Analysis of a Non-Linear System of Ordinary Differential Equations
Research Supplement – MA205 Project III

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This research supplement describes a method of analyzing a two-dimensional system of *nonlinear* ordinary differential equations (ODEs).

Symbiosis is an ecological term that describes a close relationship between two or more species in which the outcome for each is highly dependent on the other. Consider the following model of an ecological system in symbiosis

$$x'(t) = x\ \ (2 - 2x + y)$$
$$y'(t) = y\ \ (5 - 2y + x)$$

where $x(t)$ and $y(t)$ are the populations at some time level t for two species co-existing in some ecosystem. For this particular model, the two species are related by mutual stereotypic behaviors and their interaction is *facultative* – the relationship between the two species is useful, but not vital for each species' survival. We would like to determine the ***long-term behavior*** of this system, i.e. find the values x and y at which the populations of the two species will stabilize in the long run. If this stable population level exists, it will occur at one of the ***fixed points*** (also known as ***equilibrium points***) of the system.

The fixed points for the system occur at the points (x^*, y^*) where $x'(t)$ and $y'(t)$ both equal zero. Using the Solve command in *Mathematica*, we determine that the fixed points for the system are $(x^*, y^*) = (0,\ 0),\ \ (0,\ 2.5),\ \ (1,\ 0),$ and $(3,\ 4)$:

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In[1]:= Solve[{x * (2 - 2 * x + y) == 0, y * (5 - 2 * y + x) == 0}, {x, y}] // N
Out[1]= {{x → 0., y → 0.}, {x → 0., y → 2.5}, {x → 1., y → 0.}, {x → 3., y → 4.}}
```

Now that we have the fixed points, we must determine their stability. Recall from our study of one-dimensional linear and nonlinear ODEs that a solution trajectory exists in one of two states: it is either at a fixed point and stays there; or, if it is not at a fixed point, it is migrating towards a stable fixed point (and, by extension, away from an unstable fixed point) as t increases. So, our hope for this particular system is that one of the four fixed points is stable; if it is, then we have determined the long-term behavior of the system.

In order to classify the stability of each fixed point, we must look at the ***eigenvalues*** of the ***Jacobian matrix*** for the system evaluated at each fixed point. Although this terminology is formidable, these are simple constructs that in turn lead to an easy classification method as provided by a theorem to be presented later in this supplement. If we let $x'(t) = f(x, y)$ and $y'(t) = g(x, y)$, then the Jacobian matrix for this system is the 2 by 2 matrix

$$J(x,\ \ y) = \begin{bmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{bmatrix}$$

Although the eigenvalues of the Jacobian matrix evaluated at each fixed point can be calculated by hand, the technique requires the introduction of linear algebra concepts that are unnecessarily complex for the purposes of this supplement. As such, we appeal to *Mathematica*:

```
In[1]:= f[x_, y_] = x * (2 - 2 * x + y);
        g[x_, y_] = y * (5 - 2 * y + x);
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$$\text{In[3]:= } J[x\_, y\_] = \begin{pmatrix} \partial_x f[x, y] & \partial_y f[x, y] \\ \partial_x g[x, y] & \partial_y g[x, y] \end{pmatrix}$$

```
Out[3]= {{2 - 4 x + y, x}, {y, 5 + x - 4 y}}

In[4]:= Eigenvalues[J[0, 0]]

Out[4]= {5, 2}

In[5]:= Eigenvalues[J[0, 2.5]]

Out[5]= {-5., 4.5}

In[6]:= Eigenvalues[J[1, 0]]

Out[6]= {6, -2}

In[7]:= Eigenvalues[J[3, 4]] // N

Out[7]= {-10.6056, -3.39445}
```

So, we see from our *Mathematica* results that the Jacobian matrix (***Output 3***) for our system is

$$J(x,\ \ y) = \begin{bmatrix} 2-4x+y & x \\ y & 5+x-4y \end{bmatrix}$$

and the eigenvalues (from ***Outputs 4-7***) corresponding to each fixed point are

Fixed Point	***Eigenvalues***
(0, 0)	$\lambda_1 = 5,\ \ \lambda_2 = 2$
(0, 2.5)	$\lambda_1 = -5,\ \ \lambda_2 = 4.5$
(1, 0)	$\lambda_1 = 6,\ \ \lambda_2 = -2$
(3, 4)	$\lambda_1 = -10.6056,\ \ \lambda_2 = -3.39445$

Now, on to the classification theorem we referenced earlier. This theorem establishes that the stability of each fixed point is predicated on the sign (+ or -) of the eigenvalues:

<u>Theorem</u>. Let (x^*, y^*) be a fixed point of the first order system of differential equations

$$x'(t) = f(x, y)$$
$$y'(t) = g(x, y)$$

a. If each eigenvalue of the Jacobian matrix $J(x^*, y^*)$ of the system is negative or has a negative real part (in the case of complex conjugate eigenvalues), then (x^*, y^*) is an asymptotically stable steady state of the system.

b. If $J(x^*, y^*)$ has at least one positive real eigenvalue or complex conjugate eigenvalues with a positive real part, then (x^*, y^*) is an unstable steady state of the system.

So, applying this theorem to our fixed points and associated eigenvalues, we have the following results:

Fixed Point	***Eigenvalues***	***Stability***
(0, 0)	$\lambda_1 = 5, \ \lambda_2 = 2$	Unstable
(0, 2.5)	$\lambda_1 = -5, \ \lambda_2 = 4.5$	Unstable
(1, 0)	$\lambda_1 = 6, \ \lambda_2 = -2$	Unstable
(3, 4)	$\lambda_1 = -10.6056, \ \lambda_2 = -3.39445$	Asymptotically Stable

Based on these results, let us investigate the behavior of the system in the vicinity of each of the fixed points, thereby completing our analysis.

At $(0, \ 0)$, the Jacobian has eigenvalues 5 and 2. Since both eigenvalues are positive, $(0, \ 0)$ is an unstable equilibrium. Ecologically speaking, when the population of both species is at or near $(0, \ 0)$, the addition of a small population will allow both species to grow rapidly, especially when each species is sufficiently small that they are not in competition with each other for shared resources.

At $(0, \ 2.5)$, the Jacobian has eigenvalues -5 and 4.5. Since one of these eigenvalues is positive, $(0, \ 2.5)$ is an unstable equilibrium. The carrying capacity of our system for species y without the benefit provided by the presence of species x is 2.5. If we add a small population of species x, however, both species will flourish and stabilize at a higher population level.

At $(1, \ 0)$, the Jacobian has eigenvalues 6 and -2. Since one of these eigenvalues is positive, $(1, \ 0)$ is an unstable equilibrium. The carrying capacity of our system for species x without the benefit provided by the presence of species y is 1. Similar to the situation we described above, if we add a small population of species y, both species will flourish and stabilize at a higher population level.

At the co-existence equilibrium $(3,\ 4)$, the Jacobian has eigenvalues -10.6056 and -3.39445,

Since both eigenvalues are negative, $(3,\ 4)$ is an asymptotically stable equilibrium. In the environment of mutual benefit provided by the presence of both species, species x stabilizes at a population level of 3 and species y stabilizes at a population level of 4.

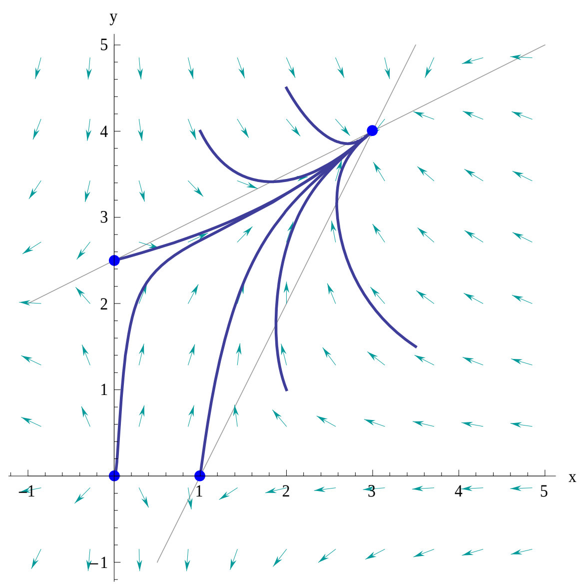
So, in the long run for our system, the populations of both species will stabilize at levels provided by the co-existence equilibrium as long as both species are initially present in the ecosystem, even in miniscule amounts. Note that this result is entirely consistent with our description of symbiosis for species in a shared ecosystem – both species had lower population levels when the complement species was not present.

To lend further weight to our analysis, let us examine the behavior of this system graphically. Our mechanism for doing this is the ***phase portrait*** for our system. The phase portrait is simply the higher-dimensional corollary to the direction field for a single ODE. We use the phase portrait to show the stability of the co-existence equilibrium $(3,\ 4)$:

```
In[4]:= << DiffEqs`DEGraphics`

PhasePlot[{x * (2 - 2 * x + y), y * (5 - 2 * y + x)}, {t, 0, 10}, {x, -1, 5}, {y, -1, 5}, ShowNullclines → True,
 ShowEquilibriumPoints → True, AspectRatio → Automatic, InitialPoints →
  {{0.01, 0.01}, {2, 1}, {1, 4}, {1, .01}, {.01, 2.5}, {3.5, 1.5}, {2, 4.5}}, AxesLabel → {"x", "y"}]
```

Out[8]=



We restrict our attention to the first quadrant in the phase space since there is no such thing as negative populations for our two species x and y. The phase portrait shows qualitatively that the co-existence equilibrium $(3,\ 4)$ is stable; solution trajectories for any initial conditions excepting the unstable equilibria $(0,\ 0)$, $(0,\ 2.5)$, and $(1,\ 0)$ "flow" into the co-existence equilibrium as t increases.