Exact Results on Determining the Contribution of a New Component Added in Parallel to a Parallel System

Bobbie Leon Foote, Ph.D.
Department of Systems Engineering
United States Military Academy
West Point, New York
bobfoote50@hotmail.com

LTC Andrew G. Glen, Ph.D.
Department of Mathematical Sciences
United States Military Academy
West Point, New York
aa1275@usma.edu

January 14, 2002

Abstract

Exact distributions for the marginal gain in reliability of adding k components in parallel to n components in parallel under the exponential assumption are computed. Expected values and variances are derived and the independence of the coefficient of variation from λ, the exponential parameter, is shown. The implications of these results for system designers is highlighted. Much of the work demonstrates the uses of A Probability Programing Language (APPL) software in complex convolution calculations. Mathematical induction is used to prove expected value formulas and generalize them.

Keywords: Computational algebra systems; Reliability Block Diagrams; Exact Distributions; Order Statistics; coefficent of variance.

1 Summary of Contents and Conclusions

Few exact distributions are known for the reliability of series and parallel systems. In this paper we show how using APPL, A Probability Programming Language, allows for the computation of exact formulas for the reliability distributions. We analyze the distributions of $Z = max(X_i, i = 1, 2, \ldots, n)$ and $Y = max(X_i, i = 1, 2, \ldots, n + k)$ representing the reliability distribution of parallel systems of n components and $n+k$ components respectively, and $W_{n,k} = Y - Z$, the distribution of the marginal gain obtained by adding the extra k components. We will abreviate $W_{n,k}$ as W, when n and k are obvious. Furthermore, we obtain exact formulas for the gain, $W_{n,k}$. Obtaining the distribution function allows for computing percentiles, probabilities, moments, random variate generation, expected values, variances and so forth, a great improvement over simply finding results based on μ and σ^2, as was often done in the past. The computations give rise to theorems on the exact form of the expected value of $W_{n,k}$, its variance, and coefficient of variation. The expected value of $W_{n,k}$ in terms of n and k turns out to have a tractable form that lends itself to mathematical modeling. Further, we show that the coefficient of variation of $W_{n,k}$ is independent of λ, which has implications about a type of risk associated with increased variance in the components. The increase in $CV(W_{n,k})$ with respect to n is "flat" for k small and shows that "risk" associated with increased parallelism is low. The results show explicitly why increasing the variance of components in parallel improves reliability. The availability of APPL shows that some reliability calculations can now be solved precisely that were formerly solved by approximation or simulation.

2 Literature Review and Motivation

The use of reliability block diagrams (RBDs) to model systems that have parallel and series components is standard. Many texts explain how the diagrams are derived (e.g., Leemis 1995). When used to model systems, it is rare that exact reliability distributions are calculated, primarily due to the intractability of the distribution functions. Kuo and Prasad state that for redundancy allocation problems, "little work is directed toward exact solutions for such problems" (Kuo and Prasad, 2000). The difficulty in finding exact solutions for RBDs lies in solving for the distribution functions, which are equivalent to using the random variable algebraic functions Minimum and Maximum. Often engineers had to settle for finding characteristics of the parallel systems, such as μ, without knowing the entire distribution, as is the case with *The Reliability Engineering Handbook* (Kececiogliu, 1991, pg. 25) who present the mean of $Y = Max(X_i)$, where $X_i \sim Exponential(\lambda)$. With the advent of A Probability Programming Language (APPL) (Glen, Leemis, Evans 2001) it is possible, among other things, to find exact distribution functions for RBDs and then find other interesting characteristics of the distributions, e.g. the mean, variance, and coefficient of variation for such systems. Since the ability to find exact reliability distributions now exists, we can also consider the exact distribution of the marginal increase in reliability of adding k extra components to the system. Using these results, or knowledge that an exact formula exists, we can then work to derive results in a simplified form.

The utility of an exact formula for the distribution of marginal gain of new components is apparent, especially when considering the obvious cost and space constraints. Space vehicles, probes and shuttles have limited capacity. Additional astronauts can be viewed as redundant components to a space shuttle system, but they are subject to supply constraints, space constraints, and fatigue failure. Back up computers have limited space for placement in airplanes and even additional servers must compete for space in office buildings. If there is an explicit law of diminishing returns for redundancy, then it is possible to construct design tradeoffs between increased reliability and constraints on space. A review of the literature revealed that there was a gap in finding general exact formulas for distribution, mean, variance and coefficient of variation for parallel systems, as well as for the difference between two such systems.

3 Problem Formulation

Systems with parallel components can be modeled with RBDs effectively. Figure 1 shows a three component parallel system with the dashed lines indicating the proposal of adding a fourth component in parallel. Let Z represent the reliability distribution of the system with

3 components and Y represent the proposed system with 4 components. The distribution of the marginal gain of the new system can be modeled by $W_{3,1} = Y - Z$. Using APPL, (Glen, et al, 2001), we can determine the exact distribution of Z, Y, and W. Consider the simple case where each component has lifetime distributed as $X_i \sim Exponential(\lambda)$. Then $Z = max(X_i,\ i = 1, 2, 3)$ and $Y = max(X_i,\ i = 1, 2, 3, 4)$ represents the reliability distributions of parallel systems of 3 components and 4 components respectively, and $W_{3,1} = Y - Z$, represents the distribution of the marginal gain obtained by adding the extra component. The software program APPL computes, among other things, distributions that result from random variable algebra. So the needed distributions are found with the APPL commands:

```
> X:= ExponentialRV(lambda);
> Y:= OrderStat(X, 4, 4);
> Z:= OrderStat(X, 3, 3);
> W:= Difference(Y,Z);
```

In this case, W is determined to have PDF as follows:

$$f(w) = \begin{cases} \frac{1}{35}\lambda\left(3\,e^{(3\lambda-1)w} - 14\,e^{(2\lambda-1)w} + 21\,e^{(\lambda-1)w}\right)e^{w} & w < 0 \\ \frac{1}{35}\lambda\left(35\,e^{3\lambda w} - 42\,e^{2\lambda w} + 21\,e^{\lambda w} - 4\right)e^{-4\lambda w} & 0 \le w \end{cases}.$$

A plot of this pdf, setting $\lambda = \frac{1}{10}$, is found in Figure 2. Note the two-piece nature of the pdf that breaks at zero. Its negative domain represents the probability that the improved system may still have a lower lifetime, even with the added component. Its expected value $E(W_{3,1}) > 0$ is evident and we will show that the expected value goes to zero as n increases, an intuitive result.

For the general case where $Z = max(X_i, i = 1, 2, \ldots, n)$ and $Y = max(X_i, i = 1, 2, \ldots, n + k)$, and $W_{n,k}$ is the distribution of adding k more components in parallel to a system of n components in parallel, the similar lines of APPL code yield the following general solution for the PDF of $W_{n,k}$:

$$f(w) = \begin{cases} -\frac{\lambda n e^{\lambda w}}{n+k+1}\texttt{hypergeom}\left([2, -n+1], [n+k+2], e^{\lambda w}\right) & w < 0 \\ \\ \int_{e^{w}}^{\infty}(n+k)\left(-x^{-\lambda}+1\right)^{n+k-1}\lambda^2 x^{-\lambda-1} n & \\ \times\left(-e^{\lambda w}\left(x^{-1}\right)^{\lambda}+1\right)^{n-1} e^{(\lambda-1)w}\left(x^{-1}\right)^{\lambda} dx e^{w} & 0 \le w \end{cases}.$$

Plots of $E(W)$, $V(W)$, $CV(W)$ as n increased led to the general theorem statements which are proved in this paper. We will investigate the properties of W, to include $E(W)$, $V(W)$, and $CV(W)$. We cover in detail the case of $k = 1$, the gain of adding one additional component and then generalize using induction. We also give interpretations of our results for any n that may help designers of parallel systems decide what number of components to put in a system.

4 Theorems and Corollaries

4.1 Theorem 1

If a component is added in parallel to a system of n components (each distributed Exponentially with parameter $\lambda > 0$) in parallel the expected gain is equal to $\frac{1}{\lambda(n+1)}$.
Proof: Let $Z = max(X_i,\ i = 1, 2, \ldots, n-1)$, $Y = max(X_i,\ i = 1, 2, \ldots, n)$ and $W_{n,1} = Y - Z$, where X_i are iid exponential random variables with parameter $\lambda > 0$. Due to independence, $F(Y) = \prod_{i=1}^{n}(1 - e^{-\lambda y})$. This product is a polynomial of the form

$$F(Y) = 1 + a_1 e^{-\lambda y} + a_2 e^{-2\lambda y} + \ldots + a_n e^{-n\lambda y}$$

where the a_j are the coefficients of the binomial expansion. So the pdf of Y has the form

$$\frac{d}{dy}F(Y) = f(y)dy = -\lambda(a_1 e^{-\lambda y} + 2a_2 e^{-2\lambda y} + \ldots + na_n e^{-n\lambda y})dy$$

and

$$\begin{aligned} E(Y) &= -\sum_{j=1}^{n} E(ja_j \lambda e^{-\lambda y} \times y) \\ &= -\sum_{j=1}^{n} ja_j (\frac{1}{j\lambda}) \\ &= -\frac{1}{\lambda}\sum_{j=1}^{n}\frac{a_j}{j}. \end{aligned}$$

Similarly

$$E(Z) = -\frac{1}{\lambda}\sum_{j=1}^{n-1}\frac{b_j}{j}.$$

Therefore,

$$E(W_{n,1}) = E(Y - Z) = -\frac{1}{\lambda}\Big(\sum_{j=1}^{n}\frac{a_j}{j} - \sum_{j=1}^{n-1}\frac{b_j}{j}\Big).$$

Renotating by making explicit the binomial expansion coefficients in terms of the parameter n, the number of factors of type $(1-x)$, we define $C_{n,j}$ as the coefficient of the j^{th} term in the expansion of $(1-x)^n$ with $C_{n,0} = 1$. For example, $C_{2,0} = 1,\ C_{2,1} = -2,\ C_{2,2} = (-1)^2$. Using this new notation, and factoring out $\frac{1}{\lambda}$, we now have

$$\sum_{j=1}^{n}\frac{C_{n,j}}{j} - \sum_{j=1}^{n-1}\frac{C_{n-1,j}}{j}.$$

The following recursion defines the binomial coefficients in terms of their predecessors:

$$C_{n,j} = -C_{n-1,j-1} + C_{n-1,j}.$$

For our induction proof, the case of $n = 3$ is easily verified and omitted. We then assume that for the case of $n-1$, the following is true:

$$\frac{1}{\lambda}(\sum_{j=1}^{n}\frac{C_{n,j}}{j} - \sum_{j=1}^{n-1}\frac{C_{n-1,j}}{j}) = \frac{-1}{\lambda n}.$$

We will now show that for the case of n, $E(W) = \frac{1}{\lambda(n+1)}$. Consider the following for $n+1$:

$$\sum_{j=1}^{n+1}\frac{C_{n+1,j}}{j} - \sum_{j=1}^{n}\frac{C_{n,j}}{j} = \frac{(-1)^{n+1}}{n+1} + \sum_{j=1}^{n}\frac{C_{n+1,j}}{j} - \sum_{j=1}^{n}\frac{C_{n,j}}{j}.$$

Substitute the induction hypothesis:

$$\begin{aligned} &= \frac{(-1)^{n+1}}{n+1} + \sum_{j=1}^{n}\frac{C_{n+1,j}}{j} - \Big(-\frac{1}{n} + \sum_{j=1}^{n-1}\frac{C_{n-1,j}}{j}\Big) \\ &= \frac{(-1)^{n+1}}{n+1} + \frac{1}{n} + \sum_{j=1}^{n}\frac{C_{n+1,j}}{j} - \sum_{j=1}^{n-1}\frac{C_{n-1,j}}{j}. \end{aligned}$$

Substitute the recursion relation:

$$= \frac{(-1)^{n+1}}{n+1} + \frac{1}{n} + \sum_{j=1}^{n}\frac{(-C_{n,j-1} + C_{n,j})}{j} - \sum_{j=1}^{n-1}\frac{C_{n-1,j}}{j}.$$

Now use the induction hypothesis:

$$
\begin{aligned}
&= \frac{(-1)^{n+1}}{n+1} + \frac{1}{n} + \sum_{j=1}^{n} \frac{-C_{n,j-1}}{j} - \frac{1}{n} \\
&= \frac{(-1)^{n+1}}{n+1} + \sum_{j=1}^{n} \frac{-C_{n,j-1}}{j}
\end{aligned}
$$

which reduces to

$$\frac{(-1)^{n+1}}{n+1} + \frac{1+(-1)^n}{n+1} = -\frac{1}{n+1}$$

and the result follows when the $\frac{-1}{\lambda}$ is multiplied back into the term.

$\square$

4.1.1 Corollary 1

$$\lim_{n\to\infty} E(W_{n,1}) = 0$$

Proof: Clearly, $\lim_{n\to\infty} \frac{1}{\lambda(n+1)} = 0$ and the result follows.

$\square$

4.1.2 Corollary 2

$E(Y)$ approaches $E(Z)$ from above.

Proof:

$$\text{Since } \lim_{n\to\infty} E(W_{n,1}) = 0, \Rightarrow E(Y) \to E(Z) \text{ as } n \to \infty$$

$$\text{also, } E(Y) > E(Z) \text{ since } E(Y) - E(Z) = \frac{1}{\lambda(n+1)}.$$

Thus $E(Y)$ approaches $E(Z)$ from above.

$\square$

4.1.3 Corollary 3

$E(W_{n,k}) = \frac{1}{\lambda}\Big(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+k}\Big)$

Proof: Since the induction hypothesis has been proven, the marginal gain is $\frac{1}{\lambda(n+1)}$ for any n and $k = 1$. As n goes from $n \to n + k - 1$, the gains are $\frac{1}{\lambda(n+1)}, \frac{1}{\lambda(n+2)}, \ldots, \frac{1}{\lambda(n+k)}$. Summing gives the result.

□

4.2 Theorem 2

$V(W_{n,1})$ is a function of $\frac{1}{\lambda^2}$

Proof:

From the first theorem, we can see that $E(Y^2) = -\frac{2}{\lambda^2}\sum_{j=1}^{n}\frac{a_j}{j^2}$ and we know that

$$\begin{aligned}
V(W) &= V(Y) + V(Z) \\
&= E(Y^2) + E(Z^2) - E(Y)^2 - E(Z)^2 \\
&= \frac{-2}{\lambda^2}\Big(\sum_{j=1}^{n}\frac{a_j}{j^2} + \sum_{j=1}^{n-1}\frac{b_j}{j^2}\Big) - (E(Y))^2 - (E(Z))^2 \\
&= \frac{1}{\lambda^2}\Big[-2\Big(\sum_{j=1}^{n}\frac{a_j}{j^2} + \sum_{j=1}^{n-1}\frac{b_j}{j^2}\Big) - \Big(\sum_{j=1}^{n}\frac{a_j}{j} + \sum_{j=1}^{n-1}\frac{b_j}{j}\Big)\Big] \qquad (1) \\
&= \frac{-1}{\lambda^2}[f(n, a_j, b_j)]
\end{aligned}$$

□

4.2.1 Corollary 1

The coefficient of variation of $W_{n,1}$ is independent of λ.

Proof:

$$CV(W) = \frac{\sigma_W}{\mu_W} = \frac{\sqrt{\frac{-1}{\lambda^2}[f(n, a_j, b_j)]}}{\frac{1}{\lambda(n+1)}} = (n+1)\sqrt{-f(n, a_j, b_j)}$$

□

4.3 Theorem 3

$E(Y^2) \to E(Z^2)$ as $n \to \infty$ and $V(Y) \to V(Z)$as $n \to \infty$, implying $V(W_{n,1}) = 2V(Y)$ in the limit.

Proof:

$$E(Y^2) - E(Z^2) \propto \Big(\sum_{j=1}^{n} \frac{a_j}{j^2} - \sum_{j=1}^{n-1} \frac{b_j}{j^2}\Big) = f.$$

Using the terminology and notation from Appendix A and simplification from Mathmatica,

$$\begin{aligned} f &= \frac{1}{2}\Big[2\gamma\texttt{Polygamma}(0,n) + \texttt{Polygamma}(0,n)^2 - 2\gamma\texttt{Polygamma}(0,n+1) \\ &\quad -\texttt{Polygamma}(0,n+1)^2 - \texttt{Polygamma}(1,n) + \texttt{Polygamma}(1,n+1)\Big]. \end{aligned}$$

Terms 1 and 3 can be written as $\psi(n) - \psi(n+1) = \sum_{i=1}^{n-1} \frac{1}{n-1} - \sum_{i=1}^{n-1} \frac{1}{n+1-i} = -\frac{1}{n} \to 0$ as $n \to \infty$. Similarly, terms 5 and 6 approach zero as do terms 2 and 4. Thus the following holds true as $n \to \infty$:

$$\begin{aligned} f &\to 0 \\ &\Rightarrow E(Y^2) \to E(Z^2) \\ &\Rightarrow V(Y) \to V(Z) \\ &\Rightarrow V(W) \to 2V(Y) \end{aligned}$$

□

4.4 Theorem 4

$V(Y)$ diverges as $n \to \infty$.

Proof: Using Mathmatica, equation (1) simplifies to

$$\begin{aligned} &\frac{1}{12}\Big[-18\gamma^2 - \pi^2 - 36\gamma\texttt{Polygamma}(0,n+1) \\ &-18\texttt{Polygamma}(0,n+1)^2 + 6\texttt{Polygamma}(1,n+1)\Big], \end{aligned} \tag{2}$$

remembering that this equation is multiplied by -$\frac{1}{\lambda^2}$ to represent $V(Y)$. From the Appendix, $\texttt{Polygamma}(0,n+1) = \sum_{i=1}^{n} \frac{1}{n+1-i}$, and $\texttt{Polygamma}(1,n+1) = \sum_{i=1}^{n} \frac{-1}{(n+1-i)^2}$.

The first sum is equivalent to $\sum_{k=1}^{n} \frac{1}{k}$, which diverges (Fulks, 1978). The second sum is equivalent to $\sum_{k=1}^{n} \frac{1}{k^2}$ which converges to $\frac{\pi^2}{6}$ in the limit. The two divergent series have the same sign and equation (2) is divergent.

$\square$

4.5 Theorem 5

$CV(W_{n,1})$ increases as a concave function and as $n \to \infty$ the $CV(W_{n,1})$ is of quadratic order.

Proof: From equation (2), the term $\texttt{Polygamma}(0, n+1)^2$ dominates $\texttt{Polygamma}(0, n+1)$ for large n and also the constant terms. Hence for large n,
$V(W) \propto \Big(\sum_{k=1}^{n} \frac{1}{k}\Big)^2$ and $CV(W) \propto \sum_{k=1}^{n} \frac{1}{k} / \frac{1}{n}$. This ratio clearly diverges and is $n\Big(\sum_{k=1}^{n} \frac{1}{k}\Big)$. The derivative is $n\,\texttt{Polygamma}(1, n+1) + \sum_{k=1}^{n} \frac{1}{k}$. The increase in the first term dominates the increase in the second term as $n \to n+1$. Hence the coefficient of variation is concave and of quadratic order, since if $f = an^2$, $\frac{df}{dn}$ is of order $2an$ or linear in n. Plots of $CV(W_{n,k})$ show that $(n, CV(W_{n,k}))$ is flat for $3 \leq n \leq 10$, see Figure 3.

$\square$

4.6 Theorem 6

Increases in reliability of W depend on increases in $\frac{1}{\lambda}$.

Proof: The marginal gain of adding each new component is $E(W_{n,1}) = \frac{1}{\lambda(n+1)}$. As $\frac{1}{\lambda}$ increases, marginal gain increases.

$\square$

5 Conclusions

These general results allow the following conclusions and design procedures to be executed.

• The ability to compute exact distributions of Y, Z, and W is of great benefit over what has been customary for complex systems, i.e., concentrating on characteristics such as μ, σ, and so forth. Now that APPL allows for calculating exact distributions, the following characteristics can be found:

- Exact Percentiles of the Y and W can be calculated, assisting statistical procedures in warranty design.

- The CDF can be computed, allowing exact probabilities and p-values to be calculated for statistical tests.

- The inverse CDF can be found or solved for numerically, useful in variate generation for simulation of systems.

- The marginal distribution of the order statistics of Y and W can be computed, which could be useful in model verification and outlier detection.

- All these functions are currently possible in the APPL environment.

• If Δ is the smallest marginal gain worthwhile due to cost and other constraints, then n is determined by $\dfrac{1}{\lambda(n+1)} = \Delta$, useful in optimization scenarios.

• Theorem 2 Corollary 1 shows reducing λ will not reduce or increase $CV(W)$. CV is a measure of a type of risk. In this special (though frequently occuring) case, we do not increase risk by increasing component variation in reliability.

• Trade-off models of reliability gain versus cost, volume, and weight constraints can now be made explicit.

• Increasing $\frac{1}{\lambda}$ increases variance for a given n which increases reliability.

• The concavity of $CV(W_{n,1})$ as a function of n implies as n increases the left hand tail of the pdf of W decreases in probability and the right hand tail increases in probability.

• The flat concave nature of the increase in $CV(W_{n,1})$ implies the series nature of the reliability loss due to adding a component representing the reliability of the physical connections of the parallel circuit need to be explored by further research.

• Since CV is a measure of a type of risk, the flatness of $CV(W_{n,k})$ shows that this risk does not increase as we increase parallelism, see Figure 3.

• We are not limited to the Exponential distribution for components, nor are we limited to strictly parallel systems:

- A $W_{3,1}$ gain distribution when all components are distributed as Weibull$(1/2, 2)$ has been computed, a much more complex distibution because the convolution $Y - Z$ is a difficult computation with a Weibull random variable. The PDF is a two page-long formula (omitted for brevity, but available from the second author). The PDF is shown in Figure 4.

- A System with one exponential(λ) component in series with two parallel Weibull(α, β) components has pdf:

$$f(x) = 2\, e^{-\lambda\, x - \alpha^{\beta} x^{\beta}} \lambda + 2\, e^{-\lambda\, x - \alpha^{\beta} x^{\beta}} \alpha^{\beta} x^{\beta - 1} \beta - e^{-\lambda\, x - 2\, \alpha^{\beta} x^{\beta}} \lambda - 2\, e^{-\lambda\, x - 2\, \alpha^{\beta} x^{\beta}} \alpha^{\beta} x^{\beta - 1} \beta$$

for $0 < x$ and $\lambda,\ \alpha,\ \beta > 0$.

- We are currently researching these more complex cases.

6 Appendix

The following functions describe the complexity of variance calculations for the case of a parallel system with the components having the memoryless property. The series involving the binomial coefficients divided by j and j^2 often simplify (using Maple and Mathmatica) to variations of the ψ function, $\psi(n) = \dfrac{d}{dn}(\ln \Gamma(n))$. For n integer, $\psi(n) = \sum_{i=1}^{n-1} \dfrac{1}{n-i} = \sum_{k=1}^{n-1} \dfrac{1}{k}$, the harmonic series. Also, let $\texttt{Polygamma}(c, n) = \dfrac{d^c}{dn^c}(\psi(n))$. Hence, for n integer,

$$\texttt{Polygamma}(1, n) = \frac{d}{dn}\Big(\sum_{i=1}^{n-1} \frac{1}{n-i} \Big) = -\sum_{i=1}^{n-1} \frac{1}{(n-i)^2}.$$

Finally, the constant $\gamma \doteq 0.577216$ (called **EulerGamma** in Mathmatica) appears in some of the simplification.

References

Fulks, W. (1978), *Advanced Calculus, Third Edition*, John Wiley and Sons .

Glen, A., L. Leemis, and D. Evans (2001), "APPL: A Probability Programming Language," *The American Statistician*, **55**, Number 2, 156–166.

Kececioglu, D., (1991), *Reliability Engineering Handbook,* Vols. 1 and 2, Prentice–Hall.

Kuo, W. and V. R. Prasad (2000), "An Annotated Overview of System-Reliability Optimization," *IEEE Transactions on Reliability*, **49**, Number 2. 176–187.

Leemis, L. M. (1995), *Reliability: Probabilistic Models and Statistical Methods*, Prentice–Hall.

Figures

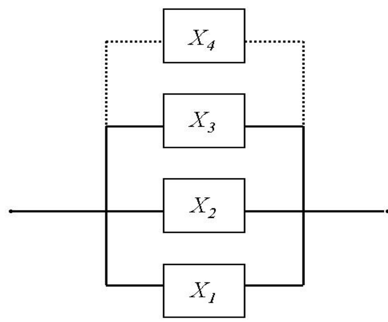

Figure 1: Reliability Block Diagram of 3 components with an optional fourth component addition being considered.

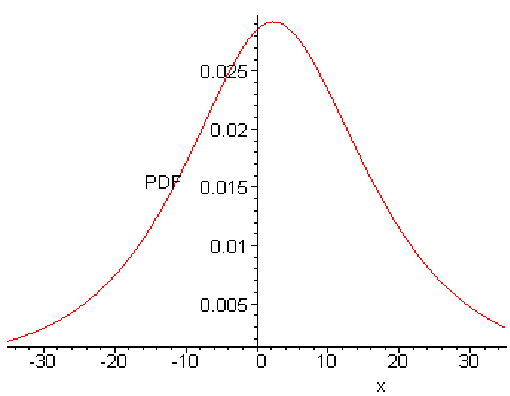

Figure 2: The PDF of $W_{3,1}$, the increase of the system from 3 to 4 components.

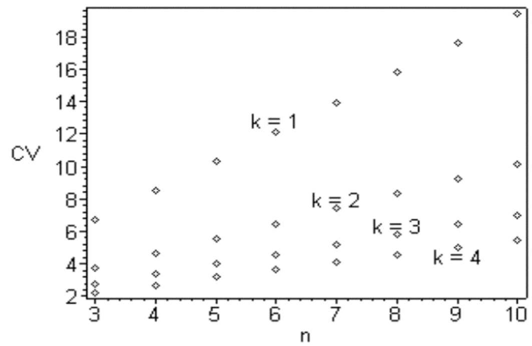

Figure 3: The CV of $W_{n,k}$, as $n = 3, \ldots, 10$ and $k = 1, \ldots, 4$, when the underlying components are exponentially distributed for any λ. Note the flatness of CV even though it is concave for $k = 1$ and convex for $k = 2, 3, 4$.

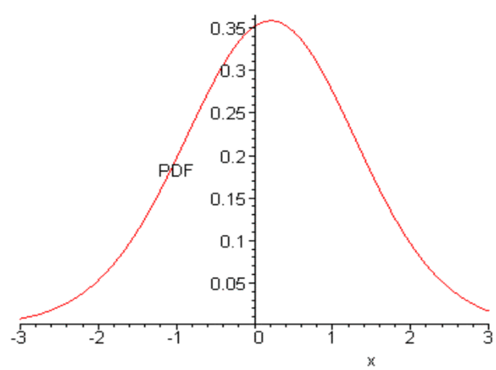

Figure 4: The pdf of $W_{3,1}$ with each component having a reliability distribution of Weibull(1/2, 2).