DYNAMIC MATHEMATICS AND PURSUIT-EVASION GAMES

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USMA D/Math, Center for Faculty Development, November 20, 2008

Overview

Goal: understand cooperation in the context of pursuit-evasion games

Davies Fellow Research with Dr. Chris Arney, Army Research Office and

- *CDT Andrew Plucker '08*
- *CDT Daniel Ball '08*
- *CDT Lucas Gebhart '09*

Cooperation in the Army

The American Army consists of teams known for their trust and autonomy… the teams may involve analysts, soldiers, computers, robots, sensors, vehicles, and more.

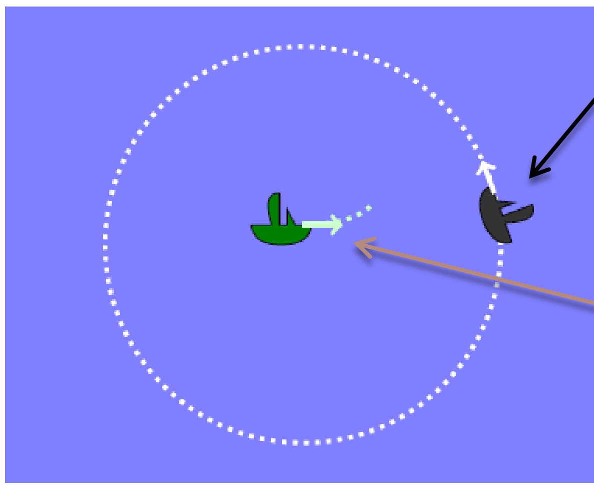
What is a Pursuit-Evasion Game?

Part I

USMA POTW #15, Spring 2008

Problem 15: Concentric Chase

Two boats are traveling in open water in circular patterns. One boat is traveling in a circle along the parametric curve $\mathbf{x}(t) = \langle 10\cos(\pi t), 10\sin(\pi t)\rangle$ (constant speed 31.41592653589793 knots). The second boat can travel at half that speed, and *always heads exactly towards the other boat.* Find the location of the two boats when $t = 200$.



$$\vec{x}_1(t) = \langle 10\cos(\pi t), 10\sin(\pi t)\rangle$$

$$\vec{x}_2{}'(t) = 5\pi \frac{\vec{x}_1(t) - \vec{x}_2(t)}{|\vec{x}_1(t) - \vec{x}_2(t)|}$$

(Note: assuming that the two boats are relatively close to begin with, you do not need to know the starting location of the second boat. A numerical solution is okay, but an answer will only receive full credit with some sort of analysis.)
This problem was inspired by my 1-year old son and a remote control Hummer.

http://www.youtube.com/watch?v=K8vuApTqIAk

Differential Equations

$$\vec{x}_2{}'(t) = 5\pi \frac{\vec{x}_1(t) - \vec{x}_2(t)}{|\vec{x}_1(t) - \vec{x}_2(t)|}$$

$$x_2{}'(t) = 5\pi \frac{x_1(t) - x_2(t)}{\sqrt{\left(x_1(t) - x_2(t)\right)^2 + \left(y_1(t) - y_2(t)\right)^2}}$$

$$= 5\pi \frac{10\cos(\pi t) - x_2(t)}{\sqrt{\left(10\cos(\pi t) - x_2(t)\right)^2 + \left(10\sin(\pi t) - y_2(t)\right)^2}}$$

$$y_2{}'(t) = 5\pi \frac{y_1(t) - y_2(t)}{\sqrt{\left(x_1(t) - x_2(t)\right)^2 + \left(y_1(t) - y_2(t)\right)^2}}$$

$$= 5\pi \frac{10\sin(\pi t) - y_2(t)}{\sqrt{\left(10\cos(\pi t) - x_2(t)\right)^2 + \left(10\sin(\pi t) - y_2(t)\right)^2}}$$

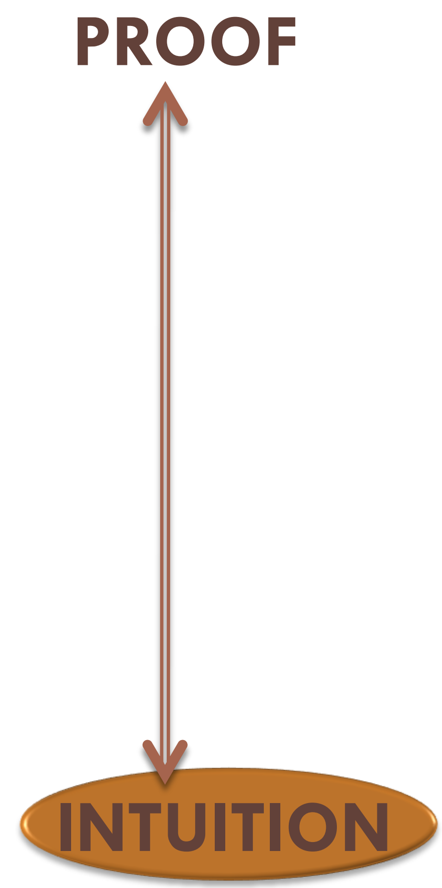
- http://www.dean.usma.edu/departments/math/people/peterson/blaise/circularchase/

What does a “Solution” to a pursuit-evasion game look like??

- Exact Solution
  - Best case
  - Some beautiful solution techniques
  - Not very robust!
  - “Impossible” outside of the simplest cases

- Approximate Solution
  - Hopefully close to the best case
  - Much easier to work with

- Any Solution
  - Good place to start
  - About the best you can do with highly complex games



PEG: a Platform for Visualization

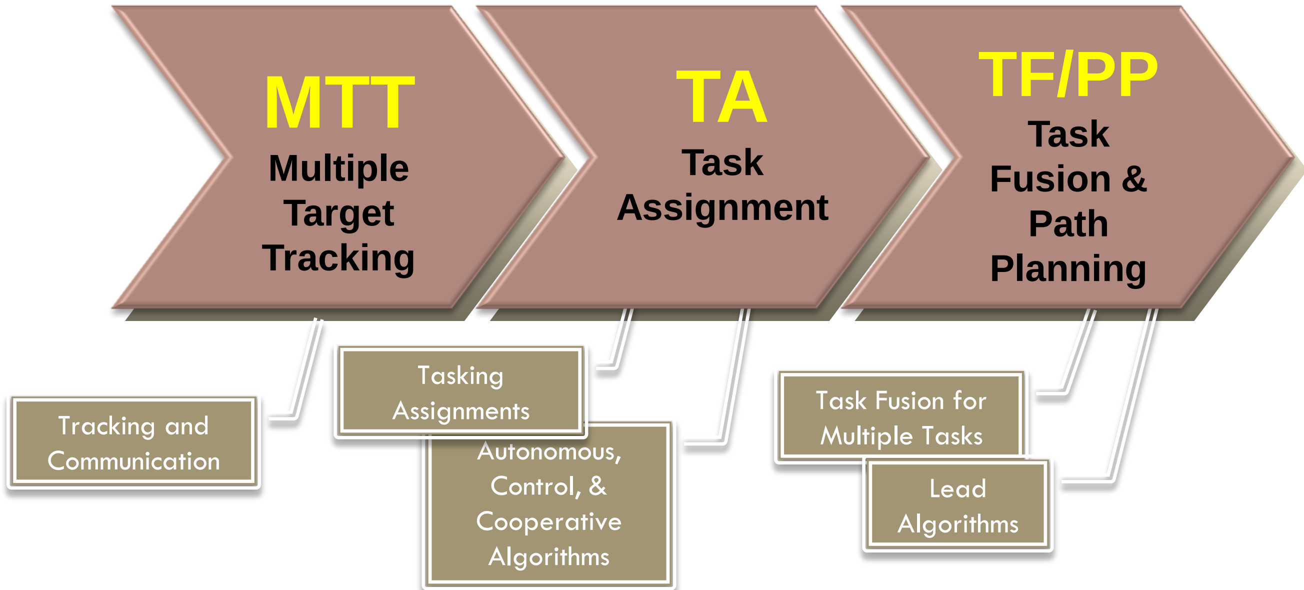
Part II

Pursuit-Evasion Platform

Using cooperation metrics to analyze and alter behavior

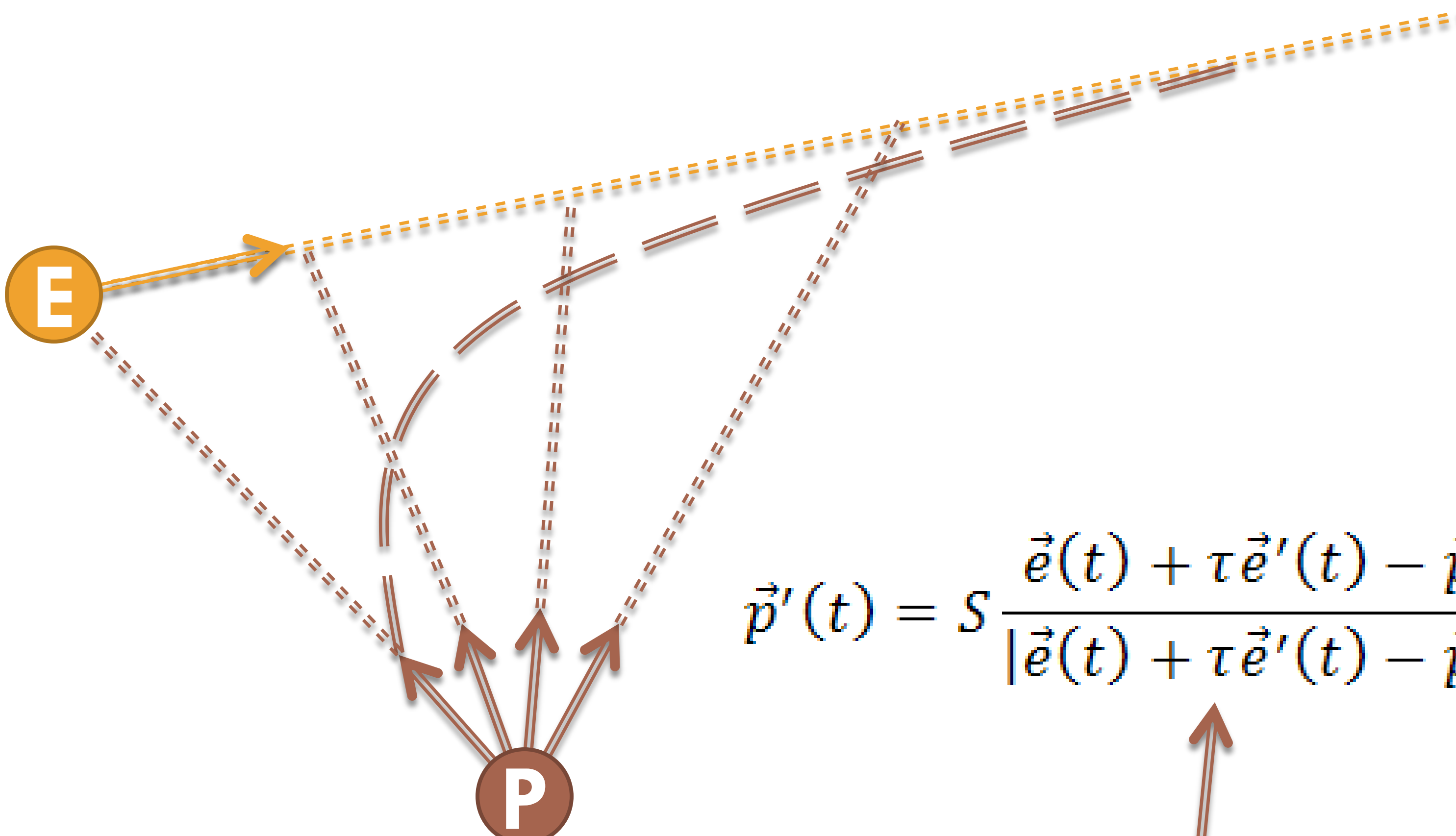
- **Approach:** Developed platform that is
  - ***Interactive*** (results of simulation immediately available)
  - ***Generic*** (supports several different behaviors, types of PEGs)
  - ***Comparative*** (between different behaviors, simulation parameters, etc.)

http://www.dean.usma.edu/departments/math/people/peterson/applets/peg2/peg2.html



Lead Factor Analysis

(CDT Andrew Plucker)

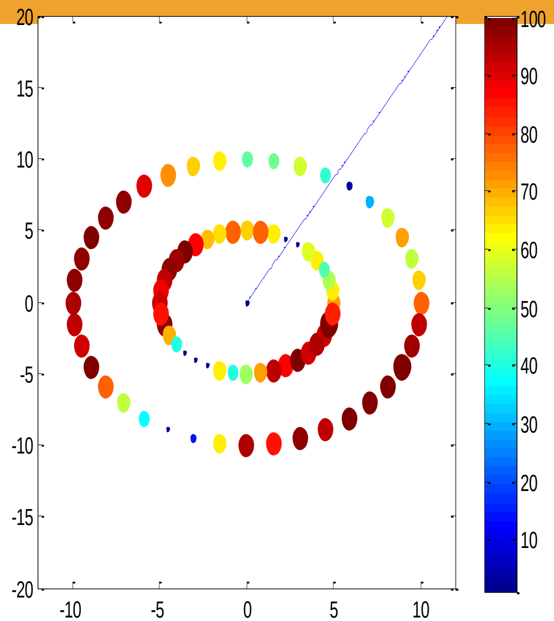


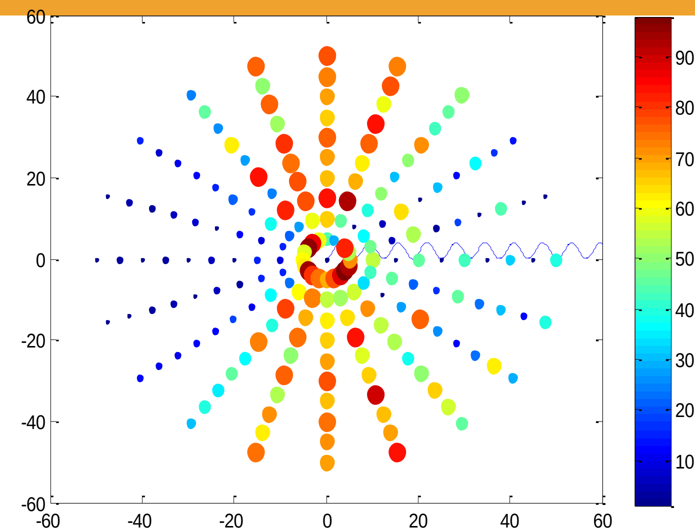
$$\vec{p}'(t) = S\,\frac{\vec{e}(t) + \tau\vec{e}'(t) - \vec{p}(t)}{|\vec{e}(t) + \tau\vec{e}'(t) - \vec{p}(t)|}$$

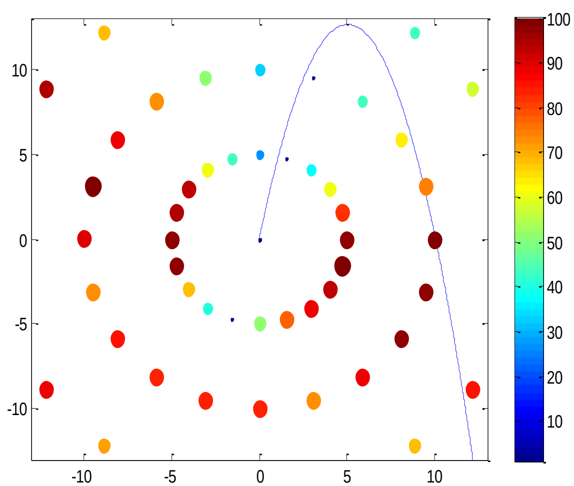
What value is best??

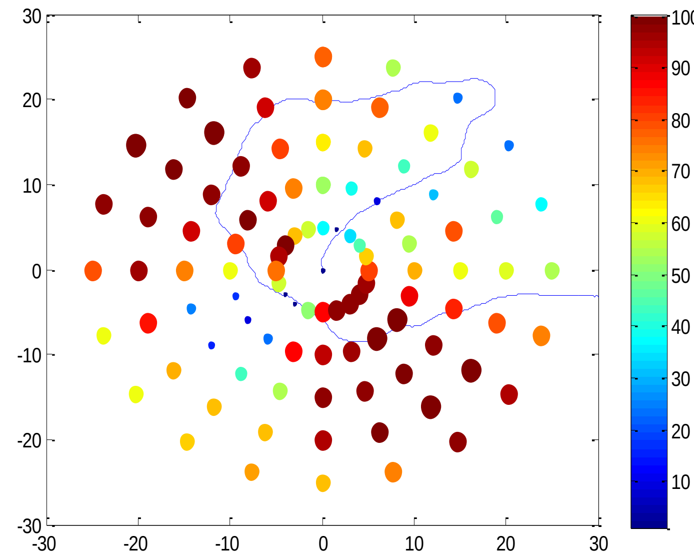
Lead Factor Analysis

(CDT Andrew Plucker)









Multiplayer Games: Task Handling

- Players need to be able to balance multiple goals
- **MA104 Problem:** *Where do you position a supply point to minimize the distance between several units?*
- **Corresponding Question:** *How do you get there?*

MA 104 Block III PSL: Multivariable Optimization Problems

Problem 1. In pursuit-evasion algorithms, the following function can be used as a measure of the "closeness" of two agents at positions (a,b) and (c,d):

$$g(a,b,c,d) = \left(1+(a-c)^2+(b-d)^2\right)^{-1}.$$

The equation is designed to approach one as the two agents approach the same position.

(a) Suppose that a velociraptor is at (a,b)=(2,3) and a goat at (c,d)=(x,y). Plot the function f(x,y)=g(2,3,x,y) in this case, and find the maximum and minimum values of g in the region $-5 \le x \le 5$ and $-5 \le y \le 5$. Does the result match your perceptions of closeness?

(b) Now, suppose the goat is at (3,0) and gets out his laptop to calculate the optimal direction to head in order to escape the raptor. In the middle of calculating the gradient, Mathematica starts beeping, and he is caught and eaten by the raptor. Hence there is no question to ask for part (b).

(c) Two days later, you stumble upon the goat's still-running laptop. UAV coverage indicates there are raptors at (-2,0), (2,-4), and (0,4). You quickly fix the syntax errors and modify the program to consider all three raptors at once, defining:

$$f(x,y) = g(-2,0,x,y) + g(3,5,x,y) + g(0,4,x,y)$$

Where should you travel to escape the raptors?

Gradient Task Handling

Define $d_i(\mathbf{x})$ to be the distance from a player $\mathbf{x}$ to another player $\mathbf{x}_i$

Proposition 6.1. *In order to maximize* $\sum_i a_i \left(d_i(\mathbf{x})\right)^{b_i} = \sum_i a_i ||\mathbf{x}_i - \mathbf{x}||_2^{b_i}$ *an agent should travel in the direction*

$$\sum_i a_i b_i \left(d_i(\mathbf{x})\right)^{b_i - 2} (\mathbf{x}_i - \mathbf{x}) .$$

More generally, the maximum of $\sum_i f_i(d_i(\mathbf{x}))$ *is obtained in the direction*

$$\sum_i f_i'(d_i(\mathbf{x})) \frac{\mathbf{x}_i - \mathbf{x}}{||\mathbf{x}_i - \mathbf{x}||} .$$

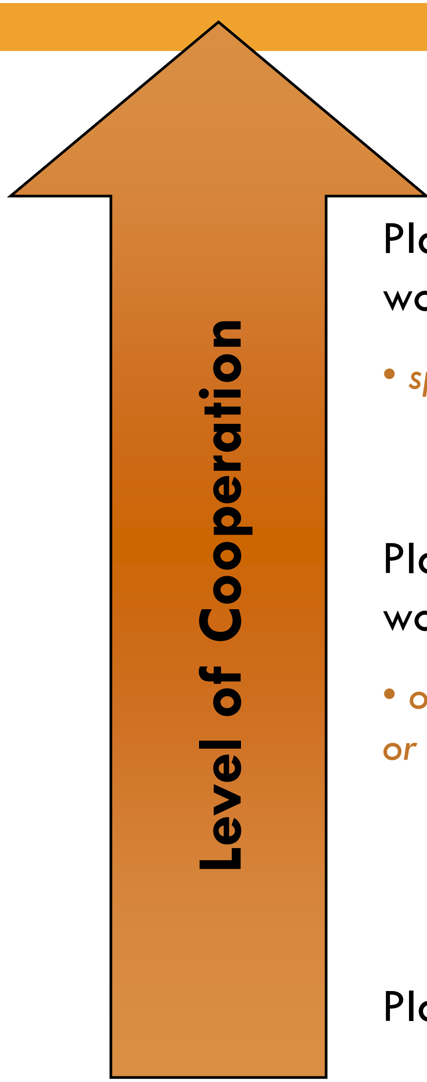
http://www.dean.usma.edu/math/people/Peterson/blaise/plotters/

Questions of Cooperation

Part III

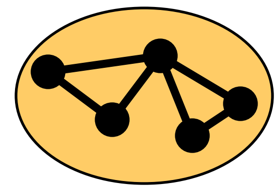
Levels of Cooperation

A simplified model



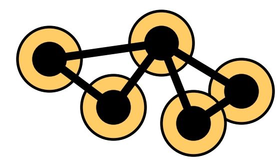
Players join together as a team, working for the team payoff

- *sports; organizations; wikipedia*

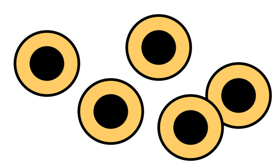


Players join together as a team, working for individual payoffs

- *oligopolies; price collusion; information or resource sharing; defensive alliances*



Players do not work together



Existing Frameworks for Cooperation I

von Neumann's **Cooperative Game Theory**

- Agents work together for a common payoff
- Mathematical theory for dividing payoff among the participants ("Shapley value")

Problem 1: *no concept of "team"*

- Agents participate only for selfish reasons
- Not applicable to most cooperative systems

Problem 2: *in practice, even selfish human behavior doesn't follow the laws of economics*

- Prisoner's Dilemma

Existing Frameworks for Cooperation II

Cooperative Systems

- Collections of agents sharing information and tasks in order to achieve a common goal

- Captures "team" aspect and the mechanics of cooperation:
  - Communication
  - Synchronization
  - Specialization
  - Trust
  - Layering

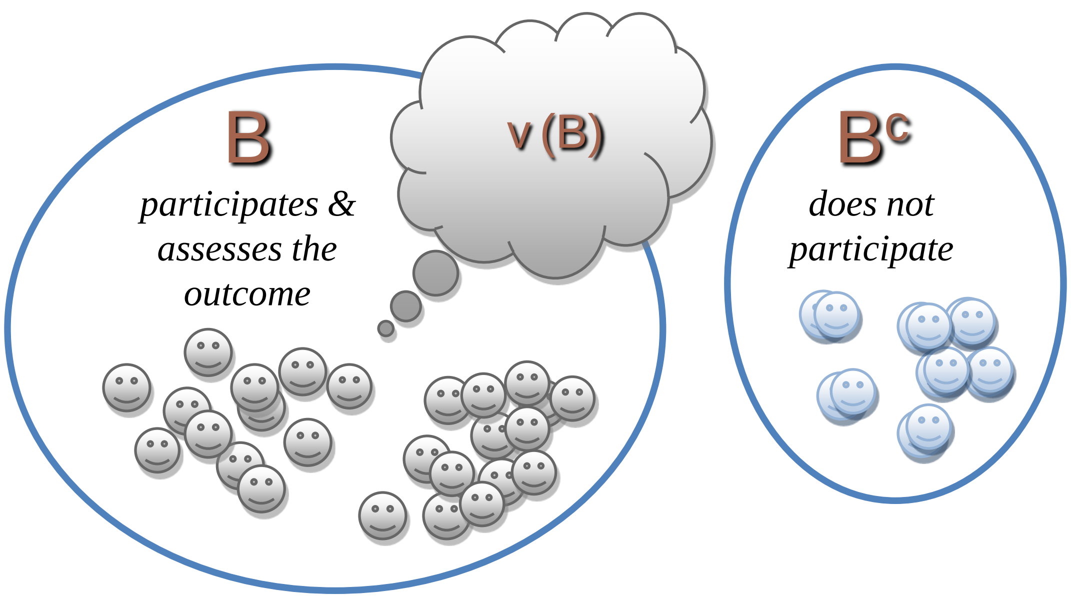
Problem: *no general mathematical notion of "team-oriented" cooperation*

Cooperative Game Theory

von Neumann, early 20th century

- A **cooperative game** is a set of players T with an **outcome** for each subset $B \subset T$ and a **payoff (utility) function** for each outcome

- Associates a value v (B) to the outcome when B participates

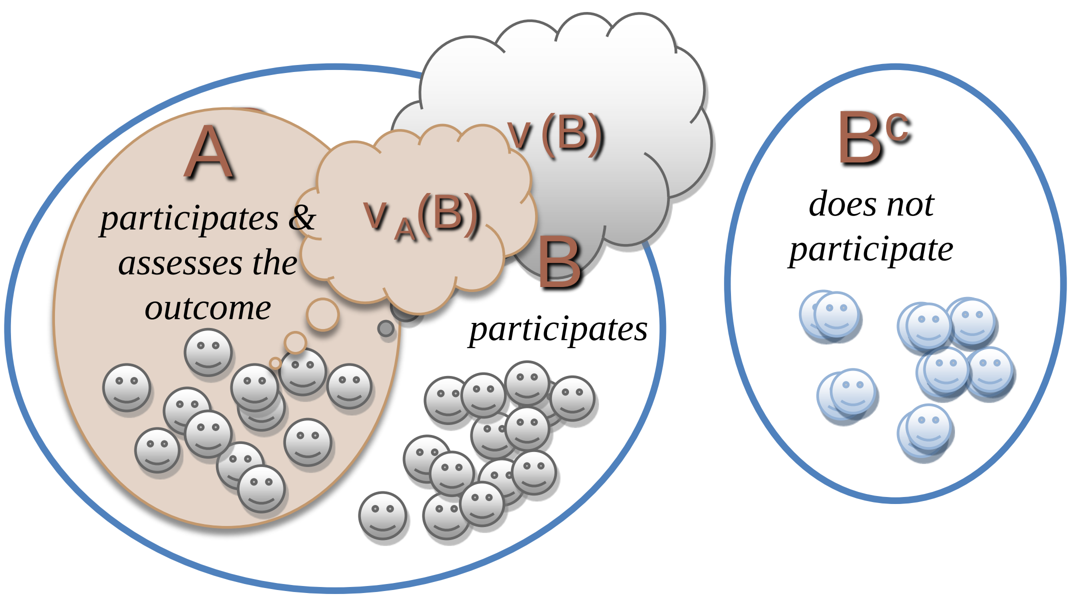


Subset Team Games

Provide unique payoffs to all subsets of players

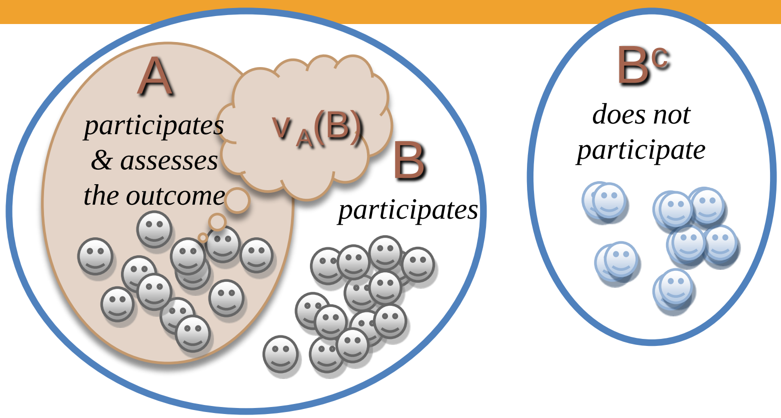
- A **subset team game** is a set of players T with an **outcome** for each subset $B \subset T$ and a **payoff (utility) function** ***for each outcome and each subset*** $A \subset B$.

- Associates a value $v_A(B)$ to each subset A of participating players B



Altruistic and Selfish Cooperation

The mathematical muscle: metrics for cooperation



Recall that $v_A(B)$ is the value to the subset A when all players in B participate

- The **altruistic contribution** of A to the team is:

  $$a_A = v_{A^c}(T) - v_{A^c}(A^c)$$

  (the gain in value to A^c when A participates)

- The **competitive contribution** of A to the team is:

  $$c_A = v_T(T) - v_{A^c}(T)$$

  (the portion of the value to T 'perceived by' A)

- The **total marginal contribution** of A to the team is:

  $$m_A = a_A + c_A = v_T(T) - v_{A^c}(A^c)$$

  (the total value added by the addition of A)

Example: Basketball

Identifying the most team-oriented players

Define $v_A(B)$ to be the number of points scored by the players in A when all of B participates.

- The *competitive contribution* $c_A = v_T(T) - v_{A^c}(T)$ is a player's *points per game*.
- The *altruistic contribution* measures the increase in team points scored by other players when A is put into the game (e.g. *assists*)

Example: Network Science

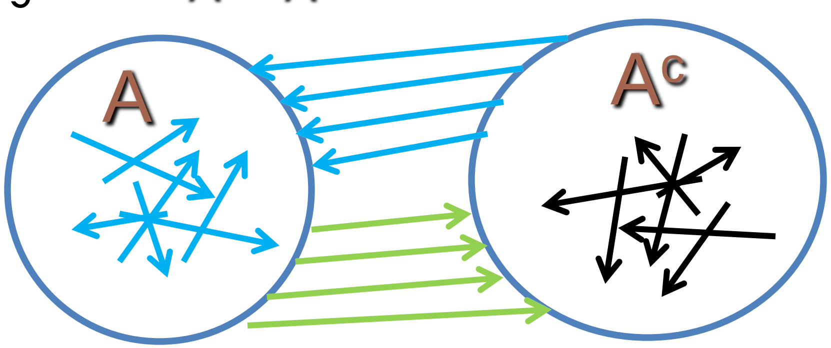
Identifying the most cooperative subsets

Given a weighted, directed graph with vertex set T:

- For players a, b, define $v_a(b)$ to be the weighting along the edge from b to a.
- Let $v_A(B)$ be the *sum* of weights on all edges from vertex set B to vertex set A.

- The *competitive contribution* $c_A = v_T(T) - v_{A^c}(T)$ equals $v_A(T)$, represents the *weighted indegree of* A.

- The *altruistic contribution* $a_A = v_{A^c}(T) - v_{A^c}(A^c)$ equals $v_{A^c}(A)$, the weighted sum of edges from A to A^c.



Future Work

(CDT Lucas Gebhart)

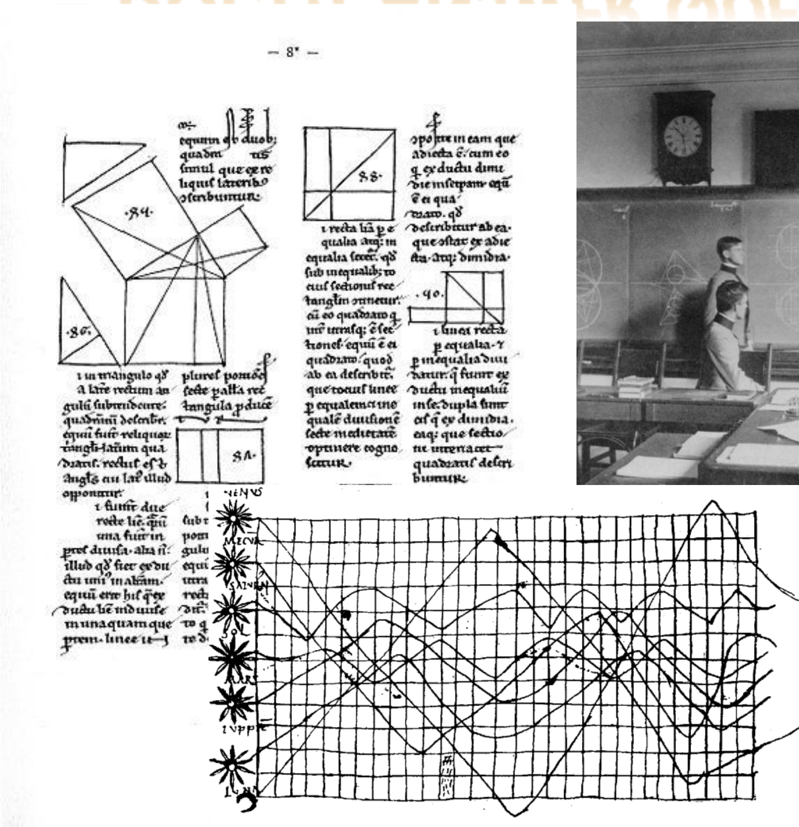
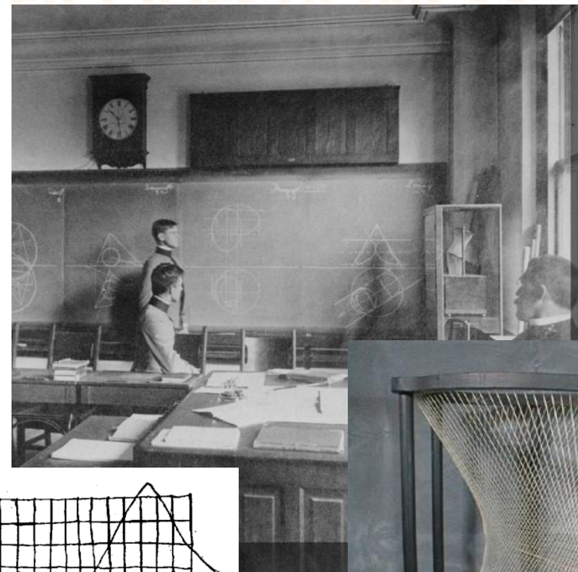
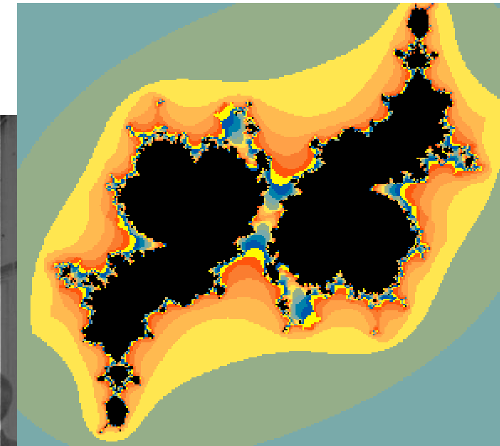
- Use **Pursuit-Evasion platform** to answer:
  - Is a more altruistic team always better, or are there situations in which some degree of competitive cooperation is beneficial?
  - How can cooperation metrics be used to build up trust over time?
  - What happens when we optimize altruism in agents?
- Find more connections with **network science**
  - Use network science invariants to generate more cooperation metrics.
  - Apply theory to real-world examples.

Interactive & Dynamic Mathematics

Part IV

What makes a good visualization?

- **DOES IT ANSWER QUESTIONS??**

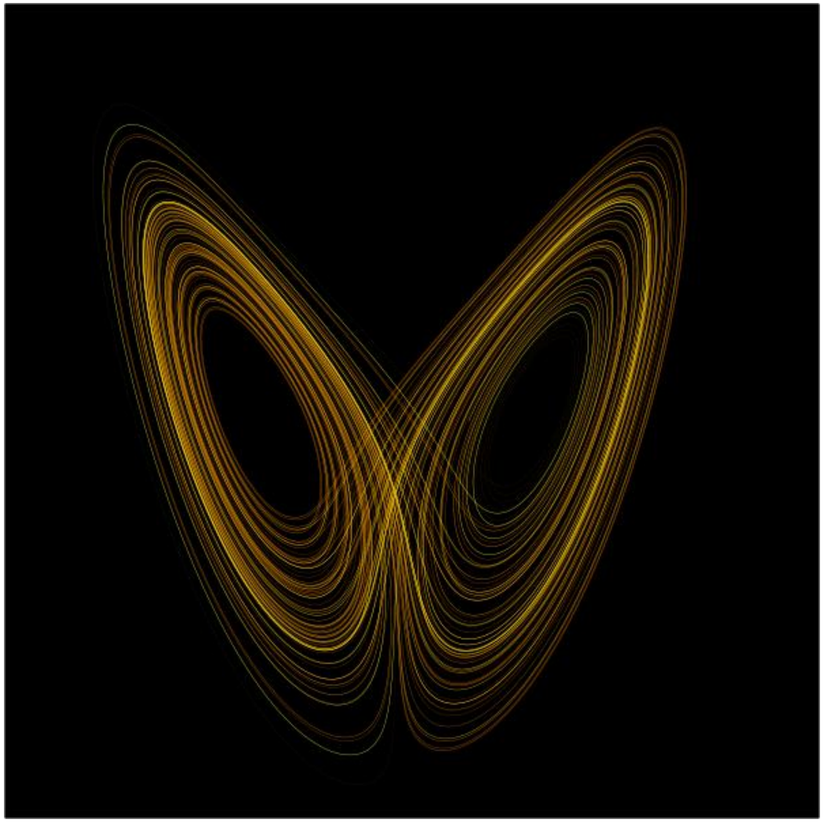
- **DOES IT INSPIRE QUESTIONS?**

The Lorenz Attractor

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$



http://www.dean.usma.edu/departments/math/people/peterson/blaise/plotters/vectorfield3d.html

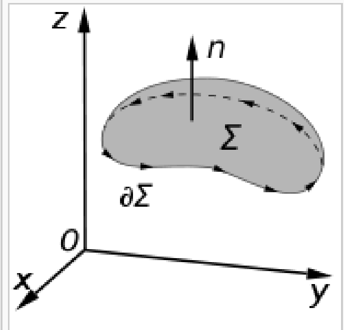
Stoke's Theorem

From Wikipedia:

Kelvin-Stokes theorem

[edit]

This is the (dualized) 1+1 dimensional case, for a 1-form (dualized because it is a statement about vector fields). This special case is often just referred to as the *Stokes' theorem* in many introductory university vector calculus courses. It is also sometimes known as the **curl theorem**.



An illustration of Kelvin-Stokes theorem with surface Σ, its boundary $\partial\Sigma$, and orientation *n*.

The classical Kelvin-Stokes theorem:

$$\int_\Sigma \nabla \times \mathbf{F} \cdot d\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r},$$

which relates the surface integral of the curl of a vector field over a surface Σ in Euclidean three-space to the line integral of the vector field over its boundary, is a special case of the general Stokes theorem (with *n* = 2) once we identify a vector field with a 1 form using the metric on Euclidean three-space. The curve of the line integral (∂Σ) must have positive orientation, meaning that *d **r*** points counterclockwise when the surface normal (**d Σ**) points toward the viewer, following the right-hand rule.

It can be rewritten for the student acquainted with forms as

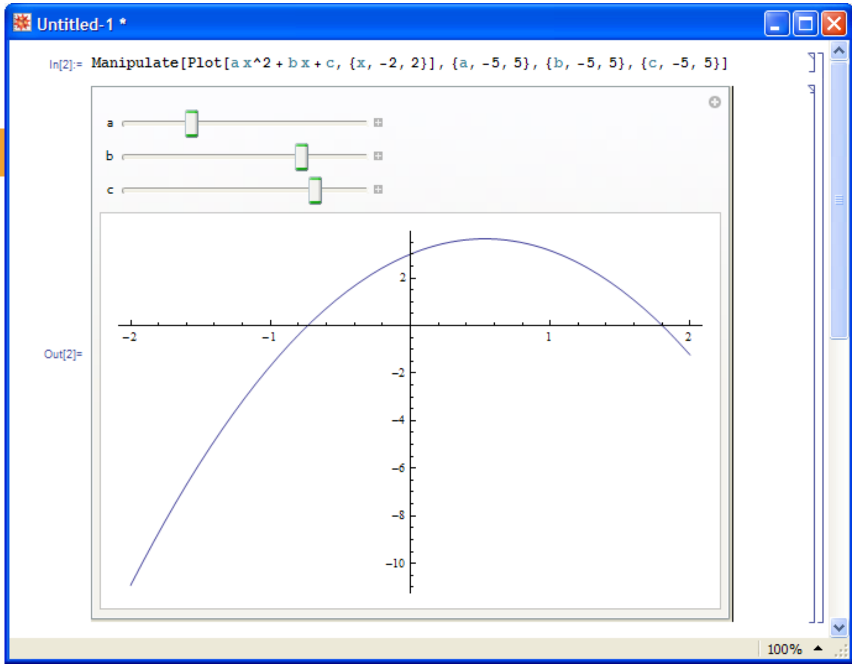
$$\iint_\Sigma \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\, dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\, dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_{\partial\Sigma} P\, dx + Q\, dy + R\, dz$$

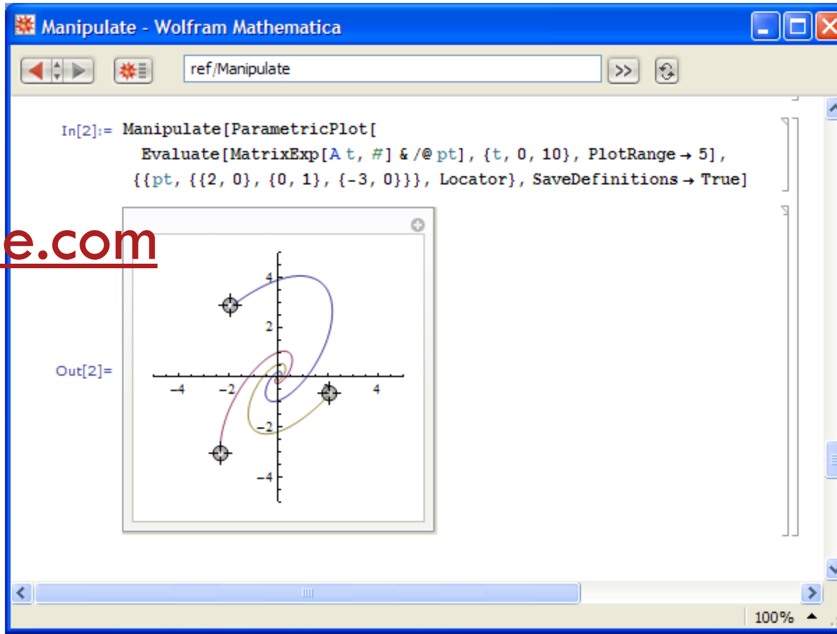
where *P*, *Q* and *R* are the components of F.

http://www.dean.usma.edu/departments/math/people/peterson/blaise/plotters/stokes.html

Some Tools

- Mathematica **Manipulate**
- Geogebra
  - www.geogebra.org
- Blaise
  - http://blaisemath.googlecode.com





Geogebra Mathlets

(www.geogebra.org)

http://www.dean.usma.edu/departments/math/people/peterson/geogebra/mathlets.html

- Cobwebs: http://www.dean.usma.edu/math/people/Peterson/geogebra/CobwebIteration_Sequence_2.html
- Continuity Epsilon-Delta: http://www.dean.usma.edu/departments/math/people/peterson/geogebra/continuity.html
- Lagrange Multipliers Problem: http://www.dean.usma.edu/departments/math/people/peterson/geogebra/farmer1.html
- Ballistics: http://www.dean.usma.edu/departments/math/people/peterson/geogebra/parametric3d-ballistic.html

Blaise Mathlets (http://blaisemath.googlecode.com)

http://www.dean.usma.edu/math/people/Peterson/blaise/plotters/

http://www.dean.usma.edu/math/people/Peterson/blaise/

- Circular Chase: http://www.dean.usma.edu/departments/math/people/peterson/blaise/circularchase/
- Pursuit Evasion: http://www.dean.usma.edu/math/people/Peterson/applets/peg2/peg2.html

Questions?