid A_1) \cdot P(A_1) = .125$. The answer to question 2 is

$$\begin{aligned} P(B) &= P[(\text{brand 1 and repair}) \text{ or } (\text{brand 2 and repair}) \text{ or } (\text{brand 3 and repair})] \\ &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\ &= .125 + .060 + .020 = .205 \end{aligned}$$

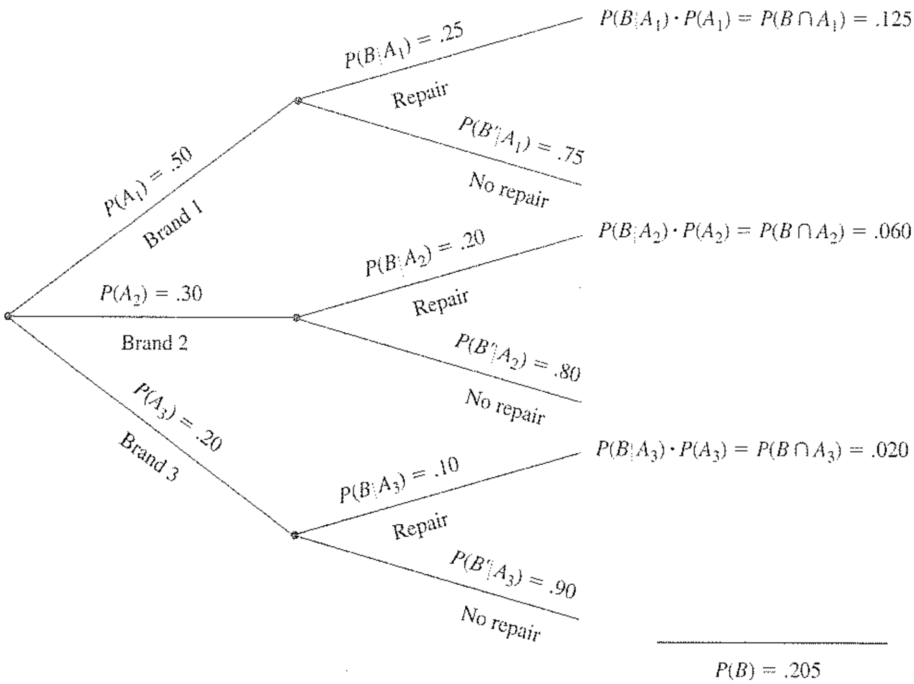


Figure 2.10 Tree diagram for Example 2.29

Finally,

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{.125}{.205} = .61$$

$$P(A_2 \mid B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{.060}{.205} = .29$$

and

$$P(A_3 \mid B) = 1 - P(A_1 \mid B) - P(A_2 \mid B) = .10$$

Notice that the initial or *prior probability* of brand 1 is .50, whereas once it is known that the selected VCR needed repair, the *posterior probability* of brand 1 increases to .61. This is because brand 1 VCRs are more likely to need warranty repair than are the other brands. The posterior probability of brand 3 is $P(A_3 \mid B) = .10$, which is much less than the prior probability $P(A_3) = .20$.

Bayes' Theorem

The computation of a posterior probability $P(A_j \mid B)$ from given prior probabilities $P(A_i)$ and conditional probabilities $P(B \mid A_i)$ occupies a central position in elementary probability. The general rule for such computations, which is really just a simple application of the multiplication rule, goes back to Reverend Thomas Bayes, who lived in the eighteenth century. To state it we first need another result. Recall that events $A_1, \ldots, A_k$ are mutually exclusive if no two have any common outcomes. The events are *exhaustive* if one A_i must occur, so that $A_1 \cup \cdots \cup A_k = \mathcal{S}$.

The Law of Total Probability

Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$P(B) = P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_k)P(A_k) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i) \tag{2.5}$$

Proof

Because the A_i's are mutually exclusive and exhaustive, if B occurs it must be in conjunction with exactly one of the A_i's. That is, $B = (A_1 \text{ and } B)$ or . . . or $(A_k \text{ and } B) = (A_1 \cap B) \cup \cdots \cup (A_k \cap B)$, where the events $(A_i \cap B)$ are mutually exclusive. This "partitioning of B" is illustrated in Figure 2.11. Thus

$$P(B) = \sum_{i=1}^{k} P(A_i \cap B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

as desired.

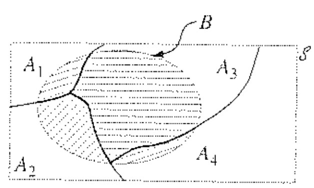


Figure 2.11 Partition of B by mutually exclusive and exhaustive A_i's

An example of the use of Equation (2.5) appeared in answering question 2 of Example 2.29, where $A_1 = \{\text{brand 1}\}$, $A_2 = \{\text{brand 2}\}$, $A_3 = \{\text{brand 3}\}$, and $B = \{\text{repair}\}$.

Bayes' Theorem

Let $A_1, A_2, \ldots, A_k$ be a collection of k mutually exclusive and exhaustive events with $P(A_i) > 0$ for $i = 1, \ldots, k$. Then for any other event B for which $P(B) > 0$,

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i) \cdot P(A_i)} \qquad j = 1, \ldots, k \qquad (2.6)$$

The transition from the second to the third expression in (2.6) rests on using the multiplication rule in the numerator and the law of total probability in the denominator.

The proliferation of events and subscripts in (2.6) can be a bit intimidating to probability newcomers. As long as there are relatively few events in the partition, a tree diagram (as in Example 2.29) can be used as a basis for calculating posterior probabilities without ever referring explicitly to Bayes' theorem.

Example 2.30 *Incidence of a rare disease.* Only 1 in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed. The test is such that when an individual actually has the disease, a positive result will occur 99% of the time, whereas an individual without the disease will show a positive test result only 2% of the time. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

To use Bayes' theorem, let A_1 = {individual has the disease}, A_2 = {individual does not have the disease}, and B = {positive test result}. Then $P(A_1) = .001$, $P(A_2) = .999$, $P(B \mid A_1) = .99$, and $P(B \mid A_2) = .02$. The tree diagram for this problem is in Figure 2.12.

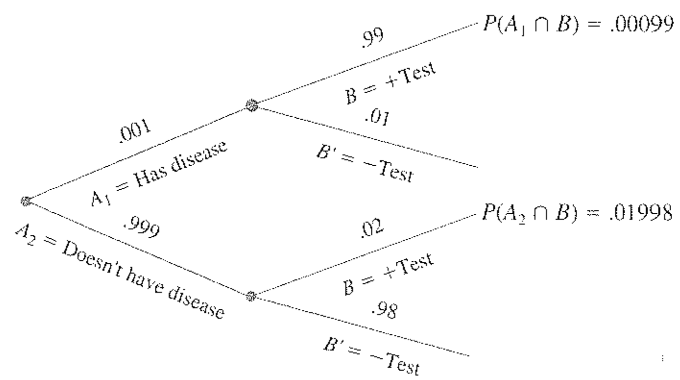


Figure 2.12 Tree diagram for the rare-disease problem

Next to each branch corresponding to a positive test result, the multiplication rule yields the recorded probabilities. Therefore, $P(B) = .00099 + .01998 = .02097$, from which we have

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{.00099}{.02097} = .047$$

This result seems counterintuitive; the diagnostic test appears so accurate we expect someone with a positive test result to be highly likely to have the disease, whereas the computed conditional probability is only .047. However, because the disease is rare and the test only moderately reliable, most positive test results arise from errors rather than from diseased individuals. The probability of having the disease has increased by a multiplicative factor of 47 (from prior .001 to posterior .047); but to get a further increase in the posterior probability, a diagnostic test with much smaller error rates is needed. If the disease were not so rare (e.g., 25% incidence in the population), then the error rates for the present test would provide good diagnoses. ■

Exercises Section 2.4 (45–67)

45. The population of a particular country consists of three ethnic groups. Each individual belongs to one of the four major blood groups. The accompanying *joint probability table* gives the proportions of individuals in the various ethnic group–blood group combinations.

		Blood Group			
		O	A	B	AB
Ethnic Group	1	.082	.106	.008	.004
	2	.135	.141	.018	.006
	3	.215	.200	.065	.020

Suppose that an individual is randomly selected from the population, and define events by A = {type A selected}, B = {type B selected}, and C = {ethnic group 3 selected}.

a. Calculate $P(A)$, $P(C)$, and $P(A \cap C)$.

b. Calculate both $P(A \mid C)$ and $P(C \mid A)$, and explain in context what each of these probabilities represents.

c. If the selected individual does not have type B blood, what is the probability that he or she is from ethnic group 1?

46. Suppose an individual is randomly selected from the population of all adult males living in the United States. Let A be the event that the selected individual is over 6 ft in height, and let B be the event that the selected individual is a professional basketball player. Which do you think is larger, $P(A \mid B)$ or $P(B \mid A)$? Why?

47. Return to the credit card scenario of Exercise 12 (Section 2.2), where A = {Visa}, B = {MasterCard}, $P(A) = .5$, $P(B) = .4$, and $P(A \cap B) = .25$. Calculate and interpret each of the following probabilities (a Venn diagram might help).

a. $P(B \mid A)$ **b.** $P(B' \mid A)$

c. $P(A \mid B)$ **d.** $P(A' \mid B)$

e. Given that the selected individual has at least one card, what is the probability that he or she has a Visa card?

48. Reconsider the system defect situation described in Exercise 26 (Section 2.2).

a. Given that the system has a type 1 defect, what is the probability that it has a type 2 defect?

b. Given that the system has a type 1 defect, what is the probability that it has all three types of defects?

c. Given that the system has at least one type of defect, what is the probability that it has exactly one type of defect?

d. Given that the system has both of the first two types of defects, what is the probability that it does not have the third type of defect?

49. If two bulbs are randomly selected from the box of lightbulbs described in Exercise 38 (Section 2.3) and at least one of them is found to be rated 75 W, what is the probability that both of them are 75-W bulbs? Given that at least one of the two selected is not rated 75 W, what is the probability that both selected bulbs have the same rating?

50. A department store sells sport shirts in three sizes (small, medium, and large), three patterns (plaid, print, and stripe), and two sleeve lengths (long and short). The accompanying tables give the proportions of shirts sold in the various category combinations.

Short-sleeved

	Pattern		
Size	Pl	Pr	St
S	.04	.02	.05
M	.08	.07	.12
L	.03	.07	.08

Long-sleeved

	Pattern		
Size	Pl	Pr	St
S	.03	.02	.03
M	.10	.05	.07
L	.04	.02	.08

a. What is the probability that the next shirt sold is a medium, long-sleeved, print shirt?

b. What is the probability that the next shirt sold is a medium print shirt?

c. What is the probability that the next shirt sold is a short-sleeved shirt? A long-sleeved shirt?

d. What is the probability that the size of the next shirt sold is medium? That the pattern of the next shirt sold is a print?

e. Given that the shirt just sold was a short-sleeved plaid, what is the probability that its size was medium?

f. Given that the shirt just sold was a medium plaid, what is the probability that it was short-sleeved? Long-sleeved?

51. One box contains six red balls and four green balls, and a second box contains seven red balls and three green balls. A ball is randomly chosen from the first box and placed in the second box. Then a ball is randomly selected from the second box and placed in the first box.

a. What is the probability that a red ball is selected from the first box and a red ball is selected from the second box?

b. At the conclusion of the selection process, what is the probability that the numbers of red and green balls in the first box are identical to the numbers at the beginning?

52. A system consists of two identical pumps, #1 and #2. If one pump fails, the system will still operate. However, because of the added strain, the extra remaining pump is now more likely to fail than was originally the case. That is, $r = P(\#2 \text{ fails} \mid \#1 \text{ fails}) > P(\#2 \text{ fails}) = q$. If at least one pump fails by the end of the pump design life in 7% of all systems and both pumps fail during that period in only 1%, what is the probability that pump #1 will fail during the pump design life?

53. A certain shop repairs both audio and video components. Let A denote the event that the next component brought in for repair is an audio component, and let B be the event that the next component is a compact disc player (so the event B is contained in A). Suppose that $P(A) = .6$ and $P(B) = .05$. What is $P(B \mid A)$?

54. In Exercise 13, $A_i = \{\text{awarded project } i\}$, for $i = 1, 2, 3$. Use the probabilities given there to compute the following probabilities:

a. $P(A_2 \mid A_1)$

b. $P(A_2 \cap A_3 \mid A_1)$

c. $P(A_2 \cup A_3 \mid A_1)$

d. $P(A_1 \cap A_2 \cap A_3 \mid A_1 \cup A_2 \cup A_3)$.
Express in words the probability you have calculated.

55. In Exercise 42, six people (three married couples) chose seats at random in a row consisting of six seats.

a. Use the multiplication rule to compute the probability that Jim and Paula sit together on the far left (event A) and that John and Mary Lou (husband and wife) sit together in the middle (event B).

b. Given that John and Mary Lou sit together in the middle, what is the probability that the two other husbands sit next to their wives?

c. Given that John and Mary Lou sit together, what is the probability that all husbands sit next to their wives?

56. If $P(B \mid A) > P(B)$, show that $P(B' \mid A) < P(B')$. (*Hint:* Add $P(B' \mid A)$ to both sides of the given inequality and then use the result of Exercise 57.)

57. For any events A and B with $P(B) > 0$, show that $P(A \mid B) + P(A' \mid B) = 1$.

58. Show that for any three events A, B, and C with $P(C) > 0$, $P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$.

59. At a certain gas station, 40% of the customers use regular unleaded gas (A_1), 35% use extra unleaded gas (A_2), and 25% use premium unleaded gas (A_3). Of those customers using regular gas, only 30% fill their tanks (event B). Of those customers using extra gas, 60% fill their tanks, whereas of those using premium, 50% fill their tanks.

a. What is the probability that the next customer will request extra unleaded gas and fill the tank ($A_2 \cap B$)?

b. What is the probability that the next customer fills the tank?

c. If the next customer fills the tank, what is the probability that regular gas is requested? Extra gas? Premium gas?

60. Seventy percent of the light aircraft that disappear while in flight in a certain country are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose a light aircraft has disappeared.

a. If it has an emergency locator, what is the probability that it will not be discovered?

b. If it does not have an emergency locator, what is the probability that it will be discovered?

61. Components of a certain type are shipped to a supplier in batches of ten. Suppose that 50% of all such batches contain no defective components, 30% contain one defective component, and 20% contain two defective components. Two components from a batch are randomly selected and tested. What are the probabilities associated with 0, 1, and 2 defective components being in the batch under each of the following conditions?

a. Neither tested component is defective.

b. One of the two tested components is defective. (*Hint:* Draw a tree diagram with three first-generation branches for the three different types of batches.)

62. A company that manufactures video cameras produces a basic model and a deluxe model. Over the past year, 40% of the cameras sold have been of the basic model. Of those buying the basic model, 30% purchase an extended warranty, whereas 50% of all deluxe purchasers do so. If you learn that a randomly selected purchaser has an extended warranty, how likely is it that he or she has a basic model?

63. For customers purchasing a full set of tires at a particular tire store, consider the events

A = {tires purchased were made in the United States}

B = {purchaser has tires balanced immediately}

C = {purchasler requests front-end alignment}

along with A', B', and C'. Assume the following unconditional and conditional probabilities:

$P(A) = .75 \quad P(B \mid A) = .9 \quad P(B \mid A') = .8$

$P(C \mid A \cap B) = .8 \quad P(C \mid A \cap B') = .6$

$P(C \mid A' \cap B) = .7 \quad P(C \mid A' \cap B') = .3$

a. Construct a tree diagram consisting of first-, second-, and third-generation branches and place an event label and appropriate probability next to each branch.

b. Compute $P(A \cap B \cap C)$.

c. Compute $P(B \cap C)$.

d. Compute $P(C)$.

e. Compute $P(A \mid B \cap C)$, the probability of a purchase of U.S. tires given that both balancing and an alignment were requested.

64. In Example 2.30, suppose that the incidence rate for the disease is 1 in 25 rather than 1 in 1000. What then is the probability of a positive test result? Given that the test result is positive, what is the probability that the individual has the disease? Given a negative test result, what is the probability that the individual does not have the disease?

65. At a large university, in the never-ending quest for a satisfactory textbook, the Statistics Department has tried a different text during each of the last three quarters. During the fall quarter, 500 students used the text by Professor Mean; during the winter quarter, 300 students used the text by Professor Median; and during the spring quarter, 200 students used the text by Professor Mode. A survey at the end of each quarter showed that 200 students were satisfied with Mean's book, 150 were satisfied with Median's book, and 160 were satisfied with Mode's book. If a student who took statistics during one of these quarters is selected at random and admits to having been satisfied with the text, is the student most likely to have used the book by Mean, Median, or Mode? Who is the least likely author? (*Hint:* Draw a tree diagram or use Bayes' theorem.)

66. A friend who lives in Los Angeles makes frequent consulting trips to Washington, D.C.; 50% of the time she travels on airline #1, 30% of the time on airline #2, and the remaining 20% of the time on airline #3. For airline #1, flights are late into D.C. 30% of the time and late into L.A. 10% of the time. For airline #2, these percentages are 25% and 20%, whereas for airline #3 the percentages are 40% and 25%. If we learn that on a particular trip she arrived late at exactly one of the two destinations, what are the posterior probabilities of having flown on airlines #1, #2, and #3? (*Hint:* From the tip of each first-generation branch on a tree diagram, draw three second-generation branches labeled, respectively, 0 late, 1 late, and 2 late.)

67. In Exercise 59, consider the following additional information on credit card usage:

70% of all regular fill-up customers use a credit card.

50% of all regular non-fill-up customers use a credit card.

60% of all extra fill-up customers use a credit card.

50% of all extra non-fill-up customers use a credit card.

50% of all premium fill-up customers use a credit card.

40% of all premium non-fill-up customers use a credit card.

Compute the probability of each of the following events for the next customer to arrive (a tree diagram might help).

a. {extra and fill-up and credit card}
b. {premium and non-fill-up and credit card}
c. {premium and credit card}
d. {fill-up and credit card}
e. {credit card}
f. If the next customer uses a credit card, what is the probability that premium was requested?

2.5 Independence

The definition of conditional probability enables us to revise the probability $P(A)$ originally assigned to A when we are subsequently informed that another event B has occurred; the new probability of A is $P(A \mid B)$. In our examples, it was frequently the case that $P(A \mid B)$ was unequal to the unconditional probability $P(A)$, indicating that the information "B has occurred" resulted in a change in the chance of A occurring. There are other situations, though, in which the chance that A will occur or has occurred is not affected by knowledge that B has occurred, so that $P(A \mid B) = P(A)$. It is then natural to think of A and B as independent events, meaning that the occurrence or nonoccurrence of one event has no bearing on the chance that the other will occur.

DEFINITION

Two events A and B are **independent** if $P(A \mid B) = P(A)$ and are **dependent** otherwise.

The definition of independence might seem "unsymmetric" because we do not demand that $P(B \mid A) = P(B)$ also. However, using the definition of conditional probability and the multiplication rule,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \mid B)P(B)}{P(A)} \tag{2.7}$$

The right-hand side of Equation (2.7) is $P(B)$ if and only if $P(A \mid B) = P(A)$ (independence), so the equality in the definition implies the other equality (and vice versa). It is also straightforward to show that if A and B are independent, then so are the following pairs of events: (1) A' and B, (2) A and B', and (3) A' and B'.

Example 2.31 Consider tossing a fair six-sided die once and define events $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$, and $C = \{1, 2, 3, 4\}$. We then have $P(A) = \frac{1}{2}$, $P(A \mid B) = \frac{1}{3}$, and $P(A \mid C) = \frac{1}{2}$. That is, events A and B are dependent, whereas events A and C are independent. Intuitively, if such a die is tossed and we are informed that the outcome was 1, 2, 3, or 4 (C has occurred), then the probability that A occurred is $\frac{1}{2}$, as it originally was, since two of the four relevant outcomes are even and the outcomes are still equally likely. ■

Example 2.32 Let A and B be any two mutually exclusive events with $P(A) > 0$. For example, for a randomly chosen automobile, let $A = \{$the car has four cylinders$\}$ and $B = \{$the car has six cylinders$\}$. Since the events are mutually exclusive, if B occurs, then A cannot possibly have occurred, so $P(A \mid B) = 0 \neq P(A)$. The message here is that *if two events are mutually exclusive, they cannot be independent.* When A and B are mutually exclusive, the information that A occurred says something about B (it cannot have occurred), so independence is precluded. ■

$P(A \cap B)$ When Events Are Independent

Frequently the nature of an experiment suggests that two events A and B should be assumed independent. This is the case, for example, if a manufacturer receives a circuit board from each of two different suppliers, each board is tested on arrival, and $A = \{$first is defective$\}$ and $B = \{$second is defective$\}$. If $P(A) = .1$, it should also be the case that $P(A \mid B) = .1$; knowing the condition of the second board shouldn't provide information about the condition of the first. Our next result shows how to compute $P(A \cap B)$ when the events are independent.

PROPOSITION A and B are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B) \tag{2.8}$$

To paraphrase the proposition, A and B are independent events iff* the probability that they both occur ($A \cap B$) is the product of the two individual probabilities. The verification is as follows:

$$P(A \cap B) = P(A \mid B) \cdot P(B) = P(A) \cdot P(B) \tag{2.9}$$

where the second equality in Equation (2.9) is valid iff A and B are independent. Because of the equivalence of independence with Equation (2.8), the latter can be used as a definition of independence.

*iff is an abbreviation for "if and only if."

Example 2.33 It is known that 30% of a certain company's washing machines require service while under warranty, whereas only 10% of its dryers need such service. If someone purchases both a washer and a dryer made by this company, what is the probability that both machines need warranty service?

Let A denote the event that the washer needs service while under warranty, and let B be defined analogously for the dryer. Then $P(A) = .30$ and $P(B) = .10$. Assuming that the two machines function independently of one another, the desired probability is

$$P(A \cap B) = P(A) \cdot P(B) = (.30)(.10) = .03$$

The probability that neither machine needs service is

$$P(A' \cap B') = P(A') \cdot P(B') = (.70)(.90) = .63$$

Example 2.34 Each day, Monday through Friday, a batch of components sent by a first supplier arrives at a certain inspection facility. Two days a week, a batch also arrives from a second supplier. Eighty percent of all supplier 1's batches pass inspection, and 90% of supplier 2's do likewise. What is the probability that, on a randomly selected day, two batches pass inspection? We will answer this assuming that on days when two batches are tested, whether the first batch passes is independent of whether the second batch does so. Figure 2.13 displays the relevant information.

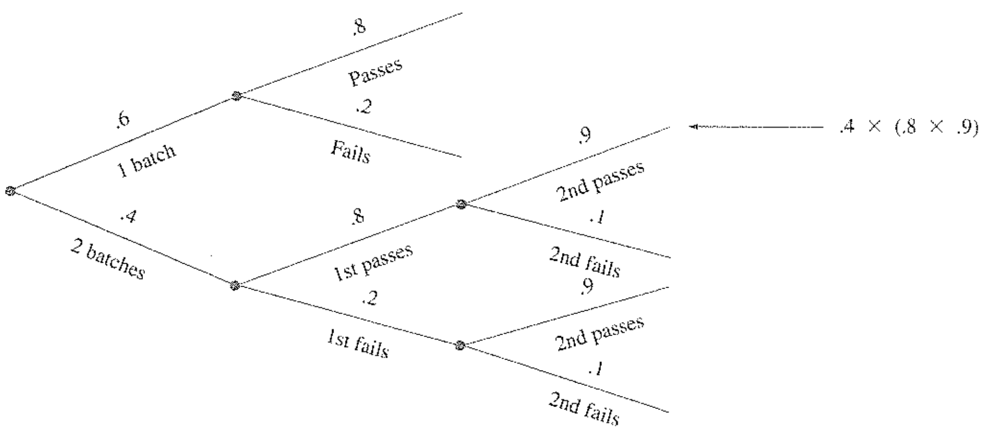


Figure 2.13 Tree diagram for Example 2.34

$$\begin{aligned} P(\text{two pass}) &= P(\text{two received} \cap \text{both pass}) \\ &= P(\text{both pass} \mid \text{two received}) \cdot P(\text{two received}) \\ &= [(.8)(.9)](.4) = .288 \end{aligned}$$

Independence of More Than Two Events

The notion of independence of two events can be extended to collections of more than two events. Although it is possible to extend the definition for two independent events by working in terms of conditional and unconditional probabilities, it is more direct and less cumbersome to proceed along the lines of the last proposition.

DEFINITION

Events $A_1, \ldots, A_n$ are **mutually independent** if for every k ($k = 2, 3, \ldots, n$) and every subset of indices $i_1, i_2, \ldots, i_k$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \cdots \cdot P(A_{i_k}).$$

To paraphrase the definition, the events are mutually independent if the probability of the intersection of any subset of the n events is equal to the product of the individual probabilities. As was the case with two events, we frequently specify at the outset of a problem the independence of certain events. The definition can then be used to calculate the probability of an intersection.

Example 2.35 The article "Reliability Evaluation of Solar Photovoltaic Arrays" (*Solar Energy*, 2002: 129–141) presents various configurations of solar photovoltaic arrays consisting of crystalline silicon solar cells. Consider first the system illustrated in Figure 2.14(a). There are two subsystems connected in parallel, each one containing three cells. In order for the system to function, at least one of the two parallel subsystems must work. Within each subsystem, the three cells are connected in series, so a subsystem will work only if all cells in the subsystem work. Consider a particular lifetime value t_0, and supose we want to determine the probability that the system lifetime exceeds t_0. Let A_i denote the event that the lifetime of cell i exceeds t_0 ($i = 1, 2, \ldots, 6$). We assume that the A_i's are independent events (whether any particular cell lasts more than t_0 hours has no bearing on whether or not any other cell does) and that $P(A_i) = .9$ for every i since the cells are identical. Then

$$\begin{aligned} P(\text{system lifetime exceeds } t_0) &= P[(A_1 \cap A_2 \cap A_3) \cup (A_4 \cap A_5 \cap A_6)] \\ &= P(A_1 \cap A_2 \cap A_3) + P(A_4 \cap A_5 \cap A_6) \\ &\quad - P\,[(A_1 \cap A_2 \cap A_3) \cap (A_4 \cap A_5 \cap A_6)] \\ &= (.9)(.9)(.9) + (.9)(.9)(.9) - (.9)(.9)(.9)(.9)(.9)(.9) = .927 \end{aligned}$$

Alternatively,

$$\begin{aligned} P(\text{system lifetime exceeds } t_0) &= 1 - P(\text{both subsystem lives are} \le t_0) \\ &= 1 - [P(\text{subsystem life is} \le t_0)]^2 \\ &= 1 - [1 - P(\text{subsystem life is} > t_0)]^2 \\ &= 1 - [1 - (.9)^3]^2 = .927 \end{aligned}$$

Next consider the total-cross-tied system shown in Figure 2.14(b), obtained from the series-parallel array by connecting ties across each column of junctions. Now the

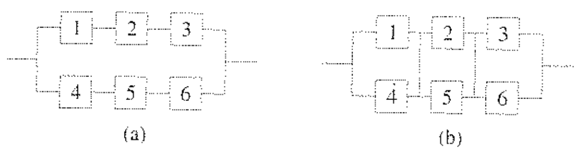


Figure 2.14 System configurations for Example 2.35: (a) series-parallel; (b) total-cross-tied

system fails as soon as an entire column fails, and system lifetime exceeds t_0 only if the life of every column does so. For this configuration,

$$\begin{aligned} P(\text{system lifetime is at least } t_0) &= [P(\text{column lifetime exceeds } t_0)]^3 \\ &= [1 - P(\text{column lifetime is} \le t_0)]^3 \\ &= [1 - P(\text{both cells in a column have lifetime} \le t_0)]^3 \\ &= [1 - (1 - .9)^2]^3 = .970 \end{aligned}$$

Exercises | Section 2.5 (68–87)

68. Reconsider the credit card scenario of Exercise 47 (Section 2.4), and show that A and B are dependent first by using the definition of independence and then by verifying that the multiplication property does not hold.

69. An oil exploration company currently has two active projects, one in Asia and the other in Europe. Let A be the event that the Asian project is successful and B be the event that the European project is successful. Suppose that A and B are independent events with $P(A) = .4$ and $P(B) = .7$.

a. If the Asian project is not successful, what is the probability that the European project is also not successful? Explain your reasoning.

b. What is the probability that at least one of the two projects will be successful?

c. Given that at least one of the two projects is successful, what is the probability that only the Asian project is successful?

70. In Exercise 13, is any A_i independent of any other A_j? Answer using the multiplication property for independent events.

71. If A and B are independent events, show that A' and B are also independent. [*Hint:* First establish a relationship between $P(A' \cap B)$, $P(B)$, and $P(A \cap B)$.]

72. Suppose that the proportions of blood phenotypes in a particular population are as follows:

A	B	AB	O
.42	.10	.04	.44

Assuming that the phenotypes of two randomly selected individuals are independent of one another, what is the probability that both phenotypes are O? What is the probability that the phenotypes of two randomly selected individuals match?

73. One of the assumptions underlying the theory of control charting (see Chapter 16) is that successive plotted points are independent of one another. Each plotted point can signal either that a manufacturing process is operating correctly or that there is some sort of malfunction. Even when a process is running correctly, there is a small probability that a particular point will signal a problem with the process. Suppose that this probability is .05. What is the probability that at least one of 10 successive points indicates a problem when in fact the process is operating correctly? Answer this question for 25 successive points.

74. The probability that a grader will make a marking error on any particular question of a multiple-choice exam is .1. If there are ten questions and questions are marked independently, what is the probability that no errors are made? That at least one error is made? If there are n questions and the probability of a marking error is p rather than .1, give expressions for these two probabilities.

75. An aircraft seam requires 25 rivets. The seam will have to be reworked if any of these rivets is defective. Suppose rivets are defective independently of one another, each with the same probability.

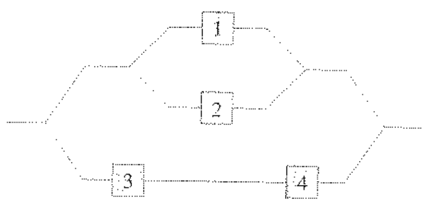
a. If 20% of all seams need reworking, what is the probability that a rivet is defective?

b. How small should the probability of a defective rivet be to ensure that only 10% of all seams need reworking?

76. A boiler has five identical relief valves. The probability that any particular valve will open on demand is .95. Assuming independent operation of the valves, calculate P(at least one valve opens) and P(at least one valve fails to open).

77. Two pumps connected in parallel fail independently of one another on any given day. The probability that only the older pump will fail is .10, and the probability that only the newer pump will fail is .05. What is the probability that the pumping system will fail on any given day (which happens if both pumps fail)?

78. Consider the system of components connected as in the accompanying picture. Components 1 and 2 are connected in parallel, so that subsystem works iff either 1 or 2 works; since 3 and 4 are connected in series, that subsystem works iff both 3 and 4 work. If components work independently of one another and P(component works) = .9, calculate P(system works).



79. Refer back to the series-parallel system configuration introduced in Example 2.35, and suppose that there are only two cells rather than three in each parallel subsystem [in Figure 2.14(a), eliminate cells 3 and 6, and renumber cells 4 and 5 as 3 and 4]. Using $P(A_i) = .9$, the probability that system lifetime exceeds t_0 is easily seen to be .9639. To what value would .9 have to be changed in order to increase the system lifetime reliability from .9639 to .99? (*Hint:* Let $P(A_i) = p$, express system reliability in terms of p, and then let $x = p^2$.)

80. Consider independently rolling two fair dice, one red and the other green. Let A be the event that the red die shows 3 dots, B be the event that the green die shows 4 dots, and C be the event that the total number of dots showing on the two dice is 7. Are these events pairwise independent (i.e., are A and B independent events, are A and C independent, and are B and C independent)? Are the three events mutually independent?

81. Components arriving at a distributor are checked for defects by two different inspectors (each component is checked by both inspectors). The first inspector detects 90% of all defectives that are present, and the second inspector does likewise. At least one inspector does not detect a defect on 20% of all defective components. What is the probability that the following occur?

a. A defective component will be detected only by the first inspector? By exactly one of the two inspectors?

b. All three defective components in a batch escape detection by both inspectors (assuming inspections of different components are independent of one another)?

82. Seventy percent of all vehicles examined at a certain emissions inspection station pass the inspection. Assuming that successive vehicles pass or fail independently of one another, calculate the following probabilities:

a. P(all of the next three vehicles inspected pass)

b. P(at least one of the next three inspected fails)

c. P(exactly one of the next three inspected passes)

d. P(at most one of the next three vehicles inspected passes)

e. Given that at least one of the next three vehicles passes inspection, what is the probability that all three pass (a conditional probability)?

83. A quality control inspector is inspecting newly produced items for faults. The inspector searches an item for faults in a series of independent fixations, each of a fixed duration. Given that a flaw is actually present, let p denote the probability that the flaw is detected during any one fixation (this model is discussed in "Human Performance in Sampling Inspection," *Human Factors*, 1979: 99–105).

a. Assuming that an item has a flaw, what is the probability that it is detected by the end of the second fixation (once a flaw has been detected, the sequence of fixations terminates)?

b. Give an expression for the probability that a flaw will be detected by the end of the nth fixation.

c. If when a flaw has not been detected in three fixations, the item is passed, what is the probability that a flawed item will pass inspection?

d. Suppose 10% of all items contain a flaw [P(randomly chosen item is flawed) = .1]. With the assumption of part (c), what is the probability that a randomly chosen item will pass inspection (it will automatically pass if it is not flawed, but could also pass if it is flawed)?

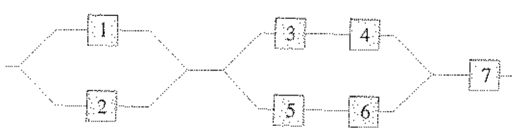
e. Given that an item has passed inspection (no flaws in three fixations), what is the probability that it is actually flawed? Calculate for $p = .5$.

84. a. A lumber company has just taken delivery on a lot of 10,000 2 × 4 boards. Suppose that 20% of these boards (2,000) are actually too green to be used in first-quality construction. Two boards are selected at random, one after the other. Let A = {the first board is green} and B = {the second board is green}. Compute $P(A)$, $P(B)$, and $P(A \cap B)$ (a tree diagram might help). Are A and B independent?

b. With A and B independent and $P(A) = P(B) = .2$, what is $P(A \cap B)$? How much difference is there between this answer and $P(A \cap B)$ in part (a)? For purposes of calculating $P(A \cap B)$, can we assume that A and B of part (a) are independent to obtain essentially the correct probability?

c. Suppose the lot consists of ten boards, of which two are green. Does the assumption of independence now yield approximately the correct answer for $P(A \cap B)$? What is the critical difference between the situation here and that of part (a)? When do you think that an independence assumption would be valid in obtaining an approximately correct answer to $P(A \cap B)$?

85. Refer to the assumptions stated in Exercise 78 and answer the question posed there for the system in the accompanying picture. How would the probability change if this were a subsystem connected in parallel to the subsystem pictured in Figure 2.14(a)?



86. Professor Stan der Deviation can take one of two routes on his way home from work. On the first route, there are four railroad crossings. The probability that he will be stopped by a train at any particular one of the crossings is .1, and trains operate independently at the four crossings. The other route is longer but there are only two crossings, independent of one another, with the same stoppage probability for each as on the first route. On a particular day, Professor Deviation has a meeting scheduled at home for a certain time. Whichever route he takes, he calculates that he will be late if he is stopped by trains at at least half the crossings encountered.

a. Which route should he take to minimize the probability of being late to the meeting?

b. If he tosses a fair coin to decide on a route and he is late, what is the probability that he took the four-crossing route?

87. Suppose identical tags are placed on both the left ear and the right ear of a fox. The fox is then let loose for a period of time. Consider the two events C_1 = {left ear tag is lost} and C_2 = {right ear tag is lost}. Let $\pi = P(C_1) = P(C_2)$, and assume C_1 and C_2 are independent events. Derive an expression (involving π) for the probability that exactly one tag is lost given that at most one is lost ("Ear Tag Loss in Red Foxes," *J. Wildlife Mgmt.*, 1976: 164–167). (*Hint:* Draw a tree diagram in which the two initial branches refer to whether the left ear tag was lost.)

Supplementary Exercises (88–111)

88. A small manufacturing company will start operating a night shift. There are 20 machinists employed by the company.

a. If a night crew consists of 3 machinists, how many different crews are possible?

b. If the machinists are ranked 1, 2, . . . , 20 in order of competence, how many of these crews would not have the best machinist?

c. How many of the crews would have at least 1 of the 10 best machinists?

d. If one of these crews is selected at random to work on a particular night, what is the probability that the best machinist will not work that night?

89. A factory uses three production lines to manufacture cans of a certain type. The accompanying table gives percentages of nonconforming cans, categorized by type of nonconformance, for each of the three lines during a particular time period.

	Line 1	Line 2	Line 3
Blemish	15	12	20
Crack	50	44	40
Pull-Tab Problem	21	28	24
Surface Defect	10	8	15
Other	4	8	2

During this period, line 1 produced 500 nonconforming cans, line 2 produced 400 such cans, and line 3 was responsible for 600 nonconforming cans. Suppose that one of these 1500 cans is randomly selected.

a. What is the probability that the can was produced by line 1? That the reason for nonconformance is a crack?

b. If the selected can came from line 1, what is the probability that it had a blemish?

c. Given that the selected can had a surface defect, what is the probability that it came from line 1?

90. An employee of the records office at a certain university currently has ten forms on his desk awaiting processing. Six of these are withdrawal petitions and the other four are course substitution requests.

a. If he randomly selects six of these forms to give to a subordinate, what is the probability that only one of the two types of forms remains on his desk?

b. Suppose he has time to process only four of these forms before leaving for the day. If these four are randomly selected one by one, what is the probability that each succeeding form is of a different type from its predecessor?

91. One satellite is scheduled to be launched from Cape Canaveral in Florida, and another launching is scheduled for Vandenberg Air Force Base in California. Let A denote the event that the Vandenberg launch goes off on schedule, and let B represent the event that the Cape Canaveral launch goes off on schedule. If A and B are independent events with $P(A) > P(B)$ and $P(A \cup B) = .626$, $P(A \cap B) = .144$, determine the values of $P(A)$ and $P(B)$.

92. A transmitter is sending a message by using a binary code, namely, a sequence of 0's and 1's. Each transmitted bit (0 or 1) must pass through three relays to reach the receiver. At each relay, the probability is .20 that the bit sent will be different from the bit received (a reversal). Assume that the relays operate independently of one another.

Transmitter $\longrightarrow$ Relay 1 $\longrightarrow$ Relay 2 $\longrightarrow$ Relay 3 $\longrightarrow$ Receiver

a. If a 1 is sent from the transmitter, what is the probability that a 1 is sent by all three relays?

b. If a 1 is sent from the transmitter, what is the probability that a 1 is received by the receiver? (*Hint:* The eight experimental outcomes can be displayed on a tree diagram with three generations of branches, one generation for each relay.)

c. Suppose 70% of all bits sent from the transmitter are 1's. If a 1 is received by the receiver, what is the probability that a 1 was sent?

93. Individual A has a circle of five close friends (B, C, D, E, and F). A has heard a certain rumor from outside the circle and has invited the five friends to a party to circulate the rumor. To begin, A selects one of the five at random and tells the rumor to the chosen individual. That individual then selects at random one of the four remaining individuals and repeats the rumor. Continuing, a new individual is selected from those not already having heard the rumor by the individual who has just heard it, until everyone has been told.

a. What is the probability that the rumor is repeated in the order B, C, D, E, and F?

b. What is the probability that F is the third person at the party to be told the rumor?

c. What is the probability that F is the last person to hear the rumor?

94. Refer to Exercise 93. If at each stage the person who currently "has" the rumor does not know who has already heard it and selects the next recipient at random from all five possible individuals, what is the probability that F has still not heard the rumor after it has been told ten times at the party?

95. A chemical engineer is interested in determining whether a certain trace impurity is present in a product. An experiment has a probability of .80 of detecting the impurity if it is present. The probability of not detecting the impurity if it is absent is .90. The prior probabilities of the impurity being present and being absent are .40 and .60, respectively. Three separate experiments result in only two detections. What is the posterior probability that the impurity is present?

96. Each contestant on a quiz show is asked to specify one of six possible categories from which questions will be asked. Suppose $P(\text{contestant requests category } i) = \frac{1}{6}$ and successive contestants choose their categories independently of one another. If there are three contestants on each show and all three contestants on a particular show select different categories, what is the probability that exactly one has selected category 1?

97. Fasteners used in aircraft manufacturing are slightly crimped so that they lock enough to avoid loosening during vibration. Suppose that 95% of all fasteners pass an initial inspection. Of the 5% that fail, 20% are so seriously defective that they must be scrapped. The remaining fasteners are sent to a recrimping operation, where 40% cannot be salvaged and are discarded. The other 60% of these fasteners are corrected by the recrimping process and subsequently pass inspection.

a. What is the probability that a randomly selected incoming fastener will pass inspection either initially or after recrimping?

b. Given that a fastener passed inspection, what is the probability that it passed the initial inspection and did not need recrimping?

98. One percent of all individuals in a certain population are carriers of a particular disease. A diagnostic test for this disease has a 90% detection rate for carriers and a 5% detection rate for noncarriers. Suppose the test is applied independently to two different blood samples from the same randomly selected individual.

a. What is the probability that both tests yield the same result?

b. If both tests are positive, what is the probability that the selected individual is a carrier?

99. A system consists of two components. The probability that the second component functions in a satisfactory manner during its design life is .9, the probability that at least one of the two components does so is .96, and the probability that both components do so is .75. Given that the first component functions in a satisfactory manner throughout its design life, what is the probability that the second one does also?

100. A certain company sends 40% of its overnight mail parcels via express mail service E_1. Of these parcels, 2% arrive after the guaranteed delivery time (denote the event "late delivery" by L). If a record of an overnight mailing is randomly selected from the company's file, what is the probability that the parcel went via E_1 and was late?

101. Refer to Exercise 100. Suppose that 50% of the overnight parcels are sent via express mail service E_2 and the remaining 10% are sent via E_3. Of those sent via E_2, only 1% arrive late, whereas 5% of the parcels handled by E_3 arrive late.

a. What is the probability that a randomly selected parcel arrived late?

b. If a randomly selected parcel has arrived on time, what is the probability that it was not sent via E_1?

102. A company uses three different assembly lines—A_1, A_2, and A_3—to manufacture a particular component. Of those manufactured by line A_1, 5% need rework to remedy a defect, whereas 8% of A_2's components need rework and 10% of A_3's need rework. Suppose that 50% of all components are produced by line A_1, 30% are produced by line A_2, and 20% come from line A_3. If a randomly selected component needs rework, what is the probability that it came from line A_1? From line A_2? From line A_3?

103. Disregarding the possibility of a February 29 birthday, suppose a randomly selected individual is equally likely to have been born on any one of the other 365 days.

a. If ten people are randomly selected, what is the probability that all have different birthdays? That at least two have the same birthday?

b. With k replacing ten in part (a), what is the smallest k for which there is at least a 50–50 chance that two or more people will have the same birthday?

c. If ten people are randomly selected, what is the probability that either at least two have the same birthday or at least two have the same last three digits of their Social Security numbers? [*Note:* The article "Methods for Studying Coincidences" (F. Mosteller and P. Diaconis, *J. Amer. Stat. Assoc.*, 1989: 853–861) discusses problems of this type.]

104. One method used to distinguish between granitic (G) and basaltic (B) rocks is to examine a portion of the infrared spectrum of the sun's energy reflected from the rock surface. Let R_1, R_2, and R_3 denote measured spectrum intensities at three different wavelengths; typically, for granite $R_1 < R_2 < R_3$, whereas for basalt $R_3 < R_1 < R_2$. When measurements are made remotely (using aircraft), various orderings of the R_i's may arise whether the rock is basalt or granite. Flights over regions of known composition have yielded the following information:

	Granite	Basalt
$R_1 < R_2 < R_3$	60%	10%
$R_1 < R_3 < R_2$	25%	20%
$R_3 < R_1 < R_2$	15%	70%

Suppose that for a randomly selected rock in a certain region, $P(\text{granite}) = .25$ and $P(\text{basalt}) = .75$.

a. Show that $P(\text{granite} \mid R_1 < R_2 < R_3) > P(\text{basalt} \mid R_1 < R_2 < R_3)$. If measurements yielded $R_1 <$

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$R_2 < R_3$, would you classify the rock as granite or basalt?

b. If measurements yielded $R_1 < R_3 < R_2$, how would you classify the rock? Answer the same question for $R_3 < R_1 < R_2$.

c. Using the classification rules indicated in parts (a) and (b), when selecting a rock from this region, what is the probability of an erroneous classification? [*Hint:* Either G could be classified as B or B as G, and $P(B)$ and $P(G)$ are known.]

d. If $P(\text{granite}) = p$ rather than .25, are there values of p (other than 1) for which one would always classify a rock as granite?

105. A subject is allowed a sequence of glimpses to detect a target. Let G_i = {the target is detected on the ith glimpse}, with $p_i = P(G_i)$. Suppose the G_i's are independent events and write an expression for the probability that the target has been detected by the end of the nth glimpse. (*Note:* This model is discussed in "Predicting Aircraft Detectability," *Human Factors*, 1979: 277–291.)

106. In a Little League baseball game, team A's pitcher throws a strike 50% of the time and a ball 50% of the time, successive pitches are independent of one another, and the pitcher never hits a batter. Knowing this, team B's manager has instructed the first batter not to swing at anything. Calculate the probability that

a. The batter walks on the fourth pitch

b. The batter walks on the sixth pitch (so two of the first five must be strikes), using a counting argument or constructing a tree diagram

c. The batter walks

d. The first batter up scores while no one is out (assuming that each batter pursues a no-swing strategy)

107. Four engineers, A, B, C, and D, have been scheduled for job interviews at 10 A.M. on Friday, January 13, at Random Sampling, Inc. The personnel manager has scheduled the four for interview rooms 1, 2, 3, and 4, respectively. However, the manager's secretary does not know this, so assigns them to the four rooms in a completely random fashion (what else!). What is the probability that

a. All four end up in the correct rooms?

b. None of the four ends up in the correct room?

108. A particular airline has 10 A.M. flights from Chicago to New York, Atlanta, and Los Angeles. Let A denote the event that the New York flight is full and define events B and C analogously for the other two flights. Suppose $P(A) = .6$, $P(B) = .5$, $P(C) = .4$ and the three events are independent. What is the probability that

a. All three flights are full? That at least one flight is not full?

b. Only the New York flight is full? That exactly one of the three flights is full?

109. A personnel manager is to interview four candidates for a job. These are ranked 1, 2, 3, and 4 in order of preference and will be interviewed in random order. However, at the conclusion of each interview, the manager will know only how the current candidate compares to those previously interviewed. For example, the interview order 3, 4, 1, 2 generates no information after the first interview, shows that the second candidate is worse than the first, and that the third is better than the first two. However, the order 3, 4, 2, 1 would generate the same information after each of the first three interviews. The manager wants to hire the best candidate but must make an irrevocable hire/no hire decision after each interview. Consider the following strategy: Automatically reject the first s candidates and then hire the first subsequent candidate who is best among those already interviewed (if no such candidate appears, the last one interviewed is hired).

For example, with $s = 2$, the order 3, 4, 1, 2 would result in the best being hired, whereas the order 3, 1, 2, 4 would not. Of the four possible s values (0, 1, 2, and 3), which one maximizes $P(\text{best is hired})$? (*Hint:* Write out the 24 equally likely interview orderings: $s = 0$ means that the first candidate is automatically hired.)

110. Consider four independent events A_1, A_2, A_3, and A_4 and let $p_i = P(A_i)$ for $i = 1, 2, 3, 4$. Express the probability that at least one of these four events occurs in terms of the p_i's, and do the same for the probability that at least two of the events occur.

111. A box contains the following four slips of paper, each having exactly the same dimensions: (1) win prize 1; (2) win prize 2; (3) win prize 3; (4) win prizes 1, 2, and 3. One slip will be randomly selected. Let A_1 = {win prize 1}, A_2 = {win prize 2}, and A_3 = {win prize 3}. Show that A_1 and A_2 are independent, that A_1 and A_3 are independent, and that A_2 and A_3 are also independent (this is *pairwise* independence). However, show that $P(A_1 \cap A_2 \cap A_3) \neq P(A_1) \cdot P(A_2) \cdot P(A_3)$, so the three events are *not* mutually independent.

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