ON SOME RESULTS PERTAINING TO HALIN GRAPHS

by

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Abstract

Halin graphs are planar graphs with edge sets that can be partitioned into a tree T with no vertex of degree 2 and a cycle C on only and all the leaves of the tree. In this paper, we describe some attributes of Halin graphs among which is their containment in a 3-terminal recursive class, their status as so- called class-1 graphs, and a kind of uniqueness result that shows that no matter how C is formed (if there is a choice), the resulting tree subgraphs are isomorphic. We also show that two problems pertaining to Halin subgraphs and supergraphs are NP- complete.

1 Introduction

A *Halin graph* $G = (V, E)$ is a planar graph having the property that its edge set can be partitioned into a tree T with no vertex of degree 2 and a cycle C which spans the pendant or degree-1 vertices of T. The graph in Figure 1 is Halin. The cycle is denoted in bold (observe that its removal leaves the required tree). Such graphs were first studied by Halin [Ha71] and exhibit an example, accordingly, of a class of edge-minimal, planar 3-connected graphs.

Despite their apparent simplicity, Halin graphs happen to possess a number of particularly interesting properties. First, they are easy to recognize. This is important since it is also known that Halin graphs are contained in a so-called 3-terminal recursive class [BPT91b] which means that numerous otherwise hard problems can be polynomially solved when instances are so restricted. The naive approach suffices for recognition in that we need only embed G in the plane (we may safely assume G is planar) and search for a face the edges of which, if removed from G, leave a tree.

All even order Halin graphs are ***bicritical*** in that the deletion of any two vertices leaves a graph possessing a 1-factor [LP75]. They are *1-Hamiltonian*; they are Hamiltonian and remain so after the removal of any single vertex [B75]. They are "almost *pancyclic*" in that for any Halin graph of order n, every cycle of length $3 \leq t \leq n$ is present except possibly for one of even length [BL85]. Immediate from this is that Halin graphs are not bipartite.

In the spirit of an earlier paper by Syslow and Proskurowski [SP81], we describe additional properties and outcomes on Halin graphs among which is their aforementioned inclusion in a so-called k-terminal recursive class and, in particular, what this implies algorithmically. We give a proof that Halin graphs are class-1 graphs, always having chromatic index equal to maximum degree and we also establish a sort of uniqueness result showing that regardless of how C is chosen (given that there are choices at all) the resulting trees formed upon the removal of C are isomorphic. We conclude the paper with a pair of complexity results involving Halin subgraphs and supergraphs.

2 Algorithms on Halin Graphs

2.1 k-Terminal, Recursive Graphs

From the complexity perspective, most problems are well-solved on Halin graphs. That is, we can assert the existence of fast algorithms for many such problems when instances are restricted to Halin graphs. That this is so follows from the recursive constructability of these graphs which in turn, is established by their membership in a particular 3-terminal graph class described in Borie et al. [BPT91b]. Following, we give enough detail to make this membership evident.

A *k-terminal graph* $G = (V, T, E)$ has a vertex set V, an edge set E, and a (possibly ordered) set of *terminal* vertices $T = \{t_1, \ldots, t_{|T|}\} \subseteq V$, where $|T| \leq k$. A *recursively constructed class* $C(B, R)$ in some universe U is specified by base graphs $B \subseteq U$ and a finite set of rules $R = \{f_1, \ldots, f_n\}$ where each $f_i : U^{m_i} \to U$ is a recursive composition operation with arity m_i; C is thus the closure of B in U by rules $f_1, \ldots, f_n$. Generally, for some (fixed) k, U is simply the set of k-terminal graphs and B is taken to be a set of connected k-terminal graphs (V, T, E) with $V = T$. However, each such base graph is trivially composed of individual edges whose vertices are terminal, so it is reasonable and convenient to use $C(R)$ to denote $C(B, R)$ where B only contains K_2. Loosely then, a k-terminal recursive graph class is one where any sufficiently large member is composed from smaller members of the same class, joined by merging the distinguished, terminal vertices. Upon merger, some vertices which were terminals in constituent graphs may loose terminal status upon composition.

Let us now consider a specific type of composition operation. In particular, let a *c-ary recursive (k,u,r)-operation* be a function $f(G_1, \ldots, G_c) = G = (V, T, E)$ on k-terminal graphs that satisfies the following conditions:

- $|T| \leq k$ and $|T_i| \leq k$ for each $G_i = (V_i, T_i, E_i)$,
- $V = \cup_{i=1}^{c} V_i$,
- $E = \cup_{i=1}^{c} E_i$,

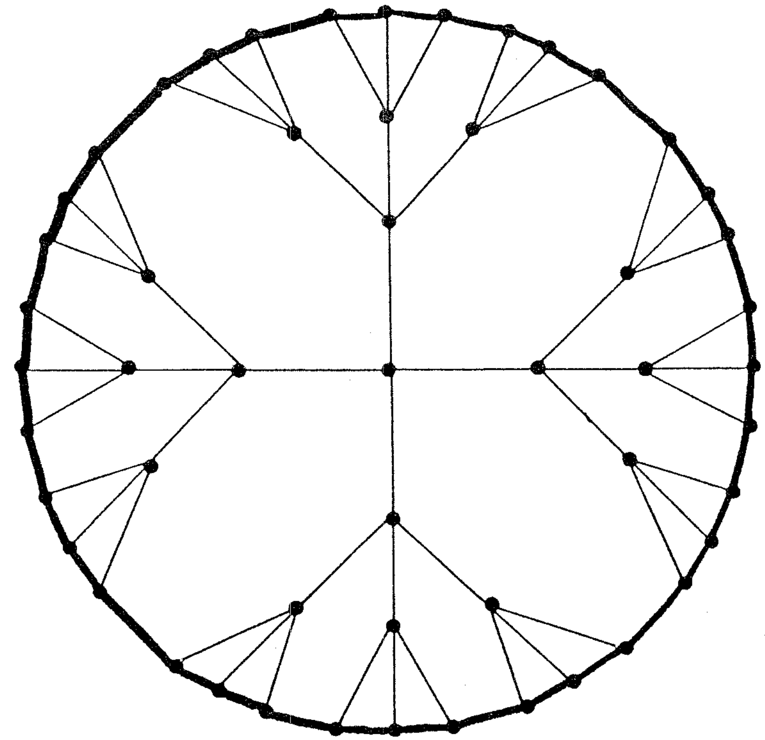


Figure 1. A Halin graph

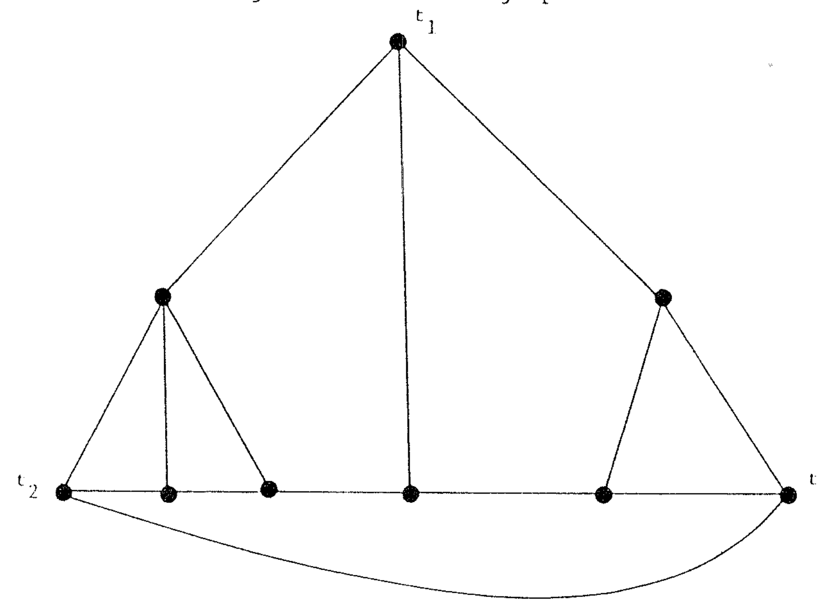


Figure 2. 3-terminal, Halin graph initialization

- $T \subseteq \cup_{i=1}^{c} T_i$,
- $|\cup_{i=1}^{c} T_i| - |T| \leq u \leq r$,
- $|\cup_{i=1}^{c} T_i| \leq r$,
- $|T_i| - |T| \leq s = k + u - r \geq 0$ for each i, and
- $V_{i_1} \cap V_{i_2} = T_{i_1} \cap T_{i_2}$ for each $i_1 \neq i_2$.

Thus, a (k, u, r)-operation joins k-terminal graphs G_i at their common terminals, producing a k-terminal graph G where there are up to r vertices in G that were terminals in the constituent G_i, and where up to u of these become undistinguished in the resultant G. Hence, r need never be greater than $k + u$. Also, none of the constituent G_i can have more than $s = k + u - r \geq 0$ more terminals than G has. For any $0 \leq k \leq r, 0 \leq u < r, k + u \geq r$, we can let $[k, u, r]$ denote the family of all (k, u, r)-operations. To illustrate, the traditional and well known series and parallel operations [D65] are both in $[2, 1, 3]$ (in fact the parallel operation is also in $[2, 0, 3]$).

But if membership in a k-terminal recursive class is good, it is reasonable to expect that recognition of such membership be easily decidable. A usual test in this regard is whether or not a member graph can be represented (efficiently) by its decomposition tree. Indeed, recursively constructed structures are often referred to simply as tree-decomposable graphs.

For a k-terminal graph $G \in C(R)$, a decomposition tree is a rooted tree with vertex labels g and f such that

- $g_v = G$ if v is the root,
- $f_v \in R$ if v is an interior node,
- $g_v = f_v(g_{v_1}, \ldots, g_{v_m})$ if interior node v has children $v_1, \ldots, v_m$, and
- $g_v \in B$ if v is a leaf.

Now, let G be Halin and assume a plane embedding as shown in Figure 2. Following, we describe a template for producing a decomposition tree for G. First, let us label terminals as t_1, t_2, and t_3 as shown. Now, G can be decomposed by applying a $(3, 0, 3)$-operation producing an edge (base graph $e \triangleq \{t_2, t_3\}$ and the graph $G \backslash e$. Then, the latter can be decomposed further by a $(3, 1, 4)$-operation in a fairly natural way. To see this, denote by h that leaf in T of $G \backslash e$ which is closest to t_3 such that it can be reached from t_2 by a path that passes through neither other leaves of T nor t_1. (Alternatively, we could choose h as that leaf which is closest to t_2 such that it can be reached from t_3 by a path that passes through neither other leaves nor t_1.) Then $G \backslash e$ is decomposed into two graphs with terminals $\{t_1, t_2, h\}$ and $\{t_1, h, t_3\}$. Finally, $(3, 1, 4)$- operations can be used to eliminate subsequent degree-1 vertices and the process repeats until only edge base graphs result.

The full decomposition tree for the graph in Figure 2 is shown in Figure 3 with the respective operations and constituent graphs indicated for each interior node.

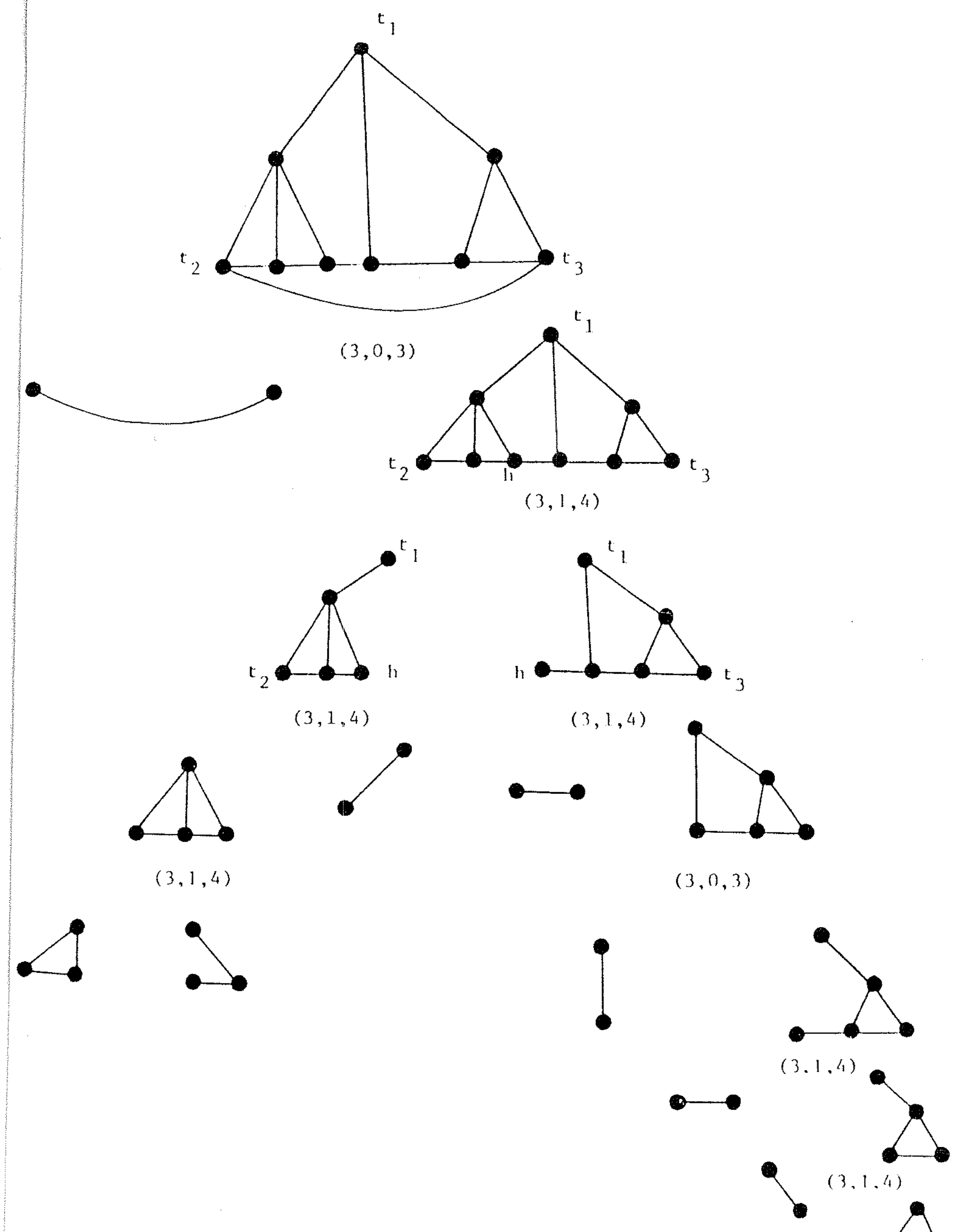


Figure 3. Halin graph decomposition

This process is valid for any Halin graph establishing, in the context of [BPT91] and as described above, their containment in $C([3,1,4])$. (Observe in Figure 3, we have terminated decomposition when graphs, possibly not edges, result with all vertices terminal).

2.2 Vertex Cover on Halin Graphs

It is easy to see that fast algorithms on these recursive graph classes are typically based on dynamic programming, so that a solution to a large member can be determined directly from solutions to the smaller members which constitute it, using a recurrence relation specific to the problem. If the number of terminals is restricted to some fixed value k, the recurrence relation can be evaluated efficiently. This in turn leads to a fast algorithm, assuming a decomposition tree for any graph in the class can be found quickly.

To illustrate, consider the *vertex cover problem*: given some $G = (V, E)$, we seek a smallest subset $V' \subseteq V$ such that V' has nonempty intersection with every edge $\{i, j\} \in E$. Now, for any Halin graph, say $H = (V, T, E)$ or subgraph of a Halin graph, and for each subset $S \subseteq T$ let us define $P_S(H)$ to be the size of a minimum vertex cover V' of H such that $S = V' \cap T$. Since Halin graphs are 3-terminal graphs, we will need to maintain 8 values for $P_S(H)$. That is, a vertex is either in or out of a cover and so there are 2^k choices for terminal subsets of $T = \{t_1, t_2, t_3\}$.

To develop appropriate recurrence relations for a dynamic programming solution, we start by constructing multiplication tables for each of the composition operations. When G_1 and G_2 are composed by an operation f to form G, the multiplication table for operation f shows which of the possible pairs of sets $S_1 \subseteq T_1$ and $S_2 \subseteq T_2$ are compatible. In addition, the table for f shows the value of the corresponding $S \subseteq T$ for each such compatible set pair.

It is now straightforward to construct the recurrence relations directly from the multiplication tables. The formulas simply compute the optimal values from among the compositions of the compatible pairs. Associated with each graph in the decomposition tree is an 8-tuple of the form $(P_\emptyset, P_{t_1}, P_{t_2}, P_{t_3}, P_{t_1t_2}, P_{t_1t_3}, P_{t_2t_3}, P_{t_1t_2t_3})$. The optimal vertex cover value for the instance graph is the minimum of these values relative to the 8-tuple associated with the root node. Initial optimal values for the components of the 8-tuple are found in the naive, brute force way for each base graph (leaf).

This yields an obvious linear time algorithm for vertex cover on any Halin graph. The time required is linear because there is only a constant amount of information to be computed for each node of the decomposition tree, and the size of this decomposition tree is linear in the cardinality of the edge set of G.

It should also be clear that this process works for any k- terminal graph class (for fixed k), assuming the decomposition tree is part of the instance. Indeed, formal models of this computation have been developed which assert the existence of polynomial if not linear time algorithms for many problems on such tree-decomposable graphs (Halin, series-parallel, partial k- trees, etc.). However, these existence results are often manifested in the form of more ad hoc strategies which pay attention to problem specific details. The work in Boric *et al.* [BPT91a,92] describes one such

formal model.

3 GRAPH THEORETIC PROPERTIES

Various interesting properties of Halin graphs were listed earlier. In this section we described some others which help to further underscore the special nature of these graphs.

3.1 Structural Properties

It is easy to see that Halin graphs are not closed under the subgraph operation. That is, no subgraph of a Halin graph is Halin which follows directly from the minimality property regarding 3- connectivity. Thus, given a decomposition tree of a Halin graph, every descendent graph is not Halin. This, of course, is typically not the case with graphs constructed in a recursive fashion. For example, Halin graphs are distinctly different from the class of series-parallel graphs in this regard.

In fact, distinctions between Halin graphs and series-parallel graphs are quite fundamental. Indeed, the two classes of graphs are incomparable. This is easy to see following the result of Dirac [Di52] which asserts that any simple graph with minimum vertex degree 3 possesses a subgraph homeomorphic to K_4 and is thus not series parallel. But a necessary condition for a graph to be Halin is that every vertex have degree at least 3 and the property follows.

If T and C are the tree and cycle edge sets, respectively, then for a Halin graph G of order p and for $|C| = k$, the corresponding total edge cardinality is $p + k - 1$. Now, since the cycle passes through only and all the pendants of T, there are exactly k such pendants. The maximum number of these is $p - 1$ which is achieved by star graphs, $K_{1,p-1}$. Hence, the largest Halin graph on p vertices has $2p - 2$ edges.

The following property will be particularly useful later.

<u>Lemma 3.1</u>: Let G be Halin with $|C| = k$. Then k is bounded as

$$p/2 + 1 \leq k \leq p - 1$$

<u>Proof</u>: The upper bound was just established. For the lower bound let k be the size of a smallest cycle. Since G is Halin, there are $p - k$ vertices in T which are not pendants and which are connected by $p - k - 1$ edges. Total degree contributed by these "tree" edges is therefore $2(p - k - 1)$. Also, there are k tree edges that are incident to the pendants of the tree. These add k to the total degree of the non-pendant vertex structure just described for a degree total of $2p - k - 2$. But since every vertex in a Halin graph has degree at least 3, there must exist degree at least $3(p - k)$ contributed by the stated $p - k$ vertices. That is, $3(p - k) \leq 2p - k - 2$ and so $k \geq p/2 + 1$ as claimed.
□

Observe that the graph in Figure 4a is not Halin which follows immediately since the graph is not even 3-connected. On the other hand, the structure in part b is Halin with C (and thus T) denoted. This graph is a supergraph of the one in part a. Finally, in Figure 4c we exhibit the graph which results by adding a single edge to the graph in part a ($\{5,11\}$). This graph remains non-Halin and moreover, cannot be made so by the addition of any edges. From Lemma 3.1 we know that the size k of a cycle in any Halin completion of the stated graph must have at least nine edges. There are only two possibilities in this regard given by edges in the cycles denoted by vertex sequences $\{1,5,9,13,14,8,4,3,2\}$ and $\{1,5,11,16,12,7,4,3,6\}$. But the removal of the respective cycle edges does not leave a tree and the addition of any new edges to the given graph can only decrease k.

Now, for a Halin graph G, it is obviously not true that the bipartition $\{T,C\}$ is unique. Alternately, there may be more than one choice for the cycle. Trivially, K_4 suffices to make the point. Of interest, however, is the following result:

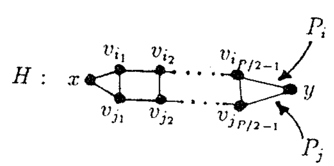
Theorem 3.2: Let $G=(V,E)$ be a Halin graph having at least two distinct decompositions. If these are given by $\{T_1,C_1\},\{T_2,C_2\},\ldots,\{T_t,C_t\}$, then for any pair C_i and C_j we have that $G\backslash C_i$ and $G\backslash C_j$ are isomorphic.

Proof: Every cycle $C_1,C_2,\ldots,C_t$ in G has the same (edge) cardinality which follows trivially since $|E|$ is fixed with value $|V|+|C|-1$. Now, let C_i and C_j be any distinct pair of cycles. Clearly, these cycles cannot be vertex disjoint since otherwise $G\backslash C_i$ (resp. $G\backslash C_j$) would not form a tree. But these cycles cannot be edge-disjoint either for then they would pass through a vertex of degree at least 4, a contradiction. Hence, they have nonempty intersection. On the other hand, C_i and C_j define faces in the plane embedding of G and since each pair of faces in a plane graph (with vertex degree at least 3) share exactly one edge, we have that C_i and C_j intersect in exactly one place accordingly.

Now, let the two cycles consist of k edges from G. We know that the number of distinct vertices in C_i and C_j is $2k-2$ which is bounded from above by $|V|=p$. Observe that the subtraction of 2 follows from the common edge in both cycles. Equivalently, k is thus bounded by $p/2+1$. But we also know from Lemma 3.1 that in any Halin graph, k must be at least $p/2+1$ and so we have that $k=p/2+1$. This implies that $|V|$ is even. But C_i and C_j pass through only vertices of degree 3 and since $|C_i|+|C_j|=p+2$, we know that G is cubic.

Let $e\in C_i\cap C_j$ be given by x,y and let us form $G\backslash e=H$. In H, $\deg_x=\deg_y=2$, with $\deg_i=3$ elsewhere. But each $C_i\backslash e$ and $C_j\backslash e$ is a path from x to y. Let us call these paths P_i and P_j where each consists of $p/2+1$ vertices. Recall that these paths are internally disjoint. But H is a graph where the vertex degree is 3 everywhere other than at x and y and thus every internal vertex (ones other than x and y) in P_i is matched to a similar internal vertex in P_j.

The only way this can be done and preserve planarity (recall that e is in G) is as follows:



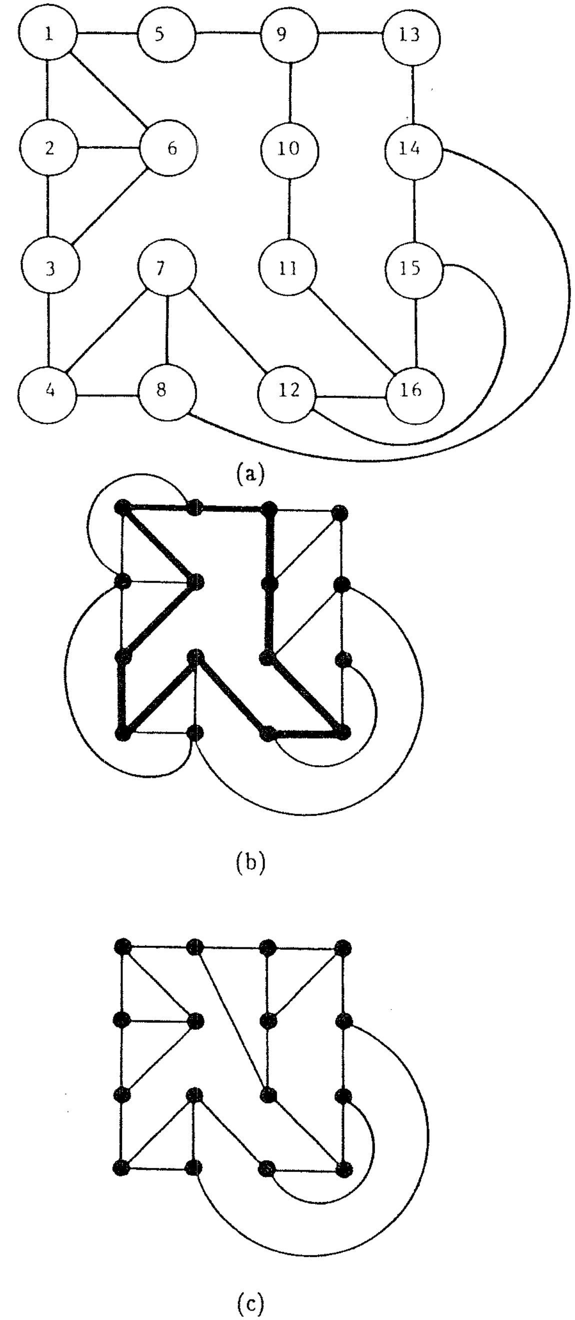


Figure 4. Sample graphs

But $G \backslash C_i$ removes P_i from H and $G \backslash C_j$ removes P_j accordingly, leaving in each case trees T_i and T_j which are isomorphic. This completes the proof.
□

As an illustration, consider G in Figure 5. There are three Halin bipartitions induced by cycles C_1, C_2, and C_3 which are indicated in bold. It is also easy to see that the value of t in the statement of the above theorem is bounded by 4. This follows since the requirement that every pair C_i and C_j have an edge in common corresponds to a complete graph K_t in the planar dual of G. Note that it is also easy to construct nonisomorphic Halin graphs on a vertex set V and which have equivalent cycle lengths.

3.2 Edge-Coloring on Halin Graphs

Recall that the chromatic index of a graph is the least number of colors needed to color the edges of the graph so that no two edges sharing a common vertex have the same color. If the maximum vertex degree in the graph is Δ then certainly the chromatic index must be at least Δ. But Vizing [V64] proved that it is never more than $\Delta + 1$; however, in general, it is hard to decide which value is correct. Moreover, graphs which always require Δ colors are referred to as class 1 graphs while those graph classes with members possibly requiring $\Delta + 1$ are called class 2. Our result is that Halin graphs are class 1 in that Δ is always the correct value. We also show how to create the coloring.

Consider first an easy lemma.

Lemma 3.3: Let $G = (V, T \cup C)$ be a Halin graph and let H be a hamiltonian cycle in G. Then the graph resulting from the removal of H from G is a forest.
Proof: Assume otherwise. If G' is the graph formed by the removal of H, then any cycle in G' corresponds to vertices in G each with degree at least 4. But then such vertices in G could not be part of C and are then interior in T which is a contradiction.
□

We now can establish the following.

Theorem 3.4: Let G be Halin with maximum degree Δ. Then G has chromatic index Δ.
Proof: We shall prove the result in two parts. For the first, let us assume that the number of vertices in G is even and suppose we have an existing hamiltonian cycle in G say H. Removing H produces G' which is, from the previous lemma, a forest and hence is bipartite. But every bipartite graph is class 1 and colorable with exactly $\Delta - 2$ colors in this case. The remaining two colors can be used to correctly color H since H has an even number of vertices.

Now, assume G has an odd number of vertices. As before let H be a hamiltonian cycle in G. There must then exist an edge $(x, y) \in C \backslash H$ since H is hamiltonian. But then since x and y are degree-3, there is an edge $(y, z) \in H$. Let us color all but (y, z) in H using two colors. This leaves $\Delta - 2$ colors overall. Again, removal

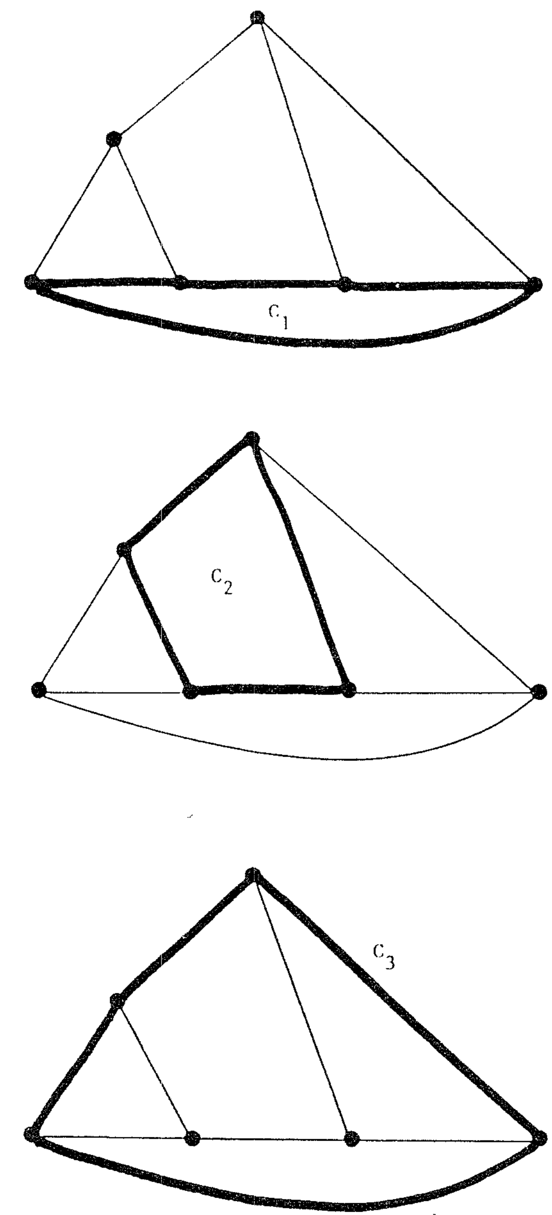


Figure 5. Alternative Halin bipartitions

of H from G yields a forest which we can color with these $\Delta - 2$ colors with the exception of edge (x, y). Note that (x, y) is an isolated component since x and y are degree-3. Now, there also exists an edge $(z, w) \notin H$. But (z, w) is colored as part of the forest. Let us color (x, y) with this color which is admissible since (x, y) is isolated. We are now left with only (y, z). But the other edges incident to y and those incident to z have been colored with 3 colors (2 from H and 1 for the (x, y) and (z, w) color) leaving $\Delta - 3 \geq 1$ colors to choose from for (y, z).
□

The graph in Figure 6 illustrates the coloring for the (more interesting) second part of the proof.

4 HALIN SUBGRAPHS AND SUPERGRAPHS: COMPLEXITY RESULTS

As suggested earlier, existing results have established that the underlying recursive structure of Halin graphs is sufficient to guarantee that most interesting graph problems can be solved accordingly. But this also holds, for a host of other recursive structures as well. As a consequence, it has been intimated in the literature that a potential use for the rich solvability status of recursive graphs is in the role of approximation. That is, given a graph G not known to be in such a class (or any well-solved class), how reasonable is it to create an approximation for G which is, then produce the (existing) solution on the latter which in turn, leads to a candidate solution for G? In [CNP83] this notion was entertained for the traveling salesman problem whereby Halin subgraphs were projected as the approximating graphs. In this section, we show that using Halin graphs in such a context might be problematic. We begin with a subgraph approximation case.

4.1 Halin Subgraphs

Let us state our problem in the following way:

P_H^-: Given a graph $G = (V, E)$ and an integer t, does G have a Halin subgraph $G^- = (V^-, E^-)$ where $V^- \subseteq V, E^- \subseteq E$ and $|E^-| \geq t$?

We have:

Theorem 4.1: P_H^- is NP-Complete.
Proof: Our reduction is from the longest cycle problem restricted to planar graphs.

Its statement is given below:

P_C: Given a planar graph $G = (V, E)$ and an integer k with $|V| \geq k \geq 3$, does there exist a cycle in G of length at least k?

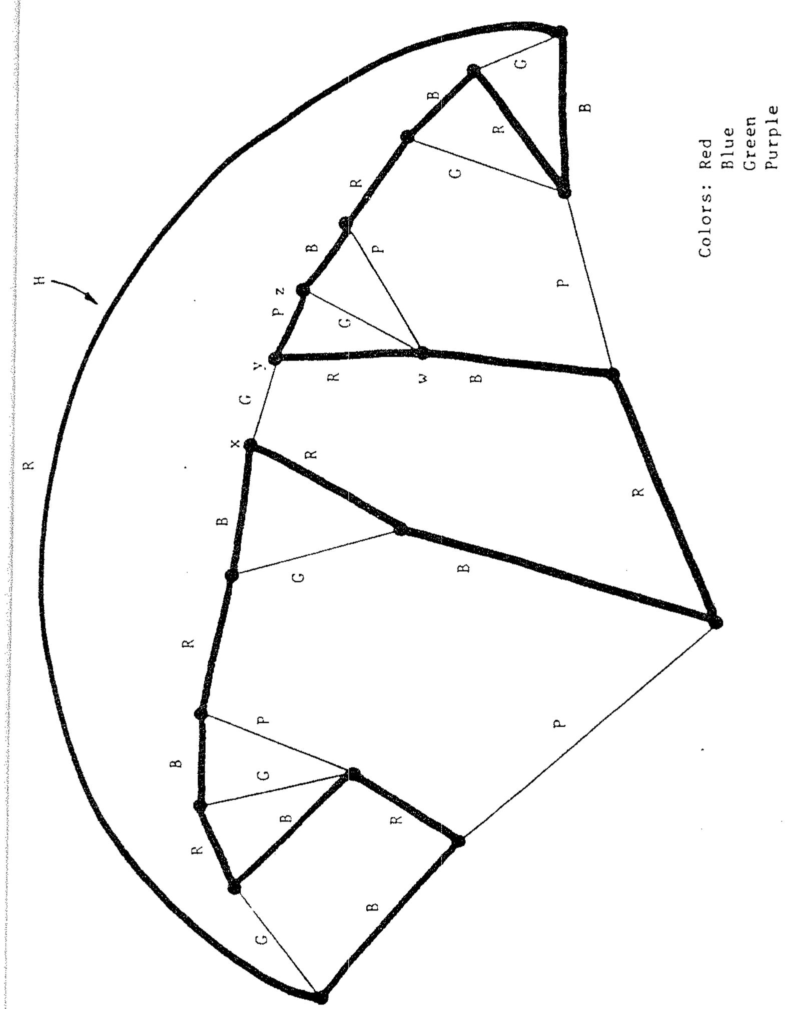


Figure 6. Edge-coloring from Theorem 3.4

From P_C we shall create an instance of P_H^- as indicated in Figure 7a. First, on every edge of G from P_C we insert a single vertex. Then, to this homeomorph, we add a single "supervertex," v_x which is connected to every other vertex. Set $t = 4k$.

Letting the constructed instance graph for P_H^- be $G' = (V', E')$, we have

$$
\begin{aligned}
V^{\bullet} &\triangleq \text{vertices inserted on edges in } E \\
V' &\triangleq V \cup V_{\bullet} \cup \{v_x\} \\
E^{\bullet} &\triangleq \text{edges formed by the subdivision induced by } V^{\bullet} \\
E_x &\triangleq \{(v_x, j) | j \in V'\} \\
E' &\triangleq E^{\bullet} \cup E_x
\end{aligned}
$$

Observe that $|V^{\bullet}| = |E|, |V'| = |V|+|E|+1, |E^{\bullet}| = 2|E|, |E_x| = |V|+|E|$, and $|E'| = 3|E| + |V|$.

Now suppose there exists a cycle in G of length at least k. By construction, a cycle of length k in G implies a cycle of length $2k$ in G', since each edge in E was split to form two $E^{\bullet}$ edges (identifiable with one $V^{\bullet}$ vertex). So this cycle in G' passes through $2k$ vertices, each of which is connected to v_x by an edge in E_x. These $2k$ edges form a star on $2k+1$ vertices with v_x at the "hub." Adding the $2k$ cycle edges produces a wheel and hence the desired Halin subgraph of size $4k$. The construction is demonstrated in Figure 7b ($k = 4$).

Conversely, let us assume there exists a Halin subgraph of G', denoted as $G^- = (V^-, E^-)$ where $V^- \subseteq V'$ and $E^- \subseteq E'$ and with edge cardinality at least $4k$. Observe that each edge in E' must be either a *cycle* edge in E^-, a *tree* edge in E^-, or *out* of E^-. In addition, we will abuse the terminology somewhat by referring to a "pendant" in a Halin graph when what we really mean is a degree-1 vertex in the tree portion of such a graph.

Now, suppose v_x is not in V^-. Then none of the E_x edges can be in the stated subgraph which leaves each $V^{\bullet}$ vertex in G' incident to only two edges. But since no vertex in a Halin graph can have degree less than three, these vertices and the edges incident to them (edges in $E^{\bullet}$) also cannot be in the stated subgraph. This leaves no edges at all, so no Halin graph is even possible, contrary to our assumption. Hence, $v_x \in V^-$.

Next we establish that v_x is not a pendant. Suppose otherwise. Then the hypothesized *cycle* would pass through v_x, implying that exactly two of the edges in E_x are *cycle edges*, and exactly one edge in E_x is a *tree* edge. But by the size of E^- and the bound on any cycle length from Lemma 3.1, we know that the assumed *cycle* must pass through at least one $V^{\bullet}$ vertex that is not adjacent to v_x by a *cycle* edge. Then, the remaining edge incident to that $V^{\bullet}$ vertex must be a *tree* edge in E^-. But such an edge connects the stated $V^{\bullet}$ vertex directly to v_x which contradicts the degree requirement on v_x and denies that G^- is Halin (see Figure 8a). Therefore, v_x cannot be a pendant, and thus has no *cycle* edges incident to it.

Now in G', v_x is connected to every other vertex by an edge in E_x. Since $v_x \in V^-$, each of these E_x edges in E^- is a *tree* edge connecting v_x to another vertex y in G'. There are four possibilities: (a) y is in V and is a pendant, (b) y is in V and is not a pendant, (c) y is in $V^{\bullet}$ and is a pendant, or (d) y is in $V^{\bullet}$ and is not a pendant. Following, we examine cases (b) and (d) and show that they are not possible.

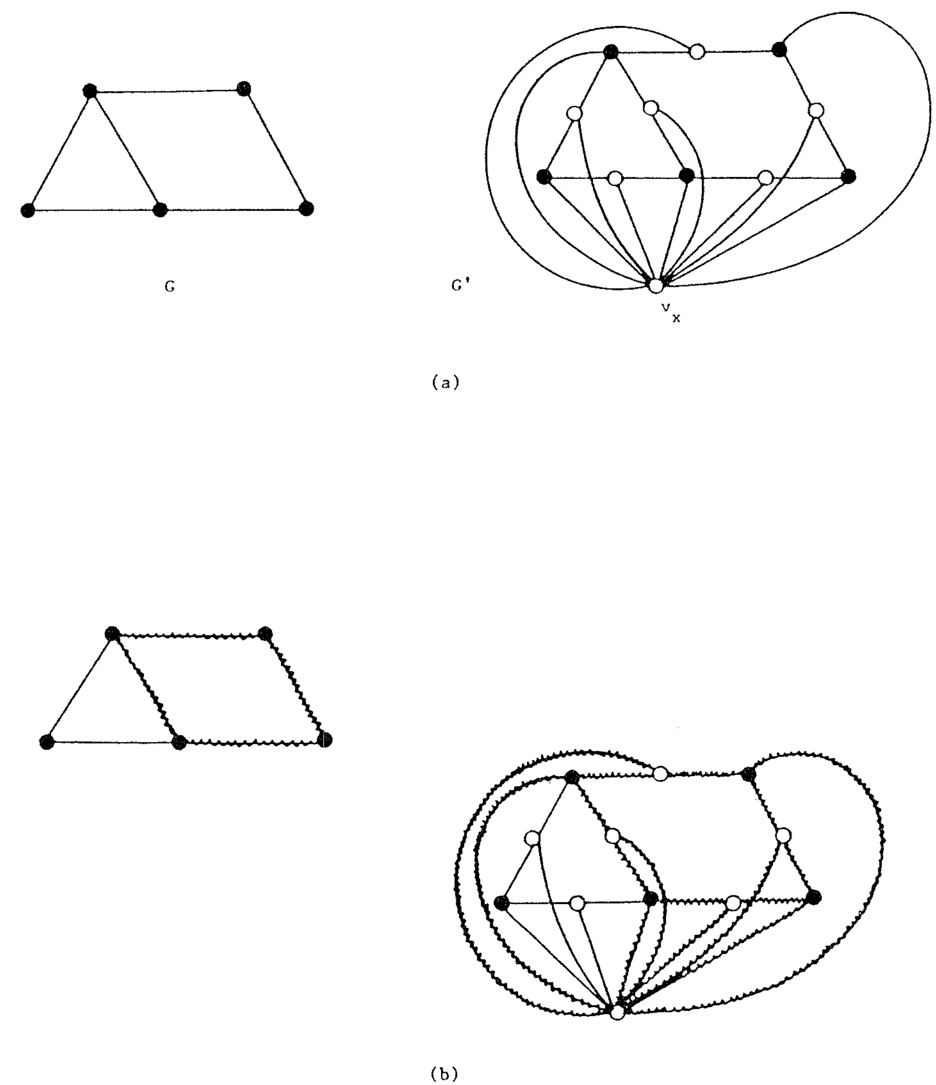


Figure 7. Construction for Theorem 4.1

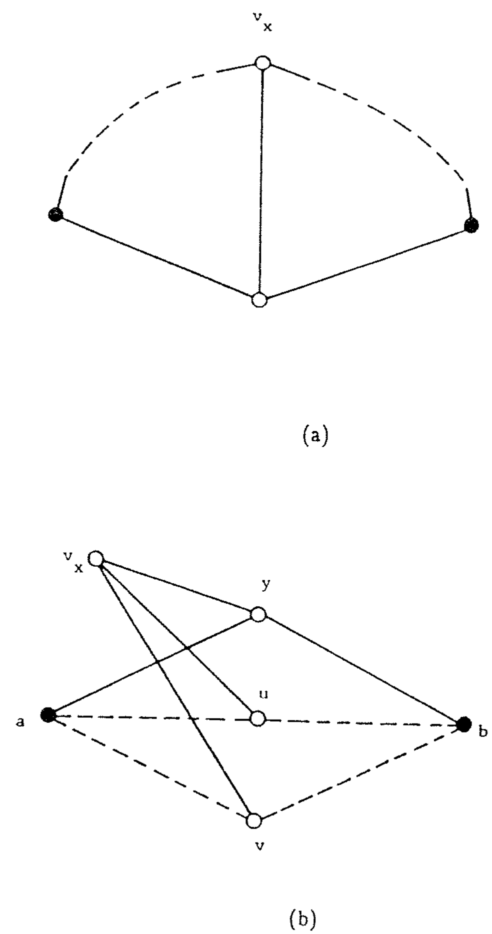


Figure 8.

- Case (b). Since we suppose that y is not a pendant, at least two $E^{\bullet}$ edges incident to y must also be *tree* edges. Both of these edges lead to degree-3 vertices in $V^{\bullet}$. Pick one of these vertices, say p. If p is a pendant, then both of the other two edges incident to it must be *cycle* edges. But one of these edges leads back to v_x which contradicts that v_x is not a pendant. If p is not a pendant, then both of the other two edges incident to it are *tree* edges, but the one that leads back to v_x forms a cycle with the edges selected thus far and G^- could not be Halin. Therefore, case (b) is impossible.

- Case (d). Since we suppose that y is not a pendant and y is in $V^{\bullet}$, both of the other two edges incident to y must also be *tree* edges. Let these two edges be incident to vertices a and b in V. Now suppose a and b are both pendants. Then the *cycle* passes through both a and b. This defines two edge-disjoint simple paths from a to b that do not pass through non-pendant vertices v_x and y. Further, since a and b are both in V, there must be at least one vertex in $V^{\bullet}$ on each of these paths. Let any $V^{\bullet}$ vertex on one path be u, and any $V^{\bullet}$ vertex on the other path be v. Now, since u is of degree three in the subgraph, the *non-cycle* edge incident to u (which leads back to v_x) must be a *tree* edge. The same is true for vertex v. However, this forms a subgraph homeomorphic to $K_{3,3}$, with bipartition $\{\{v_x, a, b\}, \{y, u, v\}\}$ (see Figure 8b). Similarly, if one or both of a and b are not pendants, then the additional *tree* edges required to connect them to pendants will again produce a $K_{3,3}$ homeomorph when added to the previous construction. In either case we deny planarity in the hypothesized subgraph so case (d) is also impossible.

Having chosen vertex y arbitrarily, we may conclude that every vertex in the hypothesized Halin subgraph with the exception of v_x must be a pendant vertex. But this can occur only with the tree edges forming a star having hub v_x. Clearly, such a (Halin) graph has exactly half of its edges in the *cycle* and half in the *tree*, all of the latter incident to v_x. Since we have supposed the existence of a Halin subgraph with $|E^-| \geq 4k$, we know that at least $2k$ edges must form the *cycle* in G^-. This cycle is represented by a vertex sequence alternating between vertices in V and those in $V^{\bullet}$. But in the construction of G', each edge of G was "split" into two edges by the insertion of a $V^{\bullet}$ vertex. Thus, the *cycle* portion of G^- having at least $2k$ edges corresponds exactly to a cycle in G of length at least k.

As indicated, it is easy to test if a graph is Halin so P_H^- is in NP and the result of the theorem follows.

□

An easy corollary results by fixing k in the above theorem. Letting $k = |V|$ we have:

Corollary 4.2: Deciding if $G = (V, E)$ possesses a *spanning* Halin subgraph is NP-Complete.

□

It is worth pointing out that results similar to Theorem 4.1 exist for other classes of graphs. Most of these are familiar in the context of so-called edge and/or vertex deletion problems among which are the results reported in Yannakakis [Ya78,Ya81a,b] and in particular, in Asano [As87] where a result analogous to Theorem 4.1 for series-parallel graphs (holding even on planar instances) is given. In this latter regard, we mention that the same result was obtained independently and reported in [Ho91].

4.2 Halin Supergraphs

Especially in view of the aforementioned references, subgraph decisions often exhibit a complexity status that is not unexpected; many are NP-Complete. On the other hand, corresponding questions regarding supergraph constructions appear to be more interesting (complexity status notwithstanding). Typical of these is the (NP-Complete) ***Hamiltonian Completion*** problem [GJ79]: Given $G = (V, E)$ and $k < |V|$ does there exist a superset $E' \subseteq E$ such that $|E' \backslash E| \leq k$ and $G' = (V, E')$ is Hamiltonian?

There are, of course, uninteresting versions of the supergraph problem. For example, if a graph is not series-parallel (indeed, not a partial k-tree for some k) then adding edges cannot make it so. This doesn't carry over for the Halin graph case however (see the graphs in Figure 4).

Let us state our supergraph problem as follows:

P_H^+: Given a graph $G = (V, E)$, does there exist a set of edges $E^+ \subseteq E$, such that the supergraph $G^+ = (V, E^+)$ is Halin?

(Equivalently: Is G a subgraph of any Halin graph?) Recall that we may assume that G is not Halin since testing for this property is easy. We may also assume that G is not 3-connected, following from the property that Halin graphs are necessarily 3-connected and minimal in this regard.

Our next result shows that resolving P_H^+ is no easier than the previous, subgraph version.

Theorem 4.3: P_H^+ is NP-Complete.
Proof: We will establish a reduction from the (strong sense) 3-Partition problem the statement of which appears below:

P_{3P}: Given a set A of $3m$ elements, an integer bound B, and an integer size $s(a)$ for each $a \in A$ such that $B/4 < s(a) < B/2$ and where $\Sigma_{a \in A} s(a) = mB$, can A be partitioned into m triples $A_1, A_2, \ldots, A_m$ such that, for $1 \leq i \leq m$, $\Sigma_{a \in A_i} s(a) = B$?

(Observe that each A_i must contain exactly three elements from A).

From an instance of P_{3P} let us create an instance of P_H^+ as indicated in Figure 9. Attached to vertex t are "tails" which correspond to the elements in A with the length of each tail related to the size of the respective element. The upper part of the graph and in particular the m "segments" correspond to the desired sets $A_1, A_2, \ldots, A_m$, each with size B where the latter is denoted by the B darkened vertices inserted within each segment. Segments are connected by the large, open vertices as shown. For ease, we shall use the terms "intrasegment" vertex and "intersegment" vertex to denote these vertices. Clearly, the graph in the figure is not Halin nor is it 3-connected.

For the first direction, assume that a suitable partition of A exists. Accordingly, we can construct a "Halin completion" of G in the following manner.

For each A_i, place the three relevant tails in an interior face bounded by intersegment vertices, intrasegment vertices, and t (we call each of these faces a "sector"). From each pendant vertex of a tail create two edges from the pendant to an adjacent pair of intrasegment vertices. For each (if any) other (degree-2) vertex on a tail, create one edge from the tail to an intrasegment vertex. In this way, we add $\Sigma_{a \in A_i}(s(a) = 1) + 3 = B$ edges. It is easy to see that planarity is maintained.

The construction is now complete yielding a Halin graph $H = (V, E_H)$ with E_H defined by E augmented with the new edges just described; the *cycle* edges are those defining the face denoted by f, say E_f, and the tree edges are given by $E_H \backslash E_f$. Figure 10 demonstrates the construction.

Conversely, suppose there exists a Halin completion of G, say $G^C = (V, E^C)$. Let $E^C = E \cup E'$. Now, vertex t must be a non-pendant vertex, since its degree in G exceeds 3 (we avoid trivialities in the statement of P_{3P}). Thus, edges x and y (in Figure 9) are *tree* edges implying that edge c is a *cycle* edge. Accordingly, the intersegment vertices incident to edge c must be pendants, which implies that edges a and b are cycle edges as well. But this means that the *cycle* subgraph in G^C is either defined by face f of G or a subface of f created in G^C. Thus, the tails of G must be part of the *tree* in G^C. Observe also that the upper portion of G (in Figure 9) inscribed by edges x, y, and c is 3-connected and we therefore may assume an embedding of it as shown in Figures 9 and 10.

If the number of cycle edges in $G^C = k$, we must have that $k \leq dim(f) \leq m(B+1)+1$, where $dim(f)$ denotes the dimension or number of edges defining face f. Also recall that k has a natural lower bound:

$$k \geq |V|/2 + 1 = (2(m(B-1)+1))/2 + 1 = m(B-1) + 2.$$

But the vertices in the tails of G have degree less than 3 and therefore have total deficiency at least mB. This deficiency has to have been satisfied by edges attached to the tail vertices but not extending between distinct tails nor between distinct vertices of the same tail which in either case denies that the tails are part of a tree. Hence, $|E'| \geq mB$.

On the other hand, the intrasegment vertices in G all have degree 2 and thus a deficiency of mB as well. But in G we have $|E| = 2mB - m + 2$ and in any Halin graph of order p with a *cycle* of length k, we know there are $p + k - 1$ edges. In our construction, G^C must then have size $2(m(B-1)+1) + k - 1$. Letting θ be the number of edges to add to G to create G^C, it is easy to see that $\theta = k - m - 1$. Let us suppose that k is different from its upper bound of $m(B+1)+1$. Then $\theta < mB$ and E' cannot be formed as required. Hence, $k = m(B+1)+1$ and the *cycle* in G^C is defined explicitly by face f. Thus all the vertices on f are pendant vertices and their deficiency is mB. Exactly B of these are required in each sector face and these edges connect precisely those vertices of the tails which must be embedded in the sectors. Moreover, if any tail vertex is connected to a vertex in a given sector so must every other vertex in that tail since G^C is planar. Thus, every one of the m sectors has exactly three tails from G embedded within it with each connected by exactly B edges to the respective vertices on face f. But then each of these sector-tail embeddings forms a triple which corresponds to a suitable 3-partition of A.

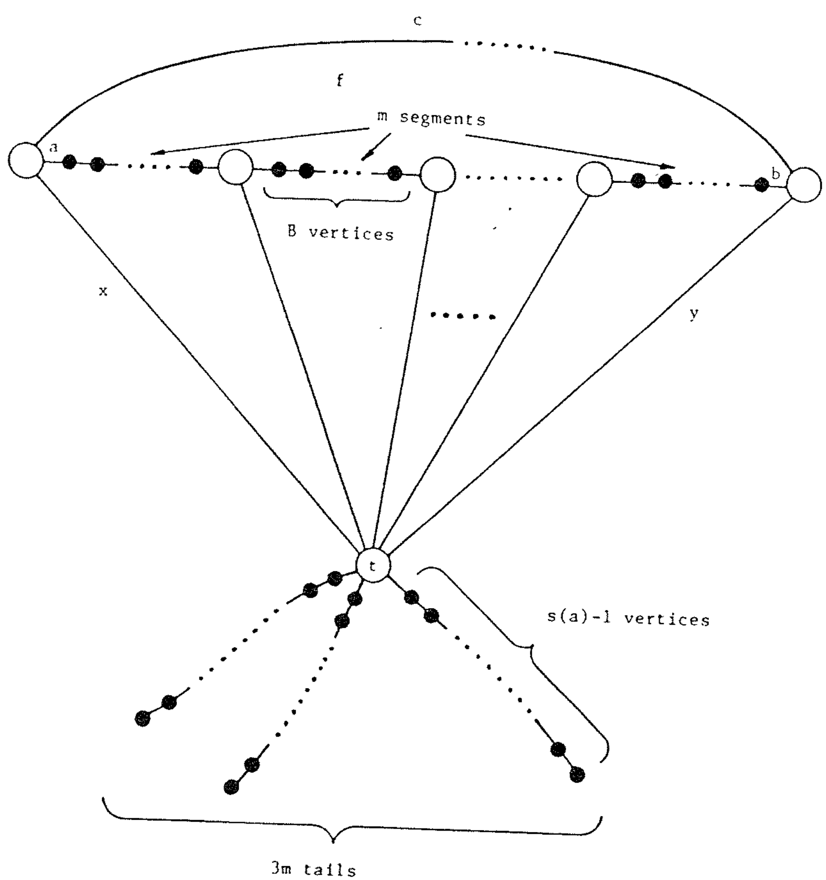


Figure 9. Construction for Theorem 4.3

The transformation from P_{3P} to form G is valid following the strong sense status of P_{3P}, which along with P_H^+'s inclusion in NP yields the desired result.
□

5 SUMMARY

The work reported in this paper reflects very little of the intent of our original research effort. Indeed, the primary aim in the latter was to examine the role of recursive graph classes in the context of approximation. As suggested at the beginning of Section 4, the recursive class of Halin graphs appeared to be an interesting model with which to proceed, largely because subgraph as well as supergraph contructions are possible for nonHalin structures.

The complexity results in Section 4 would suggest, however, that using Halin graphs in this fashion might be more complicated than one would initially be led to believe. Also, as indicated earlier, it does not appear that similar (recursive) structures such as partial 2-trees provide meaningful alternatives. Note that this pessimism is not directed at the aforementioned approximation notion in general, but rather, at the explicit role played by structures like Halin graphs in such a context.

As a theoretical matter, it would be interesting to strengthen the result of Theorem 4.1 by clarifying the issue of whether or not a graph possesses *any* Halin subgraph. On the other hand, from a practical perspective, this might be somewhat academic since, if there is any use for Halin graphs in the approximation sense described, our interest would be in large subgraphs in any event. Still, as a complexity matter, the result would be an important one from the viewpoint of completeness.

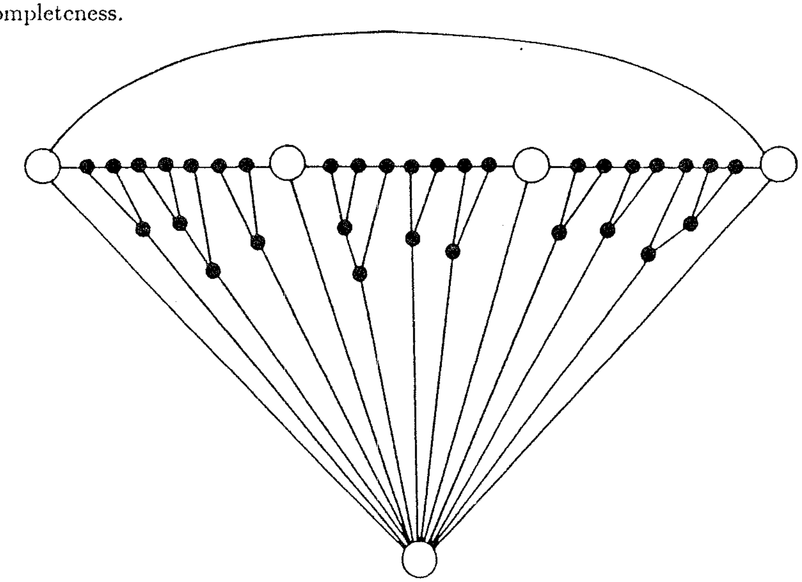


Figure 10. Halin completion

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