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On Halin subgraphs and supergraphs

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Abstract

In this paper, we present two complexity results. The first pertains to the problem of finding Halin subgraphs while the second is a supergraph version which asks if a given graph is a subgraph of any Halin graph. Both of these problems are shown to be hard which, in turn, provides somewhat damaging evidence relative to the veracity of heuristic approaches employing Halin graphs as approximations.

1. Introduction

Let $G = (V, E)$ be a finite planar graph with the property that the edge set, E, can be partitioned into a tree, T, no vertex of which has degree two, and a cycle, C, on the degree-1 or pendant vertices of T. Structures having this property are referred to as *Halin graphs* and were introduced in [8] in the context of minimal k-connected graphs. Indeed, Halin graphs yield an example of a class of edge minimal, planar 3-connected graphs. The graph in Fig. 1 is Halin.

But Halin graphs are also interesting from an algorithmic perspective. Most recent in this regard, they have been shown by several authors (i.e., [3, 13]) to be members of a so-called 3-terminal recursive class. Moreover, as shown in Borie et al. [4], this is enough to guarantee linear time algorithms for many otherwise hard problems when instances are confined to Halin graphs.

In fact, it is precisely this problem solving richness of Halin graphs that has led us, as well as others, to raise the possibility of their employment in a broader context; namely as a device useful in heuristic problem solving. For example, Cornuejols et al. [5] solved the traveling salesman problem (TSP) on Halin graphs and then suggested a heuristic strategy for the general problem as one whereby a "low cost Halin subgraph" is (heuristically) found upon which the TSP is solved optimally. Our

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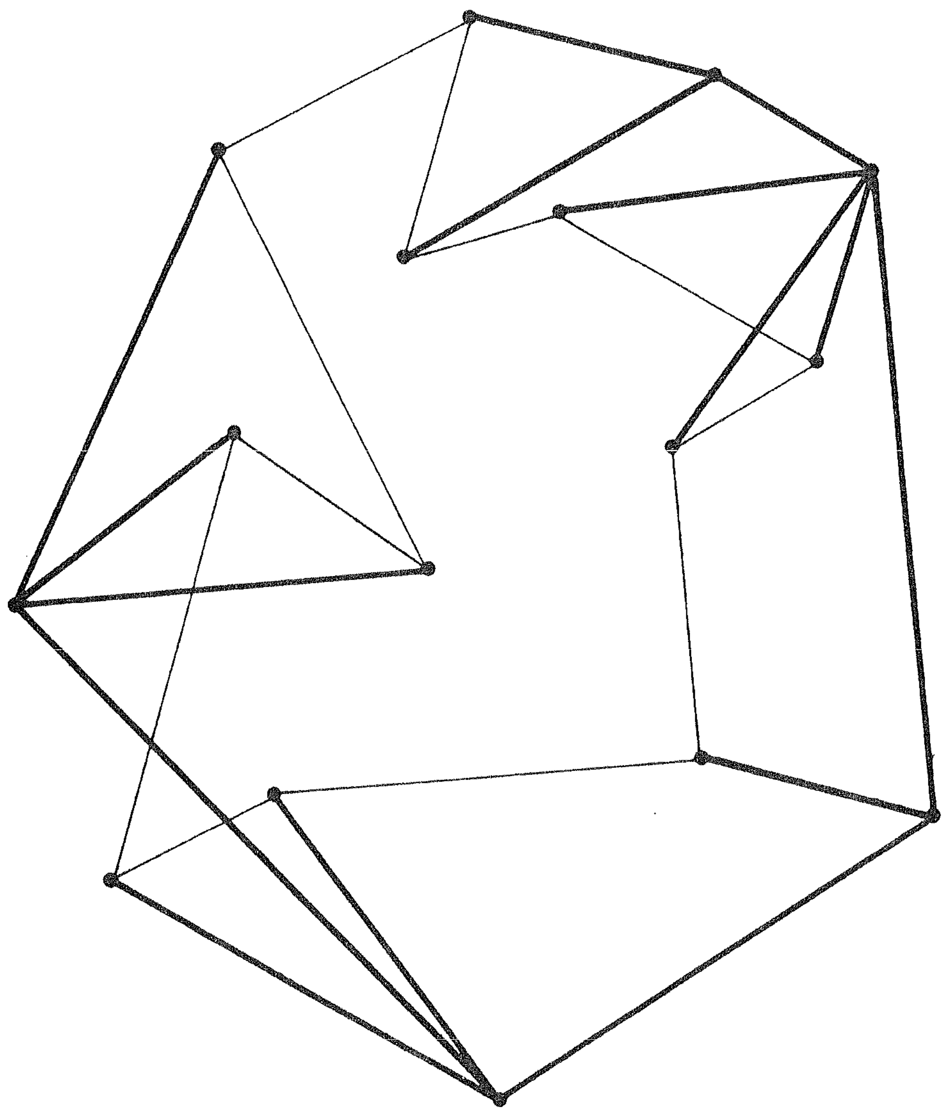


Fig. 1. A Halin graph.

general intent would aim to exploit this notion as follows: Given a graph G which is not Halin, how reasonable is it to approximate G by another graph, G', which is derived from G, is Halin and upon which a given problem is solved, thereby producing a candidate solution on G?

That we would require solutions on G' to be candidates (i.e., admissible) on G implies that we should allow G' to be either a supergraph or a subgraph of G. That

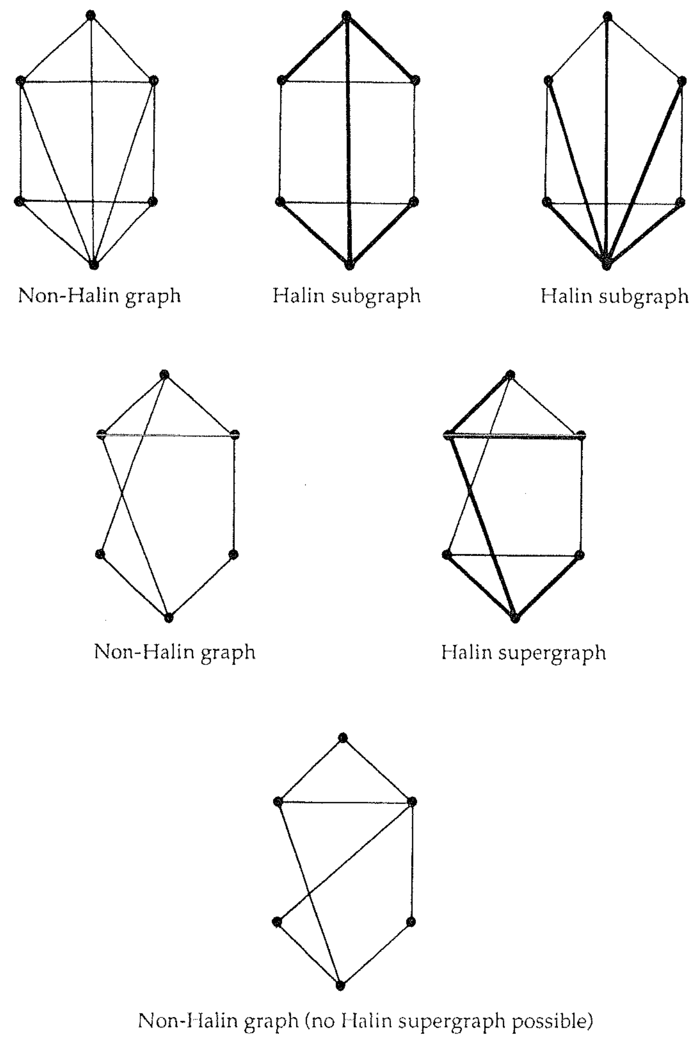


Fig. 2. Halin subgraphs and supergraphs.

is, if G' is a Halin subgraph of G and we possessed an optimal vertex coloring on G', the resultant coloring may not be admissable on G. Alternately, a Hamiltonian cycle on G' would be admissable on G and if G' were a supergraph of G, these two outcomes would be reversed. Halin subgraph and supergraph constructions are demonstrated in Fig. 2.

Regardless, in what follows, we provide evidence that this sort of Halin graph approximation strategy may be doomed or at least may require a different perspective than that heretofore suggested. We begin with some disquieting outcomes.

2. Motivation: some properties of Halin graphs and approximation issues

There are a host of interesting properties of Halin graphs, many of which have been described elsewhere. Readers are directed to references such as [11, 12] for appropriate coverage (e.g., all Halin graphs are Hamiltonian, 1-Hamiltonian, bicritical, etc.). As suggested previously, however, a key property providing some of the original motivation leading to the present work is that while Halin graphs are easy to recognize in general, they are also easily recognizable in the context of k-terminal recursive graph classes. In this regard, their so-called decomposition trees can be easily exhibited which, in turn, leads directly to the existence of the aforementioned fast algorithms for many problems when confined to Halin graphs.

But, it is not clear that Halin graphs occur very often in natural settings (any more than do various other recursive graph classes which admit fast algorithms, e.g., partial k-trees, etc.). So while a great many problems are provably solvable on Halin graphs, it would be fairly difficult to argue the merits of this attribute from other than a theoretical perspective. On the other hand, it seems natural (as apparently it was in [5]) to raise the question of their potential use in the stated context of approximation.

Let us restate the basic notion: From a given non-Halin instance, form an approximating Halin subgraph or supergraph, solve the problem of interest on the latter, and exhibit the outcome as a candidate solution on the original graph. Again, we assume the resulting subgraph or supergraph, whichever is relevant, yields a candidate accordingly and that the stated problem is solvable on Halin graphs in general. Still, the resultant heuristic strategy can be expected to be heavily linked to the quality of the approximating graphs and these, in turn, depend as much on structure as on size. The point is easy to demonstrate.

Consider the non-Halin graph G in Fig. 3(a) and suppose the problem is one of producing a minimum *dominating vertex* set (a subset of vertices having the property that every vertex in the graph is adjacent to at least one vertex in the subset). But the dominating set problem is solvable on Halin graphs and the subgraph version of our approximation tactic will certainly produce a candidate on the original instance (the supergraph case is without interest in any event since the original graph is not even planar). Now, to proceed, consider the Halin subgraph to the left in Fig. 4(a). Denoting this as H_1, it is easy to verify that a smallest dominating set (on H_1) consists of the circled vertices shown. Moreover, this minimum number would grow as the number of replications of the shaded segment indicated on the original graph in Fig. 3(a). On the other hand, if better insight had prevailed and the Halin subgraph to the right in 4(a), say H_2, had been formed, the single, circled vertex indicated there would be optimal and would remain so relative to the original graph and regardless of the number of the aforementioned segments. Clearly, subgraph H_2 is much preferred to H_1 in this case.

On the other hand, suppose we change the problem to one of finding a maximum (edge) cardinality subgraph which spans and which is Eulerian (connected with even degree). Again, this is a hard problem on arbitrary structures but is linear-time

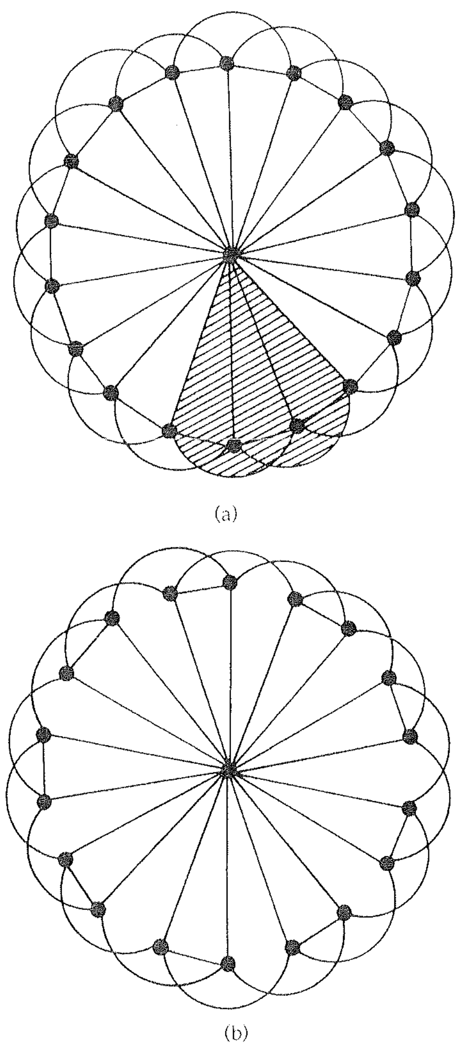


Fig. 3. Approximation illustration.

solvable on Halin graphs. Now, the subgraph form of our approximation approach remains relevant so let us again examine H_1 and H_2. For H_1 the largest spaning Eulerian subgraph is shown to the left in Fig. 4(b) while to its right we exhibit the largest such subgraph in H_2. The latter is better and can be shown to be generally so, again depending upon the number of graph segments present as indicated earlier. We have no interest in belaboring this particular point further but do note that while the subgraph candidate formed by operating on H_2 is better, even it is quite distant from an optimal spanning Eulerian subgraph of G which is shown in Fig. 3(b). As a general aside, the reader will be quick to note that if *any* Halin subgraph is found in a graph of

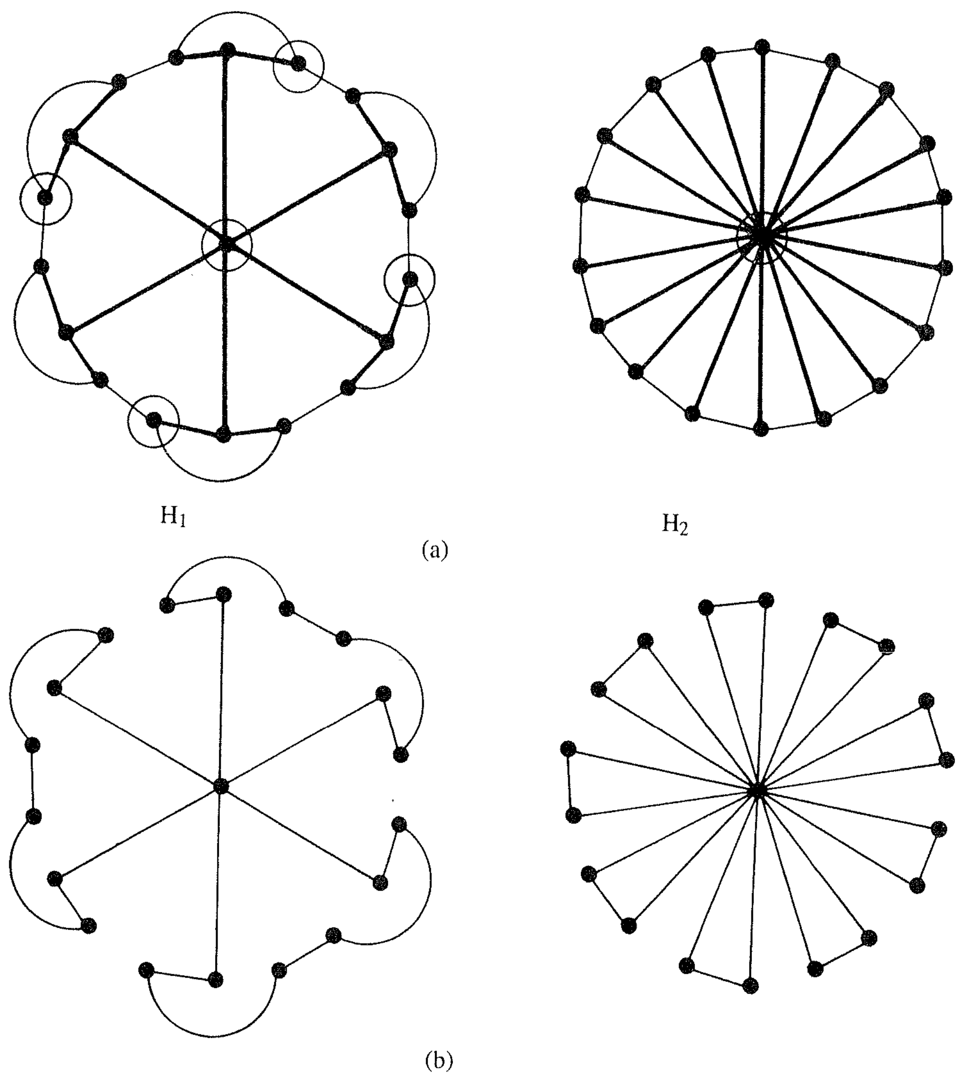


Fig. 4. Approximation outcomes for dominating set and spanning Eulerian subgraph problems.

order p, we would be guaranteed a spanning Eulerian subgraph of size at least p since all Halin graphs are Hamiltonian.

Naturally, these outcomes are circumstantial, but they do suggest potential difficulties (at least they raise issues) relative to heuristics motivated by graph approximation and, in particular, when Halin graphs play the approximating role. Not the least of these is our implicit assumption to this point that simply finding Halin subgraphs/supergraphs is less an issue than finding ones that are in some sense "good". But as intimated by the last sentence of the preceding paragraph and as we will show more directly in the next section, this assumption may be misguided.

3. The complexity of finding Halin subgraphs and supergraphs

In this section, we present the main results of the paper. In particular, we show that the problems of finding Halin subgraphs and supergraphs are both hard. Following, we take up the subgraph problem first.

3.1. *Halin subgraphs*

The problem of interest can be stated in the following way:

P_H^-: Given a graph $G = (V, E)$ and an integer t, does G have a Halin subgraph $G^- = (V^-, E^-)$ where $V^- \subseteq V$, $E^- \subseteq E$ and $|E^-| \geqslant t$?

We have the following theorem.

Theorem 3.1. P_H^- *is* NP-*complete.*

Proof. Our reduction is from the longest cycle problem restricted to planar graphs. Its statement is given below:

P_C: Given a planar graph $G = (V, E)$ and an integer k with $|V| \geqslant k \geqslant 3$, does there exist a cycle in G of length at least k?

Now, from P_C let us create an instance of P_H^- as indicated in Fig. 5(a) where on every edge of G from P_C we insert a single vertex. To this homeomorph, we add a single "supervertex" v_x and connect it to every other vertex. Set $t = 4k$.

Then, letting the constructed instance graph for P_H^- be $G' = (V', E')$, we have

$$V^* \triangleq \text{vertices inserted on edges in } E,$$

$$V' \triangleq V \cup V^* \cup \{v_x\},$$

$$E^* \triangleq \text{edges formed by the subdivision induced by } V^*,$$

$$E_x \triangleq \{(v_x, j) \mid j \in V', j \neq v_x\},$$

$$E' \triangleq E^* \cup E_x.$$

Observe that $|V^*| = |E|$, $|V'| = |V| + |E| + 1$, $|E^*| = 2|E|$, $|E_x| = |V| + |E|$, and $|E'| = 3|E| + |V|$.

Clearly, if there exists a cycle in G of length at least k, then by construction, such a cycle implies a cycle of length $2k$ in G', since each edge in E was split to form two E^* edges (identifiable with one V^* vertex). So this cycle in G' passes through $2k$ vertices, each of which is connected to v_x by an edge in E_x. These $2k$ edges form a star on $2k + 1$ vertices with v_x at the "hub". Adding the $2k$ cycle edges produces a wheel and hence the desired Halin subgraph of size $4k$. The construction is demonstrated in Fig. 5(b) ($k = 4$).

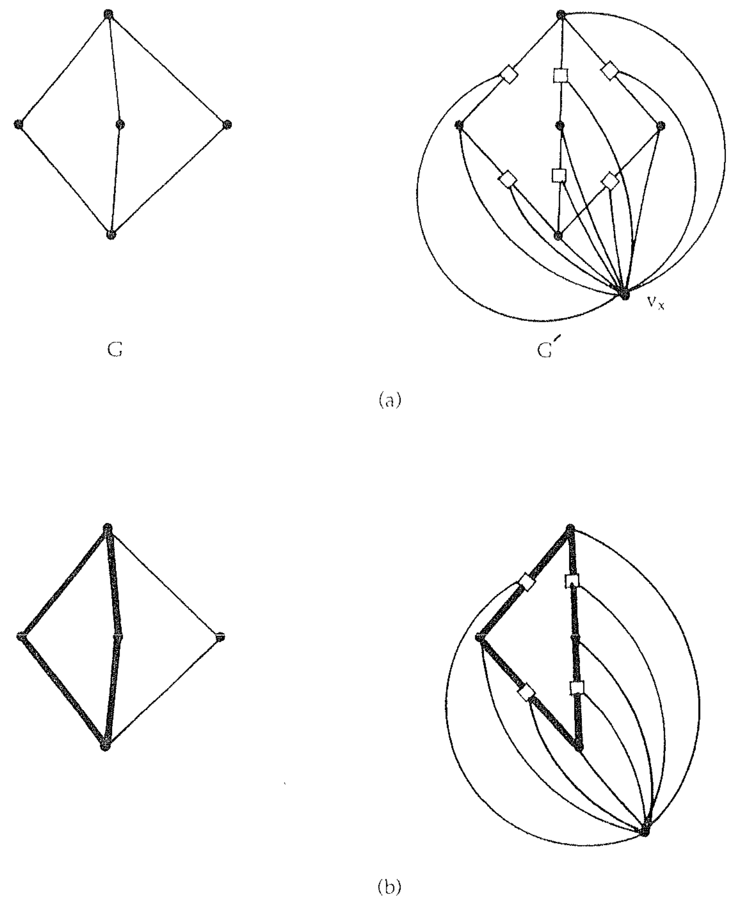
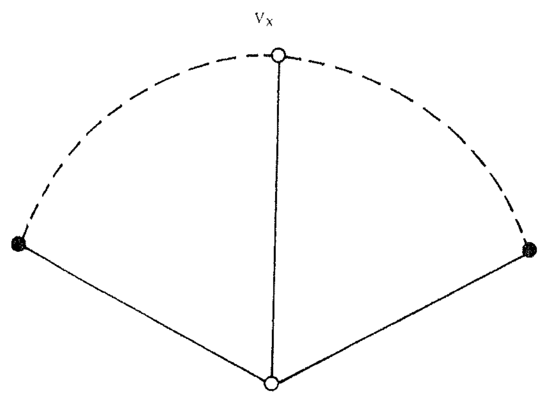


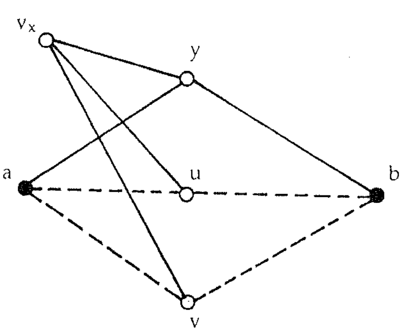
Fig. 5. Construction example for Theorem 3.1.

Conversely, let us assume there exists a Halin subgraph of G', denoted as $G^- = (V^-, E^-)$ where $V^- \subseteq V'$ and $E^- \subseteq E'$ and with edge cardinality at least $4k$. Observe that each edge in E' must be either a *cycle* edge in E^-, a *tree* edge in E^-, or *out* of E^-. In addition, we will abuse the terminology somewhat by referring to a "pendant" in a Halin graph when what we really mean is a degree-1 vertex in the tree portion of such a graph.

Now, suppose v_x is not in V^-. Then none of the E_x edges can be in the stated subgraph which leaves each V^* vertex in G' incident to only two edges. But since no vertex in a Halin graph can have degree less than three, these vertices and the edges incident to them (edges in E^*) also cannot be in the stated subgraph. This leaves no edges at all, so no Halin graph is even possible, contrary to our assumption. Hence, $v_x \in V^-$.



(a)



(b)

Fig. 6. Illustrations in proof of Theorem 3.1.

Next we establish that v_x is not a pendant. Suppose otherwise. Then the hypothesized *cycle* would pass through v_x, implying that exactly two of the edges in E_x are *cycle* edges, and exactly one edge in E_x is a *tree* edge. Now, except for trivial cases, the *cycle* must pass through at least one V^* vertex that is not connected directly to v_x by a *cycle* edge. Then, the remaining edge incident to that V^* vertex must be a *tree* edge in E^-. But such an edge connects the stated V^* vertex directly to v_x which denies that G^- is Halin (see Fig. 6(a)). Therefore, v_x cannot be a pendant, and thus has no *cycle* edges incident to it.

Now in G', v_x is connected to every other vertex by an edge in E_x. Since $v_x \in V^-$, each of these E_x edges in E^- is a *tree* edge connecting v_x to another vertex y in G'. There are four possibilities: (a) y is in V and is a pendant, (b) y is in V and is not a pendant, (c) y is in V^* and is a pendant, or (d) y is in V^* and is not a pendant. Following, we examine cases (b) and (d) and show that they are not possible.

- Case (b). Since we suppose that y is not a pendant, at least two E^* edges incident to y must also be *tree* edges. Both of these edges lead to degree-3 vertices in V^*. Pick one of these vertices, say p. If p is a pendant, then both of the other two edges incident to it must be *cycle* edges. But one of these edges leads back to v_x which contradicts that v_x is not a pendant. If p is not a pendant, then both of the other two edges incident to it are *tree* edges, but the one that leads back to v_x forms a cycle with the edges selected thus far and G^- could not be Halin. Therefore, case (b) is impossible.
- Case (d). Since we suppose that y is not a pendant and y is in V^*, both of the other two edges incident to y must also be *tree* edges. Let these two edges be incident to vertices a and b in V. Now suppose a and b are both pendants. Then the *cycle* passes through both a and b. This defines two edge-disjoint simple paths from a to b that do not pass through nonpendant vertices v_x and y. Further, since a and b are both in V, there must be at least one vertex in V^* on each of these paths. Let any V^* vertex on one path be u, and any V^* vertex on the other path be v. Now, since u is of degree three in the subgraph, the non*cycle* edge incident to u (which leads back to v_x) must be a *tree* edge. The same is true for vertex v. However, this forms a subgraph homeomorphic to $K_{3,3}$, with bipartition $\{\{v_x, a, b\}, \{y, u, v\}\}$ (see Fig. 6(b)). Similarly, if one or both of a and b are not pendants, then the additional *tree* edges required to connect them to pendants will again produce a $K_{3,3}$ homeomorph when added to the previous construction. In either case we deny planarity in the hypothesized subgraph so case (d) is also impossible.

Having chosen vertex y arbitrarily, we may conclude that every vertex in the hypothesized Halin subgraph with the exception of v_x must be a pendant vertex. But this can occur only with the tree edges forming a star having hub v_x. Clearly, such a (Halin) graph has exactly half of its edges in the *cycle* and half in the *tree*, all of the latter incident to v_x. Since we have supposed the existence of a Halin subgraph with $|E^-| \geqslant 4k$, we know that at least $2k$ edges must form the *cycle* in G^-. This cycle is represented by a vertex sequence alternating between vertices in V and those in V^*. But in the construction of G', each edge of G was "split" into two edges by the insertion of a V^* vertex. Thus, the *cycle* portion of G^- having at least $2k$ edges corresponds exactly to a cycle in G of length at least k.

We stated earlier that it is easy to test if a graph is Halin so P_H^- is in NP and the result of the theorem follows. □

An easy corollary results by fixing k in the above theorem. Letting $k = |V|$ we have Corollary 3.2.

Corollary 3.2. *Deciding if $G = (V, E)$ possesses a spanning Halin subgraph is NP-complete.*

Before moving to the supergraph case for Halin graphs, it is worth pointing out that results similar to Theorem 3.1 exist for other classes of graphs. Most of these are familiar in the context of so-called edge and/or vertex deletion problems, among which are the results reported in Yannakakis [14–16] and, in particular, in Asano [1] where a result analogous to Theorem 3.1 for series–parallel graphs (holding even on planar instances) is given. In this latter regard, we point out that the same result was obtained independently and reported in [10].

3.2. *Halin supergraphs*

Especially in view of the aforementioned references, subgraph decisions often produce a complexity status that is not unexpected. In fact, many are NP-complete. On the other hand, corresponding questions regarding supergraph constructions appear to be more interesting (complexity status notwithstanding). Typical of these is the (NP-complete) *Hamiltonian Completion* problem [7]: Given $G = (V, E)$ and $k < |V|$ does there exist a superset $E' \supseteq E$ such that $|E' \backslash E| \leqslant k$ and $G' = (V, E')$ is Hamiltonian?

There are, of course, uninteresting versions of the supergraph problem. For example, if a graph is not series–parallel (indeed, not a partial k-tree for some k) then adding edges cannot make it so. Trivially, this does not carry over for the case of Halin graphs.

Let us state our supergraph problem as follows:

$\mathrm{P}_{\mathrm{H}}^{+}$: Given a graph $G = (V, E)$, does there exist a set of edges $E^{+} \supseteq E$, such that the supergraph $G^{+} = (V, E^{+})$ is Halin?

(Equivalently: Is G a subgraph of any Halin graph?) Recall that we may assume that G is not Halin since testing for this property is easy. We may also assume that G is not 3-connected, following from the property that Halin graphs are necessarily 3-connected and minimal in this regard.

We now show that resolving $\mathrm{P}_{\mathrm{H}}^{+}$ is no easier than the previous, subgraph version. We begin with an easy lemma which will prove to be useful.

Lemma 3.3. *Let G be Halin with p vertices and with $|C| = k$. Then k is bounded as*

$$p/2 + 1 \leqslant k \leqslant p - 1.$$

Proof. The upper bound is clear. The cycle passes through only and all the pendants of T and there are exactly k of these. Accordingly, the maximum is $p - 1$ which is achieved by stars $K_{1,p-1}$. For the lower bound let k be the size of a smallest cycle. Since G is Halin, there are $p - k$ vertices in T which are not pendants and which are connected by $p - k - 1$ edges. Total degree generated by these "tree" edges is therefore $2(p - k - 1)$. Also, there are k tree edges that are incident to these pendants. These add k to the total degree of the nonpendant vertex structure just described producing a degree total of $2p - k - 2$. But since every vertex in a Halin graph has degree at least 3, there must exist degree at least of value $3(p-k)$ contributed by the stated $p-k$ vertices. That is, $3(p-k) \leqslant 2p-k-2$ and so $k \geqslant p/2+1$ as claimed. □

Our supergraph result can now be given. We have the following theorem.

Theorem 3.4. P_H^+ is *NP-complete.*

Proof: We will show a reduction from the (strong sense) 3-Partition problem the statement of which appears below:

P_{3P}: Given a set A of $3m$ elements, an integer bound B, and an integer size $s(a)$ for each $a \in A$ such that $B/4 < s(a) < B/2$ and where $\sum_{a\in A} s(a) = mB$, can A be partitioned into m triples $A_1, A_2, \ldots, A_m$ such that, for $1 \leqslant i \leqslant m$, $\sum_{a \in A_i} s(a) = B$?

(Observe that each A_i must contain exactly three elements from A).

From an instance of P_{3P} let us create an instance of P_H^+ as indicated in Fig. 7. Attached to vertex t are "tails" which correspond to the elements in A with the length of each tail related to the size of the respective element. The upper part of the graph and in particular the m "segments" correspond to the desired sets $A_1, A_2, \ldots, A_m$ each with size B where the latter is denoted by the B darkened vertices inserted within each segment. Segments are connected by the large, open vertices as shown. For ease, we shall use the terms "intrasegment" vertex and "intersegment" vertex to denote these vertices. Note that the graph in the figure is not Halin nor is it 3-connected.

Let us assume that a suitable partition of A exists. Accordingly, we can construct a Halin supergraph of G in the following manner. For each A_i, place the three relevant tails in an interior face bounded by intersegment vertices, intrasegment vertices, and t (we call each of these faces a "sector"). From each pendant vertex of a tail create two edges from the pendant to an adjacent pair of intrasegment vertices. For each (if any) other (degree-2) vertex on a tail, create one edge from the tail to an intrasegment vertex. In this way, we add $\sum_{a\in A_i}(s(a)-1)+3 = B$ edges.

It is easy to see that planarity is maintained in this construction. The construction is thus complete yielding a Halin graph $H=(V, E_H)$ with E_H defined by E augmented with the new edges just described; the *cycle* edges are those defining the face denoted by f, say E_f, and the *tree* edges are given by $E_H \setminus E_f$. Fig. 8 demonstrates the construction.

Conversely, assume there exists a Halin completion of G, say $G^C = (V, E^C)$. Let $E^C = E \cup E'$. Now, vertex t must be a nonpendant vertex, since its degree in G exceeds 3 (we avoid trivialities in the statement of P_{3P}). Thus, edges x and y (in Fig. 7) are tree edges implying that edge c is a *cycle* edge. Accordingly, the intersegment vertices incident to edge c must be pendants, which implies that edges a and b are *cycle* edges as well. But this means that the *cycle* subgraph in G^C is either defined by face f of G or a subface of f created in G^C. Thus, the tails of G must be part of the *tree* in G^C.

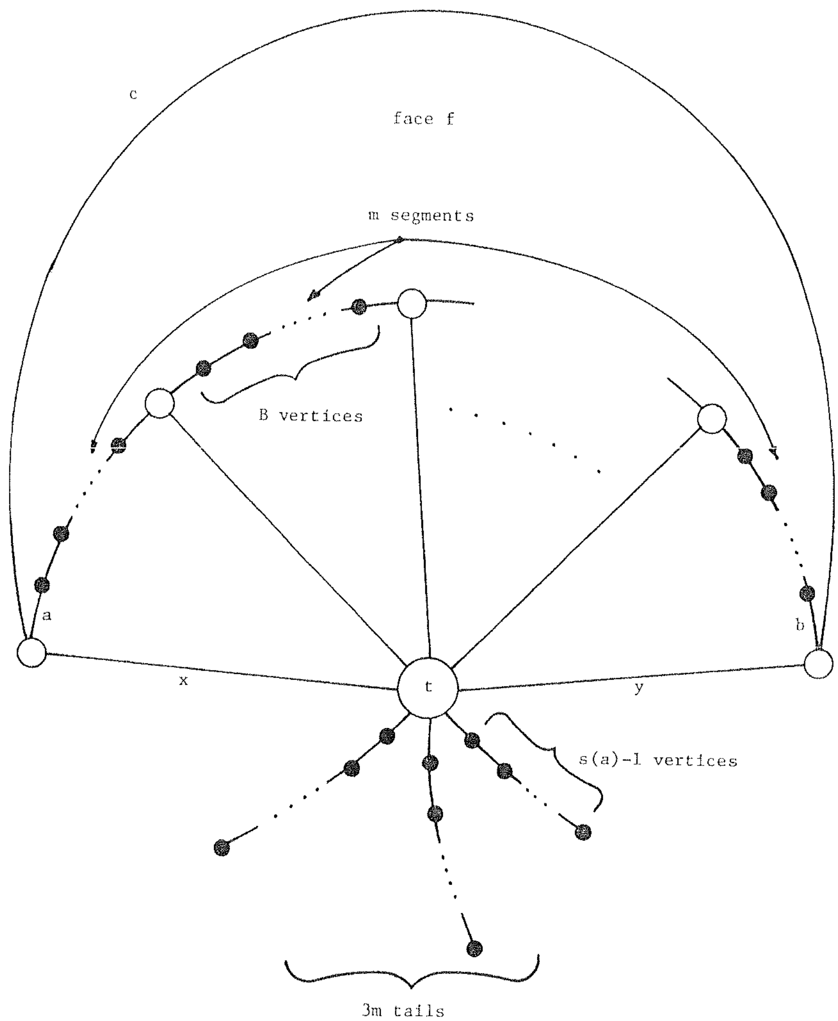


Fig. 7. Construction example for Theorem 3.4.

If the number of *cycle* edges in $G^C = k$, we must have that $k \leqslant \dim(f) = m(B+1)+1$, where $\dim(f)$ denotes the dimension or number of edges defining face f. Also recall from Lemma 3.3 that k has a natural lower bound given by

$$k \geqslant |V|/2 + 1 = (2(m(B-1)+1))/2 + 1 = m(B-1) + 2.$$

But the vertices in the tails of G have degree less than 3 and therefore have total deficiency at least mB. This deficiency has to have been satisfied by edges attached to the tail vertices but not extending between distinct tails nor between distinct vertices of the same tail which in either case denies that the tails are part of a tree. Hence, $|E'| \geqslant mB$.

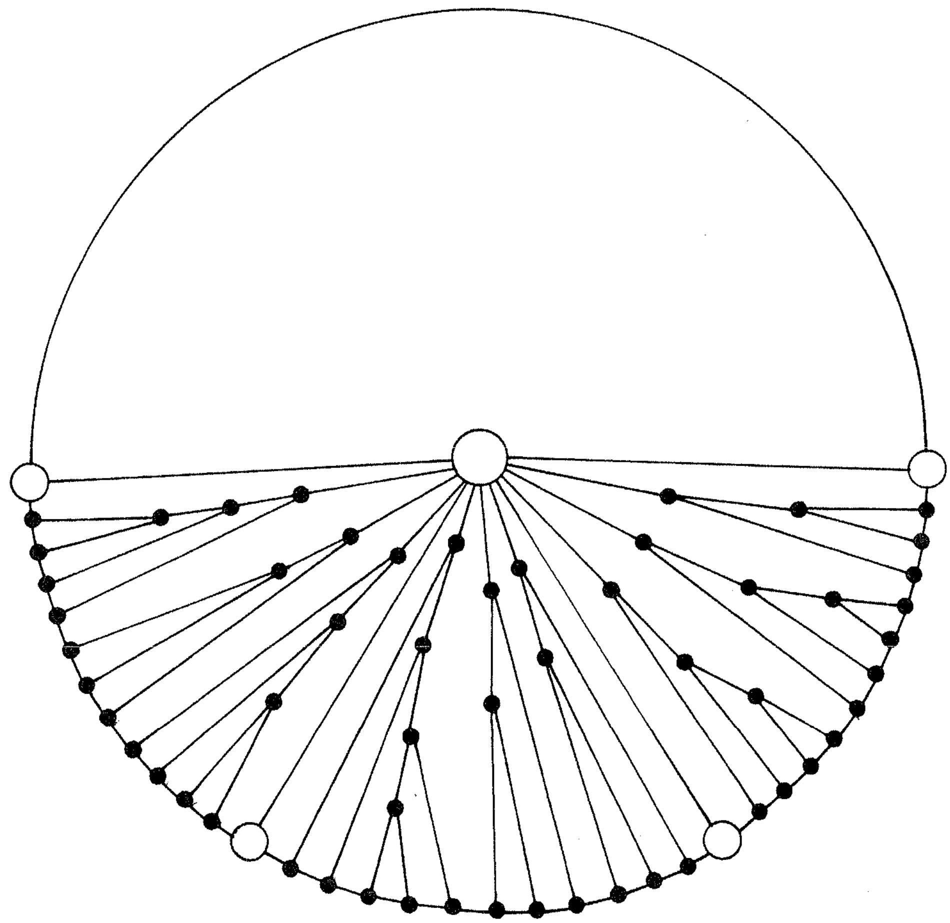


Fig. 8. Halin completion

On the other hand, the intrasegment vertices in G all have degree 2 and thus a deficiency of mB as well. But in G we have $|E| = 2mB - m + 2$ and in any Halin graph of order p with a *cycle* of length k, we have $p + k - 1$ edges. In our construction, G^C must then have size $2(m(B-1)+1) + k - 1$. Letting θ be the number of edges to add to G to create G^C, it is easy to see that $\theta = k - m - 1$. Let us suppose that k is different from its upper bound of $m(B+1)+1$. Then $\theta < mB$ and E' cannot be formed as required. Hence, $k = m(B+1)+1$ and the *cycle* in G^C is defined explicitly by face f. Thus all the vertices on f are pendant vertices and their deficiency is exactly mB. The only way this deficiency can be satisfied is by edges connected to tail vertices. Exactly B of these are required in each sector face and these edges connect exactly those vertices of the tails which must be embedded in the sectors. Moreover, if any tail vertex is connected to a vertex in a given sector so must every other vertex in that tail since G^C is planar. Thus, every one of the m sectors has exactly three tails from

G embedded within it and connected by exactly B edges to the respective vertices on face f. But then each of these sector-tail embeddings forms a triple which corresponds to a suitable 3-partition of A.

The transformation from P_{3P} to form G is valid following the strong sense status of P_{3P}. This along with P_{II}^{+}'s inclusion in NP yields the desired result. □

4. Discussion and summary

Following the complexity outcomes above, and particularly so for the subgraph case, we would not expect it to be a challenge to exhibit edge weights for a given complete graph in order that the corresponding task of finding a Halin subgraph of small total (edge) weight is difficult. Assuming these weights can be arbitrary, the point is easily demonstrated by the structure in Fig. 9 (note that in the figure, some heavily weighted edges are missing for ease of presentation). Now, there are only two possible Halin subgraphs of G and these are given by H_1 and H_2 as shown. If we weight all edges that are common to both subgraphs by 1 and then assign a weight of 1 to (x, y) and to (y, z), a slightly larger value, say $1 + \varepsilon$, then edge (x, z) can be made arbitrarily large which, in turn, makes H_2 the much preferred choice. Yet, a "greedy" strategy whereby edges are selected in nondecreasing weight order would clearly produce subgraph H_1 (note that we are not intimating that a conventional greedy strategy is even viable as a heuristic which follows by recalling that subgraphs of Halin graphs are not Halin).

But, if edge weights are restricted in some well-defined sense such as by satisfying the triangle inequality, then this sort of arbitrarily bad behavior tends to be controllable. The classic evidence in this regard is the effect such weight restrictions have on the quality of traveling salesman heuristics. In any event, we have given no serious thought to this issue in connection with heuristically generated Halin subgraphs/supergraphs but rather suggest that it might be worthy of some attention.

On the other hand, it would be particularly interesting to weaken the conditions of Theorem 3.1 to ask if a given graph admits *any* Halin subgraph. Again, such a question is often without interest in other settings such as for series–parallel graphs. (Recall that the complexity result alluded to earlier pertains to series–parallel subgraphs of at least some predetermined size (cf. [10]).) Also the sorts of complexity questions raised in this paper relative to Halin graphs could be examined in the context of other recursive structures. Among these are partial k-trees.

Still, it would seem that the more interesting pursuit would be to return to the approximation theme articulated in the introduction and, moreover, to do so with a view towards formulating different approximation strategies. That is, even if structures other than Halin graphs are employed in the approximation context herein, and the complexity status of even finding corresponding subgraphs or supergraphs notwithstanding, it is not clear that they would serve as quality approximations in any general context anyway.

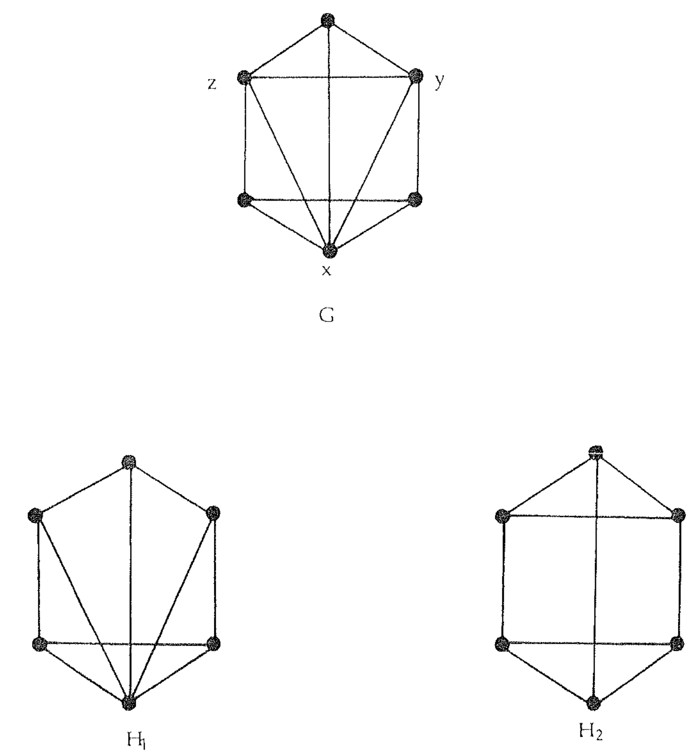


Fig. 9. Halin subgraphs on weighted, complete graphs.

Consequently, one alternative that is intriguing can be identified, loosely, as a "decomposition" approach: Given a graph G not known to be in any well-solved class, can the edges of G be partitioned into at most k subsets each of which forms a graph which is in such a class. If so, we would solve our particular problem on these pieces and try to find a way to relate the result in terms of a candidate solution for G.

It seems reasonable to require that k be small and fixed for a given graph class. In addition, we would like for the solvable classes into which these edges are partitioned to be robust in that a large number of difficult problems are well-solved accordingly. Here again, recursive structures are notable candidates.

In this regard, a useful starting point for such a tactic can be found in a recent result by Heath [9] which establishes that the edge set of every planar graph can be partitioned into at most two outerplanar graphs. But outerplanar graphs are recursively definable (they are all series–parallel) and thus by [3], most otherwise hard problems are linear-time solvable when instances are so restricted. Important also is that this outerplanar bipartition can be efficiently exhibited. So, since many problems remain hard on planar graphs, heuristic strategies are still legitimate alternatives and the decomposition notion alluded to presently may have some merit in that regard. This seems worth investigating.

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