RESEARCH STATEMENT

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1. Introduction

My area of research is number theory. In particular, I study hypergeometric functions over finite fields. I am interested in linking values of these functions to two other number theoretic objects, elliptic curves and modular forms. More recently, I have become interested in relationships between these hypergeometric functions and values of the p-adic gamma function, and how these connections can shed light on possible p-adic analogues of some classical identities. In this section, I will briefly introduce the main objects, and in the next section, I will give formal definitions, together with some recent history. Then, in Section 3, I will state some of my main results and briefly mention the ideas behind their proofs. I will conclude by discussing a current problem of interest, as well as some possible routes for future study.

Classical hypergeometric functions have been studied for centuries and enjoy many beautiful symmetries and transformation identities (see, e.g. [21]). In the 1980s, Greene [9] introduced a finite field analogue of such functions, built up out of character sums. He showed that these new functions also satisfy many transformations, in a completely analogous way to their classical counterparts. These *hypergeometric functions over* $\mathbb{F}_p$, often referred to as *Gaussian hypergeometric functions*, are the focus of my research.

Modular forms are most easily viewed as holomorphic functions on the complex upper half plane, which act in a nice way under various collections of transformations. The study of such functions and their properties encompasses a rich theory which includes work of classical mathematicians such as Poincaré, Hecke, and Ramanujan, and yet remains an active field of research still today, in number theory and other areas of mathematics.

Elliptic curves can be described as curves of genus 1, given by a cubic equation in two variables, together with a distinguished point, the *point at infinity.* These curves enjoy the special property of a group law, and they have relevance both to classical problems such as the congruent number problem (see [13]) and to current questions in cryptography, algebraic geometry, and more. Elliptic curves and modular forms have many known connections, perhaps most famously those brought to light in the proof of Fermat's Last Theorem.

The main results given in Section 3 provide explicit relationships between these classes of objects. Theorem 1 expresses values of a hypergeometric function in terms of the number of points on an elliptic curve over $\mathbb{F}_p$, while Theorems 2 and 3 provide formulas for the traces of Hecke operators on spaces of cusp forms in level 1. Perhaps most interestingly, Theorems 1 and 2 combine in a special case to give a formula for Ramanujan's τ-function $\tau(p)$ in terms of hypergeometric functions, in Corollary 3.

2. Recent History

Classical hypergeometric series have been studied by mathematicians such as Euler, Vandermonde, and Kummer. An important example of these series is defined for $a, b, c \in \mathbb{C}$ as

$$ {}_2F_1[a,b;c;z] = {}_2F_1\left(\begin{matrix}a, & b\\ & c\end{matrix}\middle| z\right) := \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, $$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$.

In 1836, Kummer showed that the above series satisfies a well known second order differential equation. The specialization ${}_2F_1[\frac{1}{2},\frac{1}{2};1;t]$ has further interesting properties, as it is closely related to elliptic curves. In fact, it is a multiple of an elliptic integral which represents a period of the lattice associated to the Legendre family of elliptic curves $y^2 = x(x-1)(x-t)$.

Connections between classical hypergeometric functions, modular forms, and elliptic curves have been investigated since the early 1900s. More recently, Stiller, Beukers, and others have continued to discover new relationships. In [22], Stiller constructed an isomorphism between the graded algebra generated by classical Eisenstein series E_4 and E_6 and one generated by powers and multiples of hypergeometric series of the form ${}_2F_1[\frac{1}{12},\frac{5}{12};\frac{1}{2};t]$. Soon afterwards, Beukers [5] gave identifications between periods of families of elliptic curves and values of particular hypergeometric series. For example, he related a period of $y^2 = x^2 - x - t$ to the values ${}_2F_1[\frac{1}{12},\frac{5}{12};\frac{1}{2};\frac{27}{4}t^2]$.

Meanwhile, as these and other similar results were being considered, Greene [9] was developing the theory of hypergeometric functions over finite fields. Let p be an odd prime, and $\mathbb{F}_p$ denote the field of p elements. Let $\widehat{\mathbb{F}_p^\times}$ denote the group of multiplicative characters on $\mathbb{F}_p^\times$, and extend such characters χ to all of $\mathbb{F}_p$ by setting $\chi(0)=0$. If $A, B \in \widehat{\mathbb{F}_p^\times}$ and J denotes the Jacobi sum, then define

$$ \binom{A}{B} := \frac{B(-1)}{p} J(A,\overline{B}) = \frac{B(-1)}{p} \sum_{x\in\mathbb{F}_p} A(x)\overline{B}(1-x). $$

Greene defined *hypergeometric functions over* $\mathbb{F}_p$ in the following way:

Definition 1. *If n is a positive integer, $x \in \mathbb{F}_p$, and $A_0, A_1, \ldots, A_n, B_1, B_2, \ldots, B_n \in \widehat{\mathbb{F}_p^\times}$, then define*

$$ {}_{n+1}F_n\left(\begin{matrix}A_0, & A_1, & \ldots, & A_n\\ & B_1, & \ldots, & B_n\end{matrix}\middle| x\right) := \frac{p}{p-1}\sum_{\chi}\binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x). $$

Greene explored the properties of the above series and showed that it satisfies many transformations analogous to those satisfied by its classical counterpart. The development of these new objects generated interest in finding connections they may have with modular forms and elliptic curves. One type of hypergeometric function over $\mathbb{F}_p$, in which all the A_i are the quadratic character (or Legendre symbol) and all the B_i are the trivial character, seemed to generate particular interest.

We will be most interested in the case when $n=1$. For ease of notation, we will write

$$ {}_2F_1[A,B;C;x] := {}_2F_1\left(\begin{matrix}A, & B\\ & C\end{matrix}\middle| x\right). $$

Let ϕ and ε denote the unique quadratic and trivial characters, respectively, on $\mathbb{F}_p$. Further, define two families of elliptic curves as follows:

$$ {}_2E_1(t) : y^2 = x(x-1)(x-t) $$
$$ {}_3E_2(t) : y^2 = (x-1)(x^2+t). $$

Then, for odd primes p and $t \in \mathbb{F}_p$, define the traces of Frobenius on the above families by

$$\begin{aligned} {}_2A_1(p,t) &= p+1-\#{}_2E_1(t)(\mathbb{F}_p),\ t \neq 0,1 \\ {}_3A_2(p,t) &= p+1-\#{}_3E_2(t)(\mathbb{F}_p),\ t \neq 0,-1. \end{aligned}$$

These families of elliptic curves turn out to be closely related to particular hypergeometric functions over $\mathbb{F}_p$. For example, ${}_2F_1[\phi,\phi;\varepsilon;t]$ arises in the formula for Fourier coefficients of a modular form associated to ${}_2E_1(t)$ ([14, 17]). Further, Koike and Ono, respectively, gave the following explicit relationships:

Theorem ((1) Koike [14], (2) Ono [17])**.** *Let p be an odd prime. Then*

(1) $p\,{}_2F_1[\phi,\phi;\varepsilon;t] = -\phi(-1){}_2A_1(p,t),\ t \neq 0,1$

(2) $p^2\,{}_3F_2\left(\begin{matrix}\phi, & \phi, & \phi \\ & \varepsilon, & \varepsilon\end{matrix}\middle| 1+\frac{1}{t}\right) = \phi(-t)({}_3A_2(p,t)^2 - p),\ t \neq 0,-1.$

Soon after, Ahlgren and Ono [3] and Ahlgren [2] exhibited formulas for the traces of Hecke operators on spaces of cusp forms in levels 8 and 4. To state their results, we require some more notation. For a positive integer N, let $\Gamma_0(N)$ denote the congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$ defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \middle| c \equiv 0 \pmod{N} \right\}.$$

Let $S_k(\Gamma_0(N))$ denote the vector space of cusp forms of weight k on $\Gamma_0(N)$. Finally, let $\mathrm{Tr}_k(\Gamma_0(N),p)$ denote the trace of the Hecke operator $\mathrm{T}(p)$ on $S_k(\Gamma_0(N))$. Additionally, we define polynomials

$$G_k(s,p) = \sum_{j=0}^{k/2-1} (-1)^j \binom{k-2-j}{j} p^j s^{k-2j-2}.$$

Theorem ((1) Ahlgren and Ono [3], (2) Ahlgren [2])**.** *Let p be an odd prime and $k \geq 4$ be an even integer. Then*

(1) $Tr_k(\Gamma_0(8),p) = -4 - \sum_{t=2}^{p-2} G_k({}_2A_1(p,t^2),p)$

(2) $Tr_k(\Gamma_0(4),p) = -3 - \sum_{t=2}^{p-1} G_k({}_2A_1(p,t),p).$

The methods for proving the above theorem involved combining the Eichler-Selberg trace formula with a theorem given by Schoof. More recently, Frechette, Ono, and Papanikolas expanded these techniques and gave a similar formula in the level 2 case:

Theorem (Frechette, Ono, and Papanikolas [6])**.** *Let p be an odd prime and $k \geq 4$ be even. When $p \equiv 1(4)$, write $p = a^2+b^2$, where a,b are nonnegative integers, with a odd. Then*

$$Tr_k(\Gamma_0(2),p) = -2 - \delta_k(p) - \sum_{t=1}^{p-2} G_k({}_3A_2(p,t),p),$$

where

$$\delta_k(p) = \begin{cases} \frac{1}{2}G_k(2a,p) + \frac{1}{2}G_k(2b,p) & \textit{if } p \equiv 1 \pmod 4 \\ (-p)^{k/2-1} & \textit{if } p \equiv 3 \pmod 4). \end{cases}$$

In addition, Frechette, Ono, and Papanikolas used relationships between counting points on varieties over $\mathbb{F}_p$ and hypergeometric functions over $\mathbb{F}_p$ to obtain further results for the traces of Hecke operators on spaces of newforms in level 8. More recently, Papanikolas [18] used the above theorem as a starting point to obtain a new formula for Ramanujan's τ function, as well as a new congruence for $\tau(p)$ modulo 11.

3. Statement of Main Results

As in the results given in Section 2 above, we connect values of a particular hypergeometric function over $\mathbb{F}_p$ to a family of elliptic curves and to traces of Hecke operators on a space of modular forms. We now set up the notation necessary to state our results. Throughout, let p be a prime such that $p \equiv 1(12)$. For $t \neq 0, 1$, let

$$E_t : y^2 = 4x^3 - \frac{27}{1-t}x - \frac{27}{1-t}.$$

Notice that E_t is a family of elliptic curves having j-invariant $\frac{1728}{t}$. Further, define

$$a(t,p) = p + 1 - \#E_t(\mathbb{F}_p)$$

to be the trace of the Frobenius endomorphism on E_t. Let $\xi \in \widehat{\mathbb{F}_p^\times}$ be a character of order 12. As in Section 2, we let ϕ denote the quadratic character (the Legendre symbol) on $\mathbb{F}_p$.

Theorem 1 (Fuselier [7])**.** *Let $p \equiv 1 \pmod{12}$ be prime. If $t \neq 0, 1$ and notation is as above, then*

$$p\,{}_2F_1[\xi, \xi^5; \varepsilon; t] = \psi(t)a(t,p),$$

where $\psi(t) = -\phi(2)\xi^{-3}(1-t)$.

Notice that as ξ has order 12, this specialization to the hypergeometric function ${}_2F_1[\xi, \xi^5; \varepsilon; t]$ bears a resemblance to the classical specialization ${}_2F_1[\frac{1}{12}, \frac{5}{12}; \frac{1}{2}; t]$ that appears in the results of Stiller and Beukers, mentioned in Section 2. The significance of this result, which becomes important in the proof of Theorem 2 below, lies in the choice of E_t. Since this family has j-invariant $\frac{1728}{t}$ and isomorphic elliptic curves over $\mathbb{F}_p$ have the same j-invariant, Theorem 1 actually gives a relationship between *every* elliptic curve having $j \neq 0, 1728$ and the value of a hypergeometric function over $\mathbb{F}_p$.

The proof of Theorem 1 requires many manipulations of character sums, together with relationships to Gauss sums. In particular, it makes repeated use of the Hasse-Davenport theorem (see [15]), which relates products of Gauss sums of characters having a specified order. The proof also makes use of some of the properties of hypergeometric functions over $\mathbb{F}_p$ that Greene proved in [9].

We now move toward the statement of the second main theorem, in which we relate traces of Hecke operators to $a(t,p)$. We denote p and $a(t,p)$ exactly as above, and we use the notation for the congruence subgroups $\Gamma_0(N)$, as defined in Section 2. For our purposes, we will focus on the full modular group $\Gamma_0(1) = \Gamma = SL_2(\mathbb{Z})$. The polynomials $G_k(s,p)$ are also defined as in Section 2.

Additionally, pick integers a, b such that $p = a^2 + b^2$ and $a + bi \equiv 1\,(2+2i)$ in $\mathbb{Z}[i]$. Also, choose integers c, d such that $p = c^2 - cd + d^2$ and $c + d\omega \equiv 2\,(3)$ in $\mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3}$.

Theorem 2 (Fuselier [7]). *Let a, b, c, and d be chosen as above. For primes $p \equiv 1 \pmod{12}$ and even $k \geq 4$,*

$$Tr_k(\Gamma, p) = -1 - \lambda(k,p) - \sum_{t=2}^{p-1} G_k(a(t,p),p),$$

where

$$\lambda(k,p) = \frac{1}{2}[G_k(2a,p) + G_k(2b,p)] + \frac{1}{3}[G_k(c+d,p) + G_k(2c-d,p) + G_k(c-2d,p)].$$

The proof of Theorem 2 relies heavily on Hijikata's version of the Eichler-Selberg trace formula [11], which we simplify to suit our scenario. Additionally, we make use of a theorem of Schoof [19], which gives a way of counting isomorphism classes of elliptic curves in terms of orders of imaginary quadratic fields.

Theorem 2 gives rise to an inductive formula for the traces $\mathrm{Tr}_k(\Gamma, p)$, in terms of hypergeometric functions. To state it, we utilize the notation for $G_k(s,p)$ and $\lambda(k,p)$ given in Theorem 2.

Theorem 3 (Fuselier [8]). *Suppose $p \equiv 1 \pmod{12}$ is prime. Let $k \geq 4$ be even and define $m = \frac{k}{2} - 1$. Then*

$$\begin{aligned}\mathrm{Tr}_{2(m+1)}(\Gamma, p) = -1 - \lambda(2m+2,p) + b_0(p-2) - \sum_{t=2}^{p-1} p^{2m}\phi^m(1-t){}_2F_1\left(\begin{matrix}\xi, & \xi^5 \\ & \varepsilon\end{matrix}\middle| t\right)^{2m} \\ - \sum_{i=1}^{m-1} b_i(1+\lambda(2i+2,p)) - \sum_{i=1}^{m-1} b_i \mathrm{Tr}_{2i+2}(\Gamma,p),\end{aligned}$$

where

$$b_i = p^{m-i}\left[\binom{2m}{m-i} - \binom{2m}{m-i-1}\right].$$

Some of the nicest results come by taking $k = 12$ and combining Theorems 1 and 2. In doing so, we obtain formulas for Ramanujan's τ function, since $S_{12}(\Gamma)$ is one dimensional and hence $\mathrm{Tr}_{12}(\Gamma, p) = \tau(p)$. For example, we can express $\tau(p)$ in terms of tenth powers of our hypergeometric function:

Corollary 4 (Fuselier [7]). *Let $p \equiv 1 \pmod{12}$ be prime and let a, b, c, and d be defined as above. Let ξ be an element of order 12 in $\widehat{\mathbb{F}_p^\times}$. Then*

$$\begin{aligned}\tau(p) = 42p^6 - 90p^4 - 75p^3 - 35p^2 - 9p - 1 - 2^9(a^{10} + b^{10}) \\ - \frac{1}{3}\left((c+d)^{10} + (2c-d)^{10} + (c-2d)^{10}\right) - \sum_{t=2}^{p-1} p^{10}\phi(1-t){}_2F_1\left(\begin{matrix}\xi, & \xi^5 \\ & \varepsilon\end{matrix}\middle| t\right)^{10}.\end{aligned}$$

4. Future Work

Recently, I have become interested in how values of Gaussian hypergeometric functions relate to sums of particular binomial coefficients, and how p-adic analysis can be used in conjunction with these formulas to generalize some classical results. We let ${}_{n+1}F_n(t)$ denote the evaluation of the Gaussian hypergeometric function for which all $A_i = \phi$ and all $B_j = \varepsilon$.

In 2001, Ahlgren produced a combinatorial formula ([1], Theorem 1) for the hypergeometric function ${}_3F_2(t)$ in terms of sums of various binomial coefficients. He proved this by using the

Gross-Koblitz formula [10] to write the hypergeometric series in terms of the p-adic gamma function. Ahlgren then used this formulation to give a new proof of a conjecture of Beukers given in [4], namely that

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \equiv \begin{cases} 0 \pmod{p^2} & if\ p \equiv 3 \pmod 4, \\ 4a^2 - 2p \pmod{p^2} & if\ p = a^2 + b^2\ and\ a\ is\ odd. \end{cases}$$

More recently, McCarthy and Osburn [16] used hypergeometric functions over $\mathbb{F}_p$ to prove supercongruences which serve as a p-adic analogue of a theorem of Ramanujan:

Theorem ([16], Theorem 1.2)**.** *If p is an odd prime, then*

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1) \binom{-\frac{1}{2}}{k}^5 \equiv \begin{cases} -\frac{p}{\Gamma_p(\frac{3}{4})^4} \pmod{p^3} & if\ p \equiv 1 \pmod 4 \\ 0 \pmod{p^3} & if\ p \equiv 3 \pmod 4 \end{cases}$$

where Γ_p is the p-adic gamma function.

With these results in mind, I am currently re-investigating the hypergeometric function ${}_2F_1[\xi, \xi^5; \varepsilon; t]$ from Theorem 1. I hope to see how writing this function in terms of sums of binomial coefficients, together with p-adic analysis, might produce new supercongruences for high powers of p. In particular, I am interested in how generalizing to characters other than ε and ϕ affects the analysis.

In addition to this ongoing project, there are multiple other avenues for future study:

- Produce analogues of Theorems 1 and 2 in the case when p fails to be congruent to 1 modulo 12.
- Specify the techniques from Theorem 1 to particular families of elliptic curves, e.g. those arising from modular curves.
- Uncover relationships between hypergeometric functions over $\mathbb{F}_p$ and classical Eisenstein series, as Stiller [22] did for the classical case.

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