MA383 Foundations in Mathematics
Lesson 23: Induction Practice Problems

Prove the following statements by induction.

1. Easy

   (a) For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

   (b) $\sum_{i=0}^{n-1} (2i+1) = n^2$ for all $n \in \mathbb{N}$.

   (First, be sure you understand why this is the same as proving that the sum of the first n odd positive integers is n^2.)

2. Medium

   (a) $3^n < n!$ for all integers $n \geq 7$.

   (b) $\sum_{i=0}^{n} 3 \cdot 5^i = \frac{3(5^{n+1} - 1)}{4}$ for all $n \in \mathbb{N}$.

   (c) For all integers $n \geq 1$, $3 | (2^{2n} - 1)$.

   (d) $\sum_{i=1}^{n+1} i2^i = n2^{n+2} + 2$ for all integers $n \geq 0$.

3. Difficult

   (a) $2^n > n^2$ whenever n is an integer greater than 4. (Hint: We proved an exercise using induction in class that you can/should use within your proof)

   (b) $n! < n^n$ for all integers $n \geq 2$.

1 A) Thm $\forall\, n \in \mathbb{N}$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Proof: BC: Let $n=1$. Then $\sum_{i=0}^{1} 2^i = 2^0 + 2^1 = 3 = 2^{1+1} - 1 = 3$. So the base case holds

IH: Assume for some $k \in \mathbb{N}$, $\sum_{i=0}^{k} 2^k = 2^{k+1} - 1$.

Show: $\sum_{i=0}^{k+1} 2^{k+1} = 2^{k+2} - 1$

So $\sum_{i=0}^{k+1} 2^{k+1} = \sum_{i=0}^{k} 2^k + 2^{(k+1)}$ by induction hypothesis.

$$
\begin{aligned}
\text{Then } \sum_{i=0}^{k+1} 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} \\
&= (2^{k+1})2 - 1 \\
&= 2^{k+2} - 1.
\end{aligned}
$$

Thus, by induction our hypothesis holds $\forall\, n \in \mathbb{N}$. □ ✓

1 B) THM: $\sum_{i=0}^{n-1} (2i+1) = n^2 \quad \forall\, n \in \mathbb{N}$.

Proof: BC: Let $n=1$. So $\sum_{i=0}^{1-1} (2i+1) = 2(0)+1 = 1 = (1)^2$. So our base case holds.

IH: Assume for some $k \in \mathbb{N}$, $\sum_{i=0}^{k-1} (2i+1) = k^2$.

Show: $\sum_{i=0}^{k} (2i+1) = (k+1)^2$

So $\sum_{i=0}^{k} (2i+1) = \sum_{i=0}^{k-1} (2i+1) + 2k + 1$ by our IH.

$$
\begin{aligned}
\text{Then } \sum_{i=0}^{k} (2i+1) &= k^2 + 2k + 1 \\
&= (k+1)^2.
\end{aligned}
$$

Thus, by induction our hypothesis holds $\forall\, n \in \mathbb{N}$. □

2a

Thm: for all integers $n \geq 7$, $3^n < n!$

Proof, by Induction:

BASE CASE, $n=7$

$3^7 = 2187$ $\quad 7! = 5040$

$2187 < 5040$

INDUCTION HYP.:

We Assume $3^k < k!$ for some $k \in \mathbb{N}$ where $k \geq 7$

We will show $3^{k+1} < (k+1)!$

$3^{k+1} = 3 \cdot 3^k < 3k!$ $\quad$, $k+1 > 3$, because $k \geq 7$

$< (k+1)!$ $\quad$ $(k+1)k! > 3k!$

$(k+1)! > 3k!$

$\therefore 3^{k+1} < (k+1)!$

$\therefore$ for all integers $n \geq 7$ $3^n < n!$ by induction.

good.

2 B) THM: $\sum_{i=0}^{n} 3\cdot 5^i = \frac{3(5^{n+1}-1)}{4}$ for all $n \in \mathbb{N}$.

Proof: BC: Let $n=1$. Then $\sum_{i=0}^{1} 3\cdot 5^i = 3(5^0)+3(5^1) = 18 = \frac{3(5^2-1)}{4}$. So our base case holds.

IH: Assume for some $k \in \mathbb{N}$, $\sum_{i=0}^{k} 3\cdot 5^i = \frac{3(5^{k+1}-1)}{4}$.

Show: $\sum_{i=0}^{k+1} 3\cdot 5^i = \frac{3(5^{k+2}-1)}{4}$

So $\sum_{i=0}^{k+1} 3\cdot 5^i = \sum_{i=0}^{k} 3\cdot 5^i + 3\cdot 5^{(k+1)}$ by our IH.

Then $\sum_{i=0}^{k+1} 3\cdot 5^i = \frac{3(5^{k+1}-1)}{4} + 3\cdot 5^{(k+1)} = \frac{3\cdot 5^{k+1} - 3 + 12\cdot 5^{k+1}}{4} = \frac{15\cdot 5^{k+1} - 3}{4}$

$= \frac{3(5)(5^{k+1}) - 3}{4} = \frac{3(5^{k+2}-1)}{4}$.

Thus, our theorem holds by induction for all $n \in \mathbb{N}$. $\square$

2c) THM: $\forall$ integers $n \geq 1$, $3 \mid (2^{2n} - 1)$.

Proof: BC: Let $n=1$. Then $2^2 - 1 = 3$ and it is true that $3 \mid 3$. So our base case holds

IH: Assume for some integer $k \geq 1$, that $3 \mid (2^{2k} - 1)$.

Show: $3 \mid (2^{2k+2} - 1)$.

So $2^{2k} - 1 = 3q$ where $q \in \mathbb{Z}$ by our IH.

Then $2^{2k} = 3q + 1$.

So $2^{2k+2} - 1 = 2^{2k}(4) - 1 = \boxed{(3q+1)4} - 1 = 12q + 4 - 1 = 12q + 3 = 3(4q+1)$.

Thus, because $2^{k+2} - 1$ is divisible by 3, our theorem holds for all integers $k \geq 1$. $\square$

2d

Thrm: for all integers $n \geq 0$ $\sum_{i=1}^{\cdots} i2^i = n2^{n+2}+2$.

Proof, by Induction:

For the statement to be true the base case, $n=0$ must at least be true

$$\sum_{i=1}^{0+1} i2^i = 1 \cdot 2^1 = 2 \qquad \cancel{0(2)^{n+2}}^{\,0} + 2 = 2$$

the base case is true, so all that must be proved is the Induction Hypothesis.

That is that

We Assume $\sum_{i=0}^{K+1} i2^i = K2^{K+2}+2$ (the Induction Hypothesis)

and must show that $\sum_{i=0}^{K+2} i2^i = (K+1)2^{K+3}+2$

observe then that

$$\sum_{i=0}^{K+2} i2^i = \sum_{i=0}^{K+1} i2^i + (K+2)2^{K+2}$$

$$= K2^{K+2} + 2 + (K+2)2^{K+2}$$ by the Induction Hypothesis

$$= 2 \cdot 2^{K+2} \cdot K + 2 \cdot 2^{K+2} + 2$$

$$= K \cdot 2^{K+3} + 2^{K+3} + 2$$

$$= 2^{K+3}(K+1) + 2$$

$$\therefore \sum_{i=0}^{K+2} i2^i = (K+1)2^{K+3} + 2$$

✓ good.

$\therefore$ $\forall$ integers $n \geq 0$ $\sum_{i=1}^{n+1} i2^i = n2^{n+2}+2$, by induction.

3a) THM: $\forall$ integers $n > 4$, $2^n > n^2$.

Proof: BC: Let $n = 5$. Then $2^5 = 32 > 5^2 = 25$. So our base case holds.

IH: Assume that for some integer $k > 4$, $2^k > k^2$.

Show: $2^{k+1} > (k+1)^2$

So $2^{k+1} = 2^k \cdot 2$
$> 2 \cdot k^2$. by our IH. ✓

From our theorem proved in class on 17 OCT during lesson 21, we know that $k^2 > 2k+1$. for all $k \geq 3$ in $\mathbb{N}$.

So $(k+1)^2 = k^2 + 2k + 1$
$< k^2 + k^2$
$< 2k^2$.

Thus, $2^{k+1} > (k+1)^2$ for all integers $k > 4$.

So our theorem holds. $\square$

✓ good

2 3 a

Prove: $2^n > n^2, \forall n \in \mathbb{Z}, n \geq 5$

Base Case: Let $n = 5$, then:

$$2^5 = 32 \text{ and } 5^2 = 25$$

Since $32 > 25$, then the base case is true. ✓

Inductive Hypothesis: Suppose $2^k > k^2$, where $k \in \mathbb{Z}, k \geq 5$. If this is true, then the $k+1$ case must also be true. Hence the objective is

$$2^{k+1} > (k+1)^2$$

Using the Inductive Hypothesis yields:

$$2^k > (k)^2$$

Using a previously established inequality:

$$\begin{gathered} k^2 > 2k+1, \text{ thus} \\ 2^k > 2k+1 \\ 2^k + 2^k > k^2 + 2k + 1 \\ 2(2^k) > (k+1)^2 \\ 2^{k+1} > (k+1)^2 \end{gathered}$$

✓ good.

Thus, the Theorem is true by induction.$_\varphi$

3b

Thrm: For all integers $n \geq 2$, $n! < n^n$

Proof, by Induction:

Base Case, $n=2$

$2! = 2 \qquad 2^2 = 4$

$2 < 4$ ∴ The base case is satisfied ✓

Induction Hypothesis

We assume $k! < k^k$ for some integer $k \geq 2$

We will show $(k+1)! < (k+1)^{k+1}$

let it be seen $k < k+1$

$\therefore k^k < (k+1)^k$ since $k \geq 2$

then

$(k+1)! = (k+1)k! < (k+1)k^k$ by the induction hypothesis

$< (k+1)(k+1)^k$

$= (k+1)^{k+1}$

$\therefore (k+1)! < (k+1)^{k+1}$

and the induction hypothesis is true

good.

∴ For all integers $n \geq 2$, $n! > n^n$, by induction.