The Character Variety's New Clothes

AMS Session on Algebra and Number Theory, JMM 2008

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United States Military Academy

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What is a trace diagram?

Definition

An *n-trace diagram* has dual interpretations as

- a graph with vertices of degree 1, 2, and n, with edges labeled by elements of $GL(\mathbb{C}^n)$;
- a multilinear function between tensor powers of $\mathbb{C}^n$.

(There is a *categorical* equivalence.)

- The simplest trace diagrams represent functions such as $\mathrm{tr}(A)$, $\det(A)$, etc.
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For a Lie group G, the *G-character variety* of a surface with fundamental group π is

$$\mathfrak{X} = \mathrm{Hom}(\pi, G)//G,$$

where $//$ represents a *categorical quotient* with respect to the action of *simultaneous conjugation*. (meaning of "categorical": throw out enough points so the quotient exists in the category of varieties).

- This is the variety whose coordinate ring is the ring of invariants:

$$\mathbb{C}[\mathfrak{X}] \cong \mathbb{C}[\mathrm{Hom}(\pi, G)]^G,$$

$$f(A,B) = f(XAX^{-1}, XBX^{-1}).$$

- The character variety contains both *moduli space* and *Teichmüller space*, and is an important object in the theory of geometric structures.

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Diagrammatic Basis

Question: How do trace diagrams relate to the character variety?

Answer: A certain class of trace diagrams (which we call "Central Functions") provides a basis for the coordinate ring of the character variety.

- Baez (1996) showed that spin networks form a basis for the moduli space of connections.
- Sikora (2001) showed that spin networks and related graphs span the coordinate ring of the character variety.
- Little had been known about how to compute the basis functions *explicitly*.

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Algebraic Underpinnings

The following is a consequence of the *Peter-Weyl Theorem*:

Theorem (Central Function Decomposition)

The coordinate ring of the character variety $\mathbb{C}[\mathfrak{X}]$ *may be decomposed*

$$\mathbb{C}[\mathfrak{X}] \cong \bigoplus_{\vec{\lambda},\psi,\phi} \mathbb{C}\chi_{\vec{\lambda}}^{\psi,\phi},$$

where the summation depends on particular injections of irreducible representations into a tensor product of representations $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$.

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The *central functions* of the G-character variety χ are the functions $\chi_{\vec{\lambda}}^{\psi,\phi}$ in the above decomposition.

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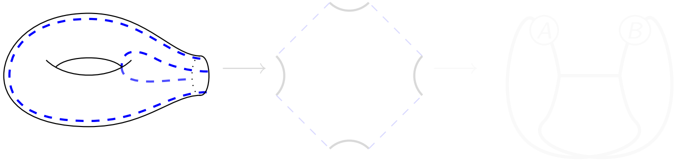
Representing Central Functions with Trace Diagrams

How do you construct these invariant functions (which live in $\mathbb{C}[Hom(\pi, G)]^G$)?



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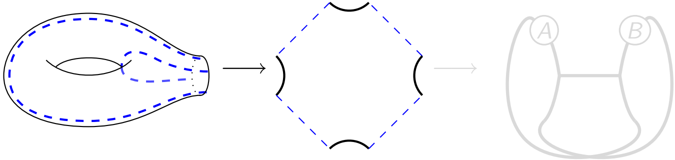

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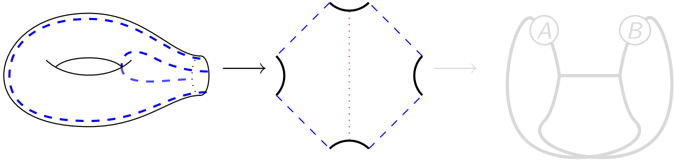

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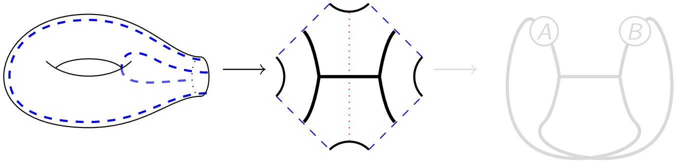

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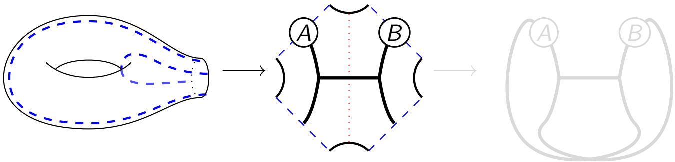

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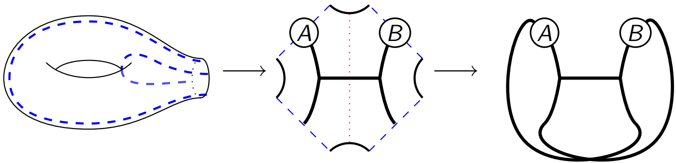

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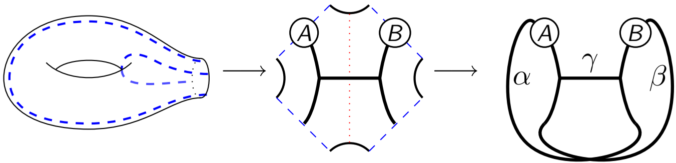

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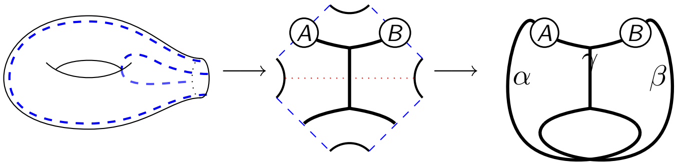

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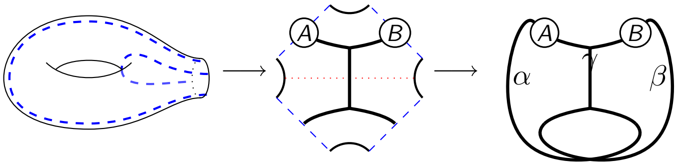

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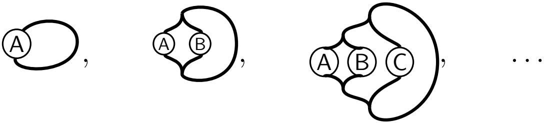
General Form of Central Functions

In general, central functions take the form:



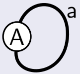
The number of labels is the *rank* of the fundamental group π.

For the remainder of the talk, assume that $G = SL(2, \mathbb{C})$. Then the nontrivial representations are indexed by $\mathbb{N}$.

Rank One

Definition

The rank one central functions (corresponding to an annulus) are

$$\chi_a(A) =$$  .

These satisfy the *Chebyshev* or *Fibonacci recurrence* $\chi_a = x \cdot \chi_{a-1} - \chi_{a-2}$ and are easily computed:

$$\chi_0(A) = 1$$
$$\chi_1(A) = x$$
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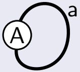
$\chi_a(A) =$ 

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Rank Two: Symmetry Property

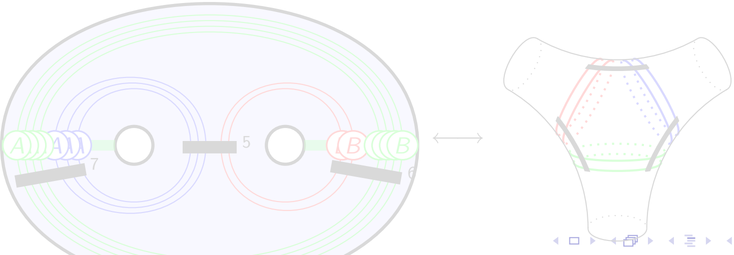
Theorem

If σ is any permutation on three letters, then

$$\chi_{\sigma(\alpha,\beta,\gamma)}(\sigma(y,x,z)) = \chi_{\alpha,\beta,\gamma}(y,x,z).$$

Fricke-Vogt Theorem: χ are polynomials in $x = \mathrm{tr}(A)$, $y = \mathrm{tr}(B)$, and $z = \mathrm{tr}(AB^{-1})$.

Proof.



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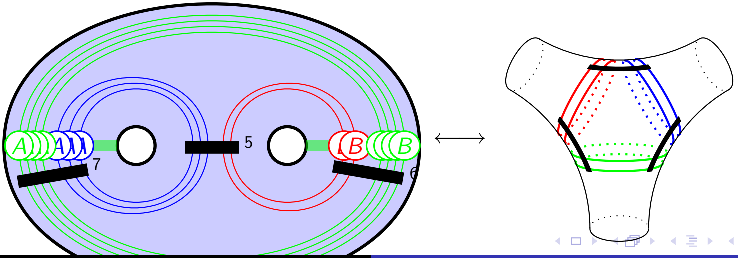

Rank Two: Recursion Property

Define $\hat{\chi}_{\alpha,\beta,\gamma} \equiv a!b!c!\chi_{\alpha,\beta,\gamma}$ and $\delta \equiv \alpha+\beta+\gamma$ (called the *rank*).
Define $\overline{\alpha} \equiv \alpha+1$ and $\underline{\alpha} \equiv \alpha-1$.

Theorem (Rank Two Recursion)

$$\hat{\chi}_{\alpha,\beta,\gamma} = x\cdot ac\hat{\chi}_{\alpha,\underline{\beta},\gamma} - \gamma^2\hat{\chi}_{\overline{\alpha},\underline{\beta},\underline{\gamma}} - \alpha^2\hat{\chi}_{\underline{\alpha},\underline{\beta},\overline{\gamma}} - \delta^2(\beta-2)^2\hat{\chi}_{\alpha,\underline{\underline{\beta}},\gamma}.$$

Notes.

- As a corollary, the leading term of $\chi_{\alpha,\beta,\gamma}$ is $x^\beta y^\alpha z^\gamma$.
- Together with the symmetry property, this provides an efficient technique for computing central functions.

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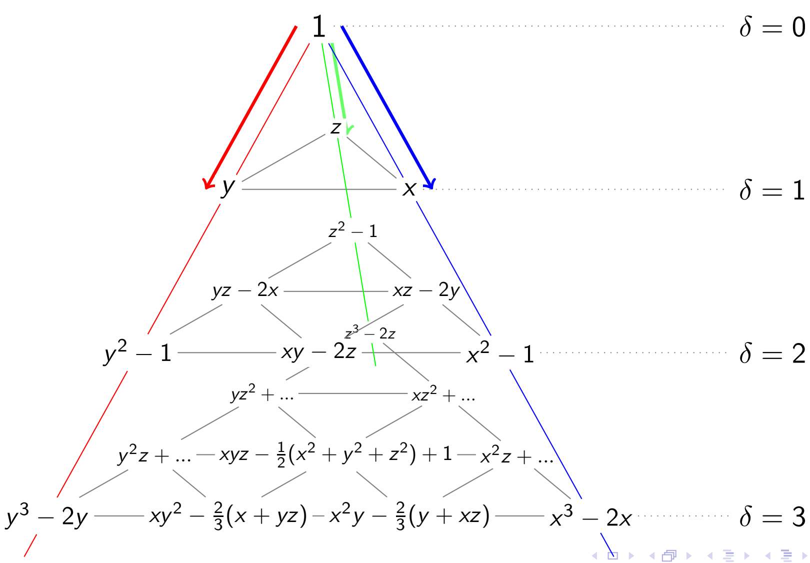
Rank Two: Table



Generalizations

There are two obvious generalizations:

- Beyond rank two: work underway;
- Beyond $SL(2, \mathbb{C})$: requires new ideas since crossings cannot be removed very easily; will be much harder!

Acknowledgments/References

Preprints regarding the $SL(2,\mathbb{C})$ rank two case are available at

- Lawton/Peterson, *Spin networks and SL(2,C)-character varieties*, **arXiv:math/0511271** (to appear, Handbook of Teichmüller Theory Vol II)
- Peterson, *The Art of Linear Algebra*, **arXiv:0712.2058**

Other References

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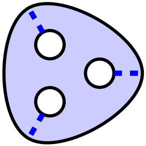
- Baez, *Spin networks in gauge theory*, **Adv. Math.** (1996)
- Sikora, *$SL(n)$-character varieties as spaces of graphs*, **Trans. AMS** (2001)

Constructing Central Functions on the Surface



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A *cut triangulation* divides the resulting polygon into triangles (with neighborhoods of vertices removed).

Choices of cut triangulations and labels provide the summation indices for central functions.

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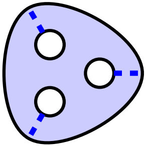

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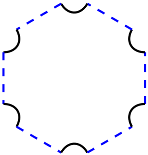

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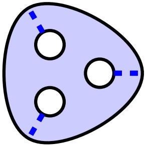

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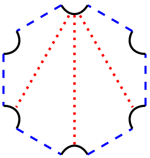

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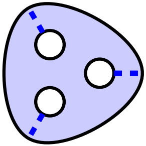

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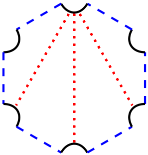

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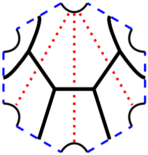

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The Diagrammatic Basis Theorem

Theorem

Let Σ be a compact surface with boundary. Given a cut triangulation extending a specified cut set, every G-admissible labelling of its dual 1-skeleton induces a trace diagram which is identified with a G-invariant function $Hom(\pi, G) \to \mathbb{C}$. Moreover, for every cut triangulation, the set of such diagrams is a basis for $\mathbb{C}[\mathfrak{X}]$.

Paraphrase

Choose a cut triangulation. Then the set of dual graphs with all possible labels provides a basis for $\mathbb{C}[\mathfrak{X}]$.

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