THE $\mathcal{ART}$ OF LINEAR ALGEBRA

ELISHA PETERSON

1. Introduction

1.1. **Motivating Questions.**

- Is there a way to generalize the cross product?
- Is there a quick way to generate inner/cross product identities such as the triple product?
- Why are the trace and determinant functions so special?
- What is the "best" way to compute the determinant?
- Why do the trace and determinant show up in the characteristic polynomial?
- What's with this *duality* thing?

1.2. **Symmetries in Linear Algebra.** Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ represent vectors, and let $\mathbf{A}$ represent a matrix.

Symmetries:

- The *inner product* $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$ satisfies $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$;
- Since $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$, it is also true that $\langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{A}^T \mathbf{v} \rangle$.
- The matrix *trace* statisfies $\mathsf{tr}(\mathbf{A}) = \mathsf{tr}(\mathbf{A}^T)$;

Anti-Symmetries:

- The *cross product* satisfies $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$;
- The matrix *determinant* satisfies $\mathsf{det}[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = -\mathsf{det}[\mathbf{v}\ \mathbf{u}\ \mathbf{w}]$;
- The *triple product identity* relates these two constructions:

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle = \mathsf{det}[\mathbf{u}\ \mathbf{v}\ \mathbf{w}].$$

1.3. **Tensor Algebra.** A *tensor product* of two vector spaces consists of pairs of elements such that

$$(\lambda \mathbf{v}, \mathbf{w}) = \lambda(\mathbf{v}, \mathbf{w}) = (\mathbf{v}, \lambda \mathbf{w})$$

where $\lambda \in \mathbb{C}$. It is usually written $\mathbf{v} \otimes \mathbf{w}$.

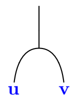
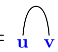
A *multilinear* function (one which is linear in each factor) can be thought of as a function on a tensor product space:

$$\langle \lambda \mathbf{u}, \mathbf{v} \rangle = \lambda \langle \mathbf{u} \times \mathbf{v} \rangle$$
$$(\lambda \mathbf{u}) \times \mathbf{v} = \lambda(\mathbf{u} \times \mathbf{v})$$
$$\mathsf{det}[(\lambda \mathbf{u})\ \mathbf{v}\ \mathbf{w}] = \lambda \mathsf{det}[\mathbf{u}\ \mathbf{v}\ \mathbf{w}].$$

So we could write $\cdot(\mathbf{u} \otimes \mathbf{v})$, $\times(\mathbf{u} \otimes \mathbf{v})$, and $\mathsf{det}(\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w})$ instead.

Date: December 5, 2007.

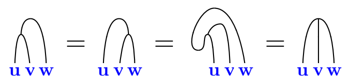
1.4. **3-Vector Diagrams.** Suppose the cross product and inner product are represented by

$\mathbf{u} \times \mathbf{v} =$  and $\langle \mathbf{u}, \mathbf{v} \rangle =$ .

Exercise 1. How can you draw the identity

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \times \mathbf{t} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle \langle \mathbf{v}, \mathbf{t} \rangle - \langle \mathbf{u}, \mathbf{t} \rangle \langle \mathbf{v}, \mathbf{w} \rangle ?$$

Exercise 2. What does this diagram represent?



1.5. **Goals of the Talk.**

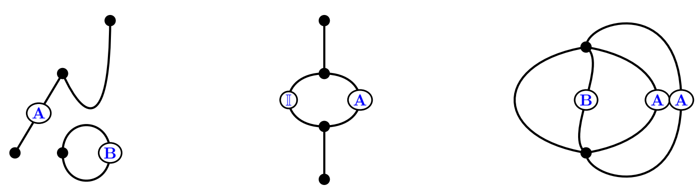
Goal 1. Describe how to create diagrams like these in *any* dimension, and how to translate them into traditional notation.

Goal 2. Be able to manipulate the diagrams topologically [as graphs], without worrying about what they represent. Make sure this manipulation doesn't change the underlying function.

2. Trace Diagrams

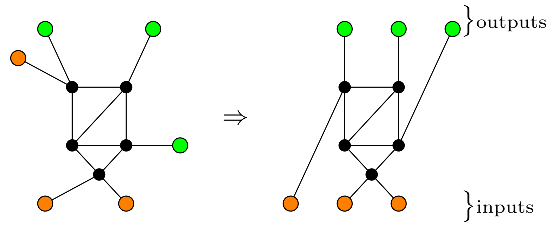
2.1. **What happens if...** Start with a graph with

- vertices of degree 1 or n;
- edges labelled by matrices,

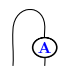
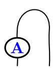


2.2. **Making it Work: Inputs and Outputs.**

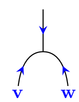
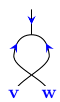
- Functions have inputs and outputs, whereas diagrams have degree 1 vertices.
- Partition these "leaves" into inputs and outputs.
- By convention, inputs are at the bottom, outputs at the top.
- Function is from $V^{\otimes |I|} \longrightarrow V^{\otimes |O|}$, where $V = \mathbb{C}^n$, $|I|$ is the number of inputs and $|O|$ the number of outputs.
- If there are no inputs, the domain is the scalars $V = \mathbb{C}^0 = \mathbb{C}$.



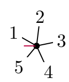
2.3. The Subtleties.

Problem 1. According to earlier definitions,  $= \langle \mathbf{u}, \mathbf{A}\mathbf{v} \rangle = \langle \mathbf{A}^T\mathbf{u}, \mathbf{v} \rangle$ while  $= \langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle$. [They are the same graph.]

- **Solution:** Orient the edges; assume all vertices are *sources* or *sinks*.

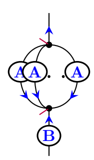
Problem 2.  $= \mathbf{v} \otimes \mathbf{w}$ and  $= \mathbf{w} \otimes \mathbf{v} = -\mathbf{v} \otimes \mathbf{w}$ are the same graph.

- **Solution:** Draw a "ciliation" on the graph to specify the order:

 implies the ordering .

2.4. The BIG Theorem.

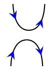
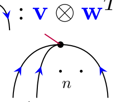
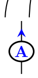
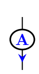
Definition 1. An *n-trace diagram* is an oriented graph with edges labeled by $n \times n$ matrices whose vertices (i) have degree 1 or n only, (ii) are sources or sinks, and (iii) are ciliated.



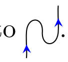
Theorem 1. *The function of a trace diagram is well-defined; in particular, every decomposition into simpler maps gives the same function.*

2.5. Making it Work: Translation.

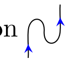
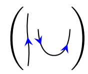
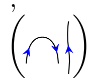
Let $\mathbf{v}$ represent a vector in $\mathbb{C}^n$, and let $\mathbf{A}$ represent an $n \times n$ matrix.

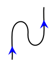
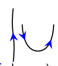
- **Identity Rule**: $: \mathbf{v} \mapsto \mathbf{v}$;
- **Cup Rule**:  $: 1 \mapsto \hat{\mathbf{e}}^{\mathbf{1}} \otimes \hat{\mathbf{e}}_{\mathbf{1}} + \cdots + \hat{\mathbf{e}}^{\mathbf{n}} \otimes \hat{\mathbf{e}}_{\mathbf{n}}$;
- **Cap Rule**:  $: \mathbf{v} \otimes \mathbf{w}^T \mapsto \langle \mathbf{v}, \mathbf{w} \rangle$;
- **Vertex Rule**:  $: \mathbf{v_1} \otimes \cdots \otimes \mathbf{v_n} \mapsto \det[\mathbf{v_1} \ \cdots \ \mathbf{v_n}]$;
- **Matrix Rule**:  $: \mathbf{v} \mapsto \mathbf{A}\mathbf{v}$,  $: \mathbf{v}^T \mapsto \mathbf{v}^T\mathbf{A}$.

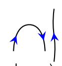
3. Some Practice

3.1. Example: Kinks. Problem. Compute the function corresponding to .

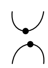
Solution.

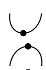
- Input and output are both $V = \mathbb{C}^n$;
- Use decomposition  =  $\circ$  to compute:

 : $\mathbf{v}$  $\longmapsto \sum_i \mathbf{v} \otimes \hat{\mathbf{e}}^{\mathbf{i}} \otimes \hat{\mathbf{e}}_{\mathbf{i}}$

 $\longmapsto \sum_i \langle \mathbf{v}, \hat{\mathbf{e}}_{\mathbf{i}} \rangle \hat{\mathbf{e}}_{\mathbf{i}}$

$$= \sum_i v_i \hat{\mathbf{e}}_{\mathbf{i}} = v.$$

3.2. The Binor Identity. Compute  for 2×2 matrices.

First, decompose it  =  $\circ$ .

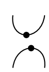
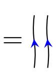
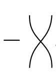
Second, use the fact that  : $1 \mapsto \hat{\mathbf{e}}_{\mathbf{1}} \otimes \hat{\mathbf{e}}_{\mathbf{2}} - \hat{\mathbf{e}}_{\mathbf{2}} \otimes \hat{\mathbf{e}}_{\mathbf{1}}$ to compute:

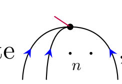
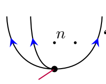
$\mathbf{v} \otimes \mathbf{w}$  $\longmapsto \mathsf{det}[\mathbf{v}\ \mathbf{w}] = v_1 w_2 - v_2 w_1$

 $\longmapsto (v_1 w_2 - v_2 w_1)(\hat{\mathbf{e}}_{\mathbf{1}} \otimes \hat{\mathbf{e}}_{\mathbf{2}} - \hat{\mathbf{e}}_{\mathbf{2}} \otimes \hat{\mathbf{e}}_{\mathbf{1}})$

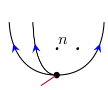
$= \mathbf{v} \otimes \mathbf{w} - \mathbf{w} \otimes \mathbf{v}.$

This proves the *binor identity*:

 =  − .

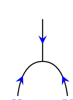
3.3. Caps and Cups. We already know how to compute , but what about ?

Proposition 1.

 $: 1 \mapsto \sum_{\sigma \in \Sigma_n} \mathsf{sgn}(\sigma) \hat{\mathbf{e}}_{\sigma(\mathbf{1})} \otimes \cdots \otimes \hat{\mathbf{e}}_{\sigma(\mathbf{n})}.$

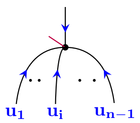
4. Answers to Questions

4.1. Generalizing the Cross Product. Given that the cross product in three dimensions is

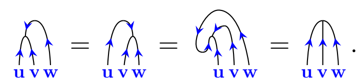
$\mathbf{u} \times \mathbf{v} =$ 

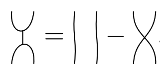
,

the natural extension to n dimensions is a product of $n-1$ vectors:

$$\mathbf{u_1} \times \cdots \times \mathbf{u_{n-1}} =$$ 

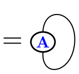
4.2. **3-Vector Identities.** The simplest identity is trivial:



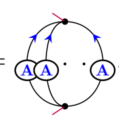
Four-vector identities depend on the relation .

4.3. **Importance of Trace and Determinant.** A *closed* diagram represents a function $\mathbb{C} \to \mathbb{C}$, or a function from a product of matrices to $\mathbb{C}$.

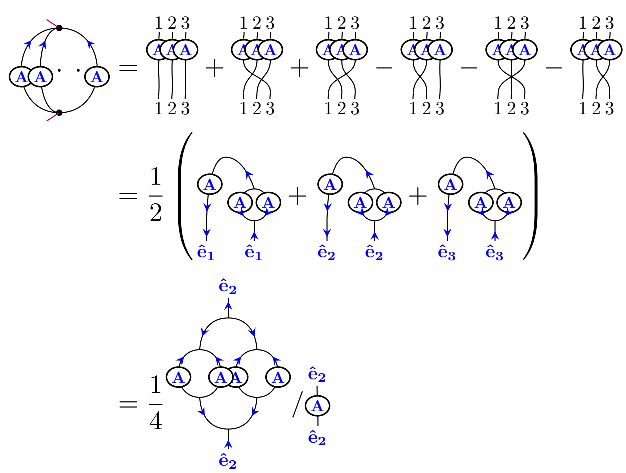
The diagrams for trace and determinant are the simplest closed diagrams:

$$\mathsf{tr}(\mathbf{A}) =$$ 

and

$$\mathsf{det}(\mathbf{A}) =$$ 

4.4. **Computing the Determinant.** Three techniques for computing the determinant:



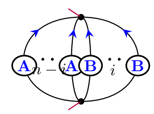
Which is the direct method? cofactor expansion? Dodgson condensation?

4.5. **The Characteristic Polynomial.** The easy answer to why $\mathsf{tr}(\mathbf{A})$, $\mathsf{det}(\mathbf{A})$ are part of the characteristic polynomial: they are the sum and product of the eigenvalues.

Diagrammatically, the n coefficients of the polynomial are the n "simplest" diagrams.

Expand $\mathsf{det}(\mathbf{A}+\mathbf{B})$ in terms of diagrams:

$$\mathsf{det}(\mathbf{A}+\mathbf{B}) = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}$$



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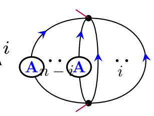
Applying to the case $\mathsf{det}(\mathbf{A}-\lambda\mathbb{I})$ gives:

$$\mathsf{det}(\mathbf{A}-\lambda\mathbb{I}) = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\lambda^i$$



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4.6. **Duality.** From a diagram's point-of-view:

- Inputs and outputs are "artificial";
- Switching orientations corresponds to transposing the whole calculation.

5. Concluding Remarks

5.1. **Application to Invariant Theory.**

- A similarity transformation of a matrix is $\mathbf{A} \mapsto \mathbf{BAB}^{-1}$ for some nonsingular matrix B.
- Both $\mathsf{tr}(\mathbf{A})$ and $\mathsf{det}(\mathbf{A})$ are invariant under this transformation: $\mathsf{tr}(\mathbf{A}) = \mathsf{tr}(\mathbf{BAB}^{-1}$ and $\mathsf{det}(\mathbf{A}) = \mathsf{det}(\mathbf{BAB}^{-1})$.
- Diagrams labeled by several matrices are invariant under this transformation *if the same matrix* $\mathbf{B}$ *is used for all transformations.*

One aspect of *Invariant Theory* is the classification of functions $M^{\times r} \to \mathbb{C}$ invariant under this simultaneous transformation. Often, the invariants can be linearly reduced to a few simple invariants.

For example, 2×2 matrices satisfy:

$$\mathbf{A}^2 = \mathsf{tr}(\mathbf{A})\mathbf{A} + \mathsf{det}(\mathbf{A})\mathbb{I},$$

and so

$$\mathsf{tr}(\mathbf{A}^2) = \mathsf{tr}(\mathbf{A})^2 + 2\mathsf{det}(\mathbf{A}).$$

For this reason, if there is only one matrix, $\mathsf{tr}(\mathbf{A})$ and $\mathsf{det}(\mathbf{A})$ are the simplest invariants.

Major Open Question: Achieve a complete understanding of all invariants of k $n\times n$ matrices (usually, restricted to either the *nonsingular* matrices or those with determinant 1).

5.2. **References.** Trace Diagrams for *any* Lie group:

- Predrag Cvitanovic, *Group Theory*, http://chaosbook.org/GroupTheory/