Research Statement

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Summary

This document describes my current research results and future plans. My first research program, *trace diagrams and their applications*, grew out of my doctoral thesis at the University of Maryland and is covered in §1. My second, *the mathematics of cooperation*, is funded by the Army Research Office and is covered in §2.

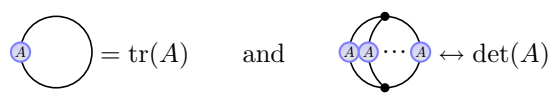
1 Trace Diagrams

1.1 The Theory of Trace Diagrams

My present work focuses on extending the theory of trace diagrams and on applying them to interesting mathematical questions. *Trace diagrams* are a certain kind of graph (with edges labeled by matrices) that can be identified with spaces of multilinear functions. This correspondence is their most useful feature, as diagrammatic proofs can be much simpler than traditional ones [MP]. Formally:

Definition 1. An *n-trace diagram* is an oriented graph with vertices of degree 1 or n such that vertices have cyclic orientations and are either sources or sinks.

Physicists introduced 2-trace as *spin networks* [Pen71, Smo97], and used them to study angular momentum. Quantized spin networks also relate to skein relations and knot theory [Kau91, BFKB99, Sik01]. The simplest closed trace diagrams are the basic matrix invariants, including

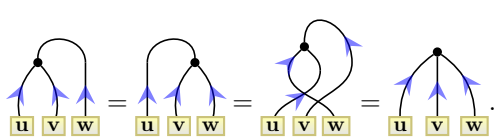


There are a few ways to describe the correspondence between trace diagrams and functions. A classical approach first describes the functions underlying pieces of the graphs, and then proves that every decomposition into these simple pieces gives the same function [CFS95, Kau91]. A more elegant approach is to describe the function as a weighted sum of n-colorings in which the colors meeting at a vertex are all different [Pet06, MP].

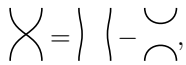
Computations in the diagrammatic category can be much easier. For example, the identities

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = \det(\mathbf{u}\,\mathbf{v}\,\mathbf{w})$$

follow from the following equivalence of graphs:



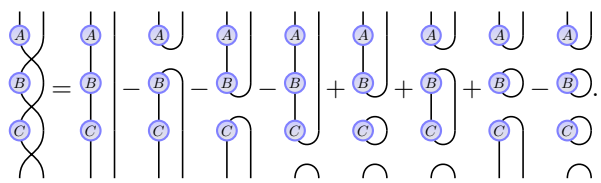
Further, the characteristic equation $X^2 - \mathrm{tr}(X)X + \det(X)I = 0$ for 2×2 matrices plays a role equivalent to the diagrammatic relation



which is usually called the *fundamental binor identity*. Consequently, finding trace relations often reduces to eliminating crossings from a diagram. For example, the relation

$$\mathrm{tr}(ABC) + \mathrm{tr}(ACB) = \mathrm{tr}(A)\mathrm{tr}(BC) + \mathrm{tr}(B)\mathrm{tr}(AC) + \mathrm{tr}(C)\mathrm{tr}(AB) - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C)$$

for three 2×2 matrices can be proven by connecting the top and bottom strands in the following relation [LPb]:



Future work on the theory of trace diagrams will involve (i) finding the fundamental relations that arise for particular values of n, (ii) further understanding how symmetry can be used to simplify calculations of the underlying functions, (iii) developing new tools for diagrams such as a "cut and paste" technique or diagrams for matrices of varying sizes, and (iv) understanding how to use representation theory to generalize the diagrams as in the case of spin networks. For partial work toward these goals, see [Pet06, Pet08, MP].

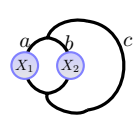
1.2 Application to Character Varieties

My doctoral thesis concerned the application of trace diagrams to character varieties. The work of Adam Sikora showed that spin networks were useful in this context, in particular due to their correspondence with the Kauffman Bracket Skein Module [Sik01]. I gave a new proof [Pet06, LP08] of the following classical theorem [Fri96, Vog89]:

Theorem 1 (Fricke-Klein-Vogt Theorem)**.** *Given $G = \mathrm{SL}(2, \mathbb{C})$, the set of regular functions on $G \times G$ which are invariant under simultaneous conjugation, meaning $f(g, h) = f(kgk^{-1}, khk^{-1})$ for all $k \in \mathrm{SL}(2, \mathbb{C})$, may be identified with the polynomial ring $\mathbb{C}[x, y, z]$, where $x = \mathrm{tr}(g)$, $y = \mathrm{tr}(h)$, and $z = \mathrm{tr}(gh^{-1})$.*

The proof uses the Peter-Weyl Theorem to express a certain coordinate ring in terms of a spin network basis, an idea first discovered by Baez [Bae96]. This theorem says a lot about the structure of a certain $\mathrm{SL}(2, \mathbb{C})$-character variety $\mathfrak{X}$ whose regular functions are those in the hypothesis [Gol03, Fri96], and it was hoped that the new proof would be more amenable to generalization. Indeed, in this new proof there are just two steps: find a linearly independent set $\{\chi_c^{a,b}\}$ in the coordinate ring $\mathbb{C}[\mathfrak{X}]$, and show that this set, consisting of what we call the *central functions*, spans $\mathbb{C}[x, y, z]$.

I showed that the functions $\chi_c^{a,b}$ may be represented diagrammatically by



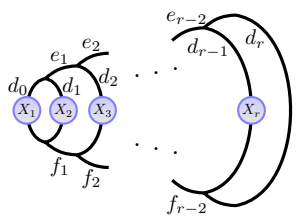
where $X_1, X_2 \in \mathrm{SL}(2, \mathbb{C})$, and used diagrammatic manipulations to prove:

1. The central functions $\chi_c^{a,b}(x,y,z)$ have a *three-fold symmetry* that relates the order of the indices (a,b,c) to that of the variables (x,y,z) [LP08].

2. There is a constructive algorithm that may be used to compute $\chi_c^{a,b}$ in terms of x,y,z, and thus a constructive proof of the Fricke-Klein-Vogt Theorem [LP08].

3. There is a formula for the product of two central functions; hence, one can write down the multiplicative structure on $\mathbb{C}[\mathfrak{X}]$ explicitly in terms of this natural additive basis [LP08].

Recent work extends this idea to the case of functions $G \times G \times \cdots \times G \to G$ invariant under simultaneous conjugation [LPa]. In this case, the corresponding coordinate ring $\mathbb{C}[\mathfrak{X}]$ is spanned by functions with the following diagrammatic representation:



Again, the diagrammatic approach simplifies computations with these basis functions. A particularly powerful result is the existence of a recurrence relation for each topological loop within the diagram [Pet09, LPa].

In future work, it may be possible to extend this work to other classical Lie groups. Toward this end, I have (i) described the Littlewood-Richardson rule for $\mathrm{SL}(n,\mathbb{C})$ in terms of diagrams, (ii) combined this with a diagrammatic description of the irreducible $\mathrm{SL}(n,\mathbb{C})$ representations [Ste90] to write down the central functions in the general case, and (iii) begun studying diagrammatic manipulations of general trace diagrams [Pet06].

1.3 Application to Invariant Theory

Trace diagrams also have application to invariant theory, which is closely linked to character varieties and has roots in Hilbert's 14th Problem. A generic goal of invariant theory is to describe algebras of elements that are invariant with respect to some action in terms of generators and relations. For matrix groups and the simultaneous conjugation action, generators and relations are known, but finding *minimal sets* of generators remains an open problem [Dre].

The $\mathrm{SL}(2,\mathbb{C})$ case is well-understood classically. However, trace diagrams have the potential to simplify the classification of trace relations required to list the minimal generators and relations. Bullock gave one step in this direction in the language of skein relations [Bul97]. My goal is to completely describe $\mathrm{SL}(2,\mathbb{C})$ trace relations using diagrams. I have shown that the most basic trace relations for $\mathrm{SL}(2,\mathbb{C})$ arise as simplifications of a family of diagrams [LPb]. The process in each case is to draw a diagram representing a trace such as $\mathrm{tr}(AB)$ with crossings, and then use the fundamental binor identity to eliminate the crossings. I am currently directing undergraduate student work on the final piece of this puzzle, a relation that arises from the Gram matrix.

The case of $\mathrm{SL}(3,\mathbb{C})$ is much harder, and in some sense represents the frontier of the field. Several papers have been published recently on this subject, even for the case of two matrices [Law07, ADS06]. The difficulty exists in part because 3-trace diagrams have been much less studied, and in part because the fundamental binor identity does not permit the removal of crossings. In

2008, my student Steven Morse outlined an algorithm for expressing arbitrary 3-trace diagrams in terms of a few simple traces of matrices, their inverses, and their transposes [Mor08]. This provides a partial characterization of $\mathrm{SL}(3, \mathbb{C})$ trace relations, and is still preliminary work. I currently have another student working to expand on this result.

1.4 Application to Linear Algebra

Diagrams are very good for representing traces, determinants, and matrix minors. I have used them to give elegant proofs of Cramer's Rule and the Cayley-Hamilton Theorem [Pet08]. Steven Morse found a diagrammatic proof of a Jacobi matrix theorem and the Dodgson condensation technique [Mor08, MP]. I believe they may also be used to generate new formulas for computing determinants, and am curious whether other notions such as Vandermonde determinants, the Gram matrix, or eigenvalues have nice diagrammatic descriptions. Exploring these ideas is in part just appreciating the beauty of mathematics, but may also lead to interesting applications. Understanding the Gram matrix diagram, for instance, may lead to a better understanding of invariant theory. These kinds of questions are very accessible for undergraduates since a first course in linear algebra is sufficient to begin research.

2 The Mathematics of Cooperation

My second area of research, funded by the Army Research Office, concerns the mathematical principles of cooperation in multi-agent systems. I am particularly interested in applications to pursuit-evasion games and network science.

2.1 The Mathematics of Cooperation

Over the past few years, I have extended the notion of a cooperative game, as introduced by von Neumann in the early 20th century [vN28], to more general situations. Classical cooperative game theory is well-suited to situations such as a group of corporations working together to raise their bottom lines, but cannot handle situations where altruism or loyalty is the motivation for cooperation. My generalization provides a framework allowing one to quantify these notions.

In my work with Chris Arney, I developed the notion of a *cooperative team game*, which consists of (i) a *team* T of *agents*, (ii) a different outcome $o(S)$ for each subset of agents $S \subset T$, and (iii) a valuation $v_R(o(S))$ of each outcome $o(S)$ by each subset of $R \subset S \subset T$ [AP08a, AP08b]. This valuation is referred to in economics as a *payoff* or *utility* function. In what follows, we will refer to it as simply $v_R(S)$, which is the utility to R of the outcome in which the players in S participate.

Within this framework, we define [AP08a, AP08b]:

- The *altruistic contribution* of a subset $A \subset T$ to the team is $a(A) \equiv v_{A^c}(T) - v_{A^c}(A^c)$, or the difference in utility to the complement A^c between the outcome when only the players in A^c participate and when all players participate.
- The *selfish contribution* of a subset $S \subset T$ to the team is $s(A) \equiv v_T(T) - v_{A^c}(T)$, which measures the difference in utilities to the entire team T and the complement A^c of the outcome when the entire team participates.

The summation $a(A) + s(A) = v_T(T) - v_{A^c}(A^c)$ corresponds to the classical notion of *marginal contribution* in cooperative game theory.

2.2 Application to Pursuit-Evasion Games

Pursuit-evasion games typically involve two teams with opposing goals. These goals vary, but in the most generic case a team of "pursuers" is trying to capture a team of "evaders". Finding the optimal paths of players in such situations leads to some beautiful differential equations when the number of players is small [Nah07]. However, there is no simple and applicable way to describe the optimal paths when the size of the teams is more than a few players [Isa65].

My research in this area has focused not so much on finding new algorithms for such games as on comparing existing algorithms and exploring the meaning of cooperation in these scenarios. I have built a highly dynamic and interactive Java simulation platform that allows the user to create and study a wide range of pursuit-evasion scenarios. The platform also allows the user to compare metrics of success with altruism, selfishness, and cooperation as defined above. We are using the platform to search for particularly cooperative behaviors, whether altruistic or selfish, and attributes that make players particularly altruistic or selfish. I am currently directing an undergraduate thesis focused on this question.

In future work, I plan to (i) investigate algorithms that optimize altruism or selfishness rather than utility, (ii) investigate and understand the role of specialization within this framework, and (iii) study the highly dynamic communications networks arising in pursuit-evasion games using the tools of network science and the cooperation framework.

2.3 Application to Graph Theory and Network Science

If the nodes of the graph correspond to the agents of a team, then one can interpret the "outcome" for a subset of vertices to be the subgraph formed from those vertices. In this case, graph invariants can frequently be used for the utility functions defined above, and so our framework provides a direct link between graph invariants and the mathematics of cooperation.

A good example to consider is the case of a weighted, directed graph. Suppose that the weighting represents the utility contribution of the source vertex to the sink vertex. Then, $v_R(S)$ can be expressed as a summation

$$v_R(S) = \sum_{r \in R, s \in S} w(s, r),$$

where $w(s, r)$ represents the weight of the edge from s to r (or 0 if there is no edge). In this case, the altruistic contribution of A reduces to $\sum_{a \in A, b \notin A} w(a, b)$, which is the "total outdegree" of the subset A, not counting edges within A. The selfish contribution reduces to $\sum_{a \in A, b \in T} w(b, a)$, which is the "total indegree" of the subset A, this time counting internal edges [AGP08].

One can also define the utility to be the size of the connected component(s) of a graph containing the given subset of vertices. In this case, one can classify those subsets of graphs that are completely altruistic, those that are completely selfish, and those that are mixed [AP08b].

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