Trace Diagrams, Surfaces, and Character Varieties

The Binor Identity in Linear Algebra and Geometry

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United States Military Academy

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Outline

1. Introduction
   - What is a Trace Diagram?
2. Working with Trace Diagrams
   - Interpreting Trace Diagrams
   - The Fundamental Binor Identity
   - Generalizations
3. Applications
   - Linear Algebra
   - Surfaces and Character Varieties
4. Concluding Remarks

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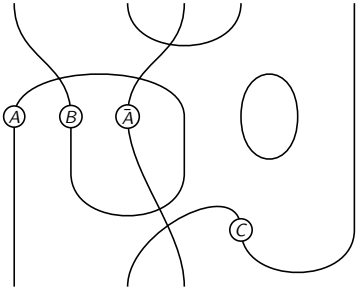
4 Concluding Remarks

What is a Trace Diagram? (informally)

- Graph marked by group elements (usually, matrix group);
- May be disconnected;
- May have several endpoints;
- May be closed (no endpoints);
- Subject to topological rules;
- Satisfies special relations;
- Generalizable to n-valent graphs;
- Each diagram corresponds to a function between tensor powers of a module/vector space;
- Closed diagrams are trace maps.



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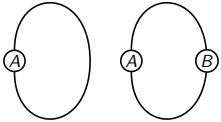

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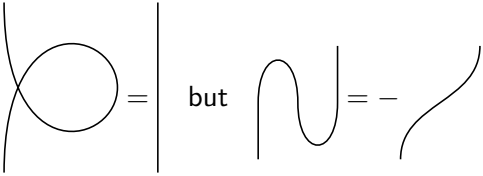

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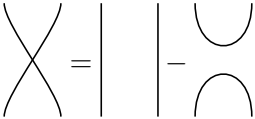

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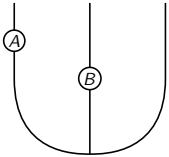

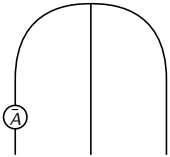

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$: V \to V \quad \text{via} \quad v \mapsto A \cdot v.$

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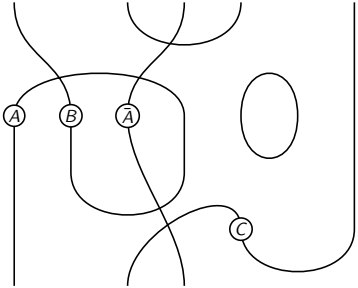

maps $V \otimes V \otimes V \to V \otimes V \otimes V \otimes V \otimes V.$

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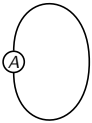

$= \mathrm{tr}(A).$

What is a Trace Diagram? (slightly more formally)

Definition

A *trace diagram* is a marked graph drawn on the plane relative to some "up" direction which provides a well-defined description of a **multilinear map** depending on the matrix markings.

- The domain and range of the corresponding multilinear map are determined by the "input" strands, by convention along the bottom edge, and "output" strands, by convention along the top edge.

Example

$: \quad V^{\otimes 3} \to V^{\otimes 5}$

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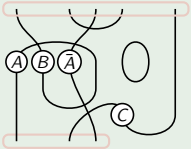

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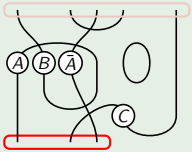

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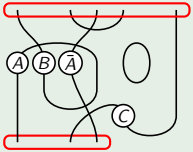

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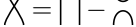
Why the Diagrammatic Approach?

Diagrams are good for:

- exhibiting mathematical structure
  - duality corresponds to turning diagrams upside-down
  - relations are often simply expressed: 
- connecting algebra with geometry
  - when placed on surfaces, trace diagrams describe the moduli space of representations of a surface group
- discovering similarities among mathematical structures
  - is both a 2×2 trace identity and the defining relation of the Poisson bracket on the coordinate ring of the character variety
- computational algorithms
  - relations used to generate recurrence equations;
  - illustrative method for generating trace identities.

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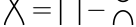
  - relations are often simply expressed: 
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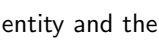
  -  is both a 2×2 trace identity and the defining relation of the Poisson bracket on the coordinate ring of the character variety
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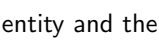
  -  is both a 2×2 trace identity and the defining relation of the Poisson bracket on the coordinate ring of the character variety
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Mathematical Ancestors and Siblings

Ancestors:

- Euler: graph theory
- Frege: Begriffsschrift c. 1890
- Feynman: Feynman diagrams
- Penrose: spin networks

Early sources:

- Stedman: group theory
- Cvitanovic: 'bird tracks'

Siblings:

- Kauffman: Kauffman bracket
- IHX, STU Relations
- Kuperberg Spiders/Webs

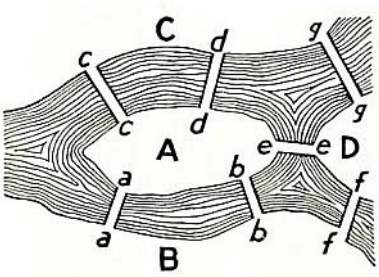

FIGURE 98. *Geographic Map: The Königsberg Bridges.*

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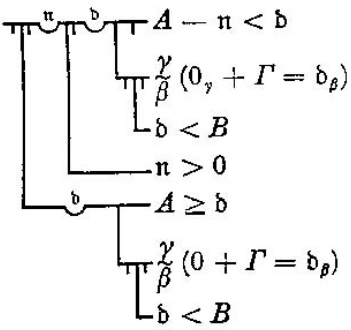

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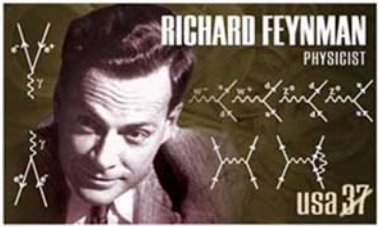

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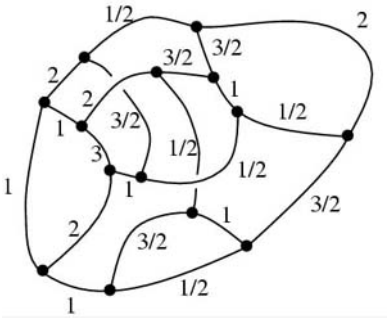

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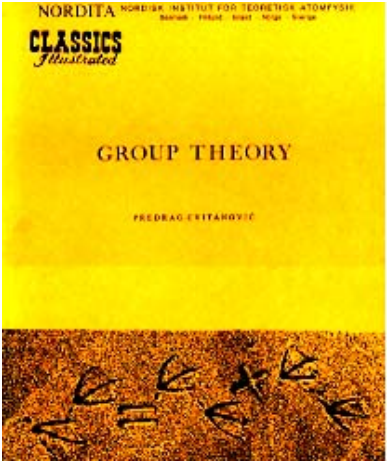

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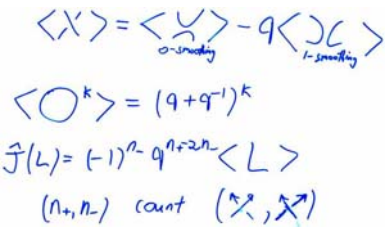

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Interpreting Trace Diagrams

How can the map of a diagram be constructed?

- tensor algebra (rigorous)
- topological relations (intuitive)

The Tensor Algebra Approach

The algebraic construction is:

Definition

If e_1, e_2 are the standard basis elements of $\mathbb{C}^2$, then

: $V \otimes V \to \mathbb{C}$ takes $e_1 \otimes e_2 \mapsto 1, e_2 \otimes e_1 \mapsto -1$ and $e_i \otimes e_i \mapsto 0$

and
: $\mathbb{C} \to V \otimes V$ takes $1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$.

Basic Multiplication

Diagram Rule (Multiplication)

For $A \in M_{n\times n}$, represents a map $A : \mathbb{C}^n \to \mathbb{C}^n$, and $=$ the identity map $I : \mathbb{C}^n \to \mathbb{C}^n$. The product of matrices AB is represented by composition of such maps: $=$.

We now restrict to 2×2 matrices, although the diagrams can be easily generalized. Additionally, the diagrams are considered to be contained in a $\mathbb{C}$-algebra, allowing finite linear combinations of diagrams.

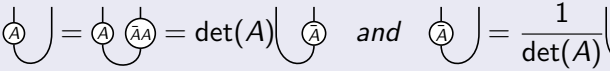
Arc Diagrams I: Maxima and Minima

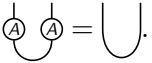
Diagram Rule (Critical Points)

Matrices pass through critical points via

Corollary

For $A \in \mathrm{GL}(2, \mathbb{C})$ and denoting the inverse of A by $\bar{A} = A^{-1}$,



Remark. When $A \in \mathrm{SL}(2, \mathbb{C})$, this rule simplifies further: moving a matrix through a max/min corresponds to inversion, and 

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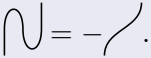
Arc Diagrams II: Wraps and Kinks

Diagram Rule (Loops and Kinks)

Arcs may be wrapped around as follows:

Kinks introduce a sign as follows:



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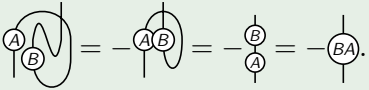
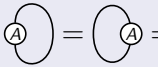


Loop Diagrams: Trace and Determinant

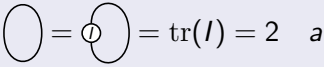
A closed diagram (loop) with matrices is a linear map $\mathbb{C} \to \mathbb{C}$ and interpreted as a complex number or a *trace polynomial*.

Diagram Rule (Trace)



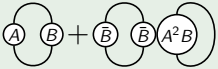
$= \mathrm{tr}(A).$

Corollary

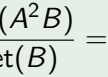


$= \mathrm{tr}(I) = 2$ *and* $= 2\det(A).$

Example



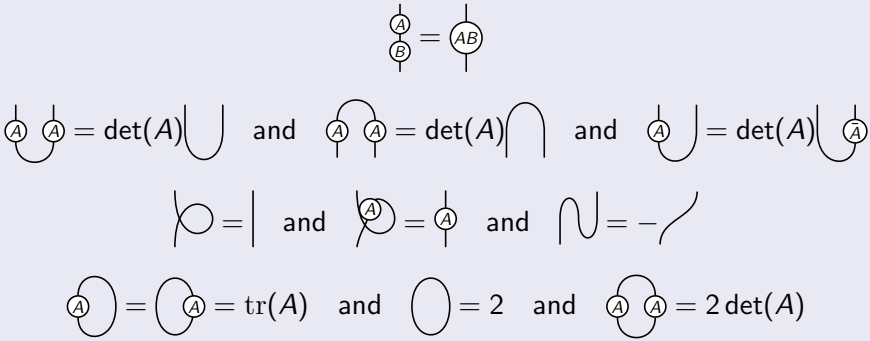
$= \det(B)$



$+ \dfrac{2\mathrm{tr}(A^2B)}{\det(B)} = \det(B)\mathrm{tr}(A\bar{B}) + \dfrac{2\mathrm{tr}(A^2B)}{\det(B)}.$

Summary of Diagram Rules

Diagram Rule (Summary)



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The Fundamental Binor Identity

Proposition (Fundamental Binor Identity)

The rules above imply the relation $=$ $-$.

Algebraic Proof.

1. represents the map $a \otimes b \mapsto b \otimes a$
2. want to show that : $a \otimes b \mapsto a \otimes b - b \otimes a$
3. verify each element of the basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$, e.g.

$$e_2 \otimes e_1 \overset{\cap}{\mapsto} -1 \overset{\cup}{\mapsto} -(e_1 \otimes e_2 - e_2 \otimes e_1) = e_2 \otimes e_1 - e_1 \otimes e_2 \qquad \checkmark$$

4. remaining basis elements work similarly $\square$

Remark. The binor identity provides a means of eliminating *all crossings* in a trace diagram!

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The Chacteristic Equation

The *characteristic polynomial* of a 2×2 matrix is

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0.$$

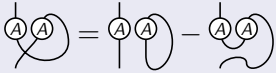
Proposition

The binor identity implies the characteristic equation

$$A^2 - \mathrm{tr}(A)A + \det(A)I = 0.$$

Proof.

By the binor identity,



This last expression is $A^2 = A * \mathrm{tr}(A) - \det(A)I$, the characteristic equation. □

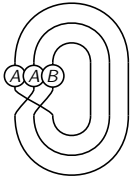
Using the Binor Identity I: Trace Relations

Example

Use the binor identity to reduce $\mathrm{tr}(A^2B) =$

.

Solution 1. Draw the diagram with crossings and apply the binor identity:



$=$

This corresponds to

$$\mathrm{tr}(A^2B) = \mathrm{tr}(A)^2\mathrm{tr}(B) - 2\det(A)\mathrm{tr}(B) - \mathrm{tr}(A)\mathrm{tr}(A\bar{B})\det(B) + \det(A)\mathrm{tr}(B).$$

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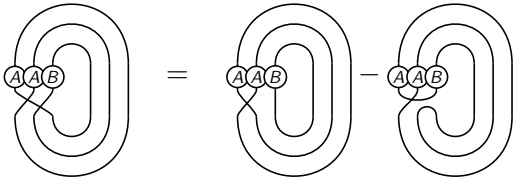

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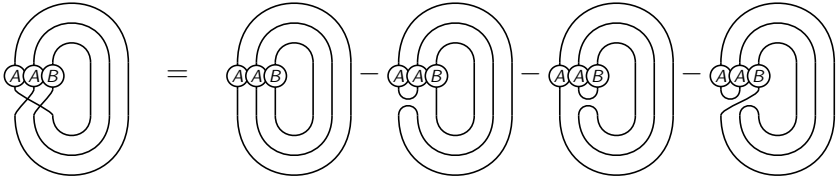

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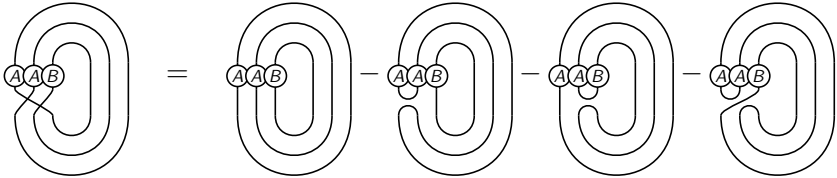

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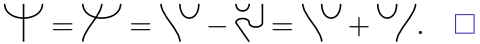
Using the Binor Identity II: Kinks and Signs

Example

Use the binor identity to evaluate .

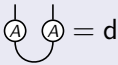
Solution. Move the crossing away from the extremum:



Remark: using would have provided the same answer.

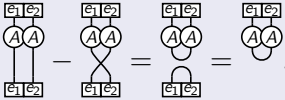
The Determinant

Proposition

The tensor algebra definition of *and* *imply*  $= \det(A)$.

Proof.

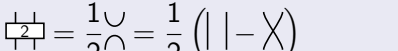
The definition $\det A = a_{11}a_{22} - a_{12}a_{21}$ becomes the diagram

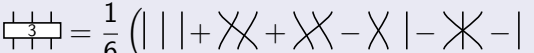
$\det(A) =$  $\square$

Generalizations: Symmetrizers and Anti-Symmetrizers

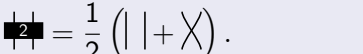
Definition

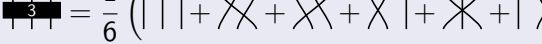
The *anti-symmetrizer* generalizes $\cup$ $\cap$ as follows:





The *symmetrizer* is



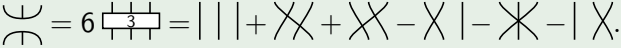


Generalizations: Higher Dimensions

To generalize to $n \times n$ matrices, generalize :

Example

Trace diagrams for 3×3 matrices are trivalent graphs, with antisymmetrizer



The local maxima and minima are defined via

$$1 \mapsto e_1 \otimes e_2 \otimes e_3 + \cdots - e_1 \otimes e_3 \otimes e_2.$$

1 Introduction
- What is a Trace Diagram?

2 Working with Trace Diagrams
- Interpreting Trace Diagrams
- The Fundamental Binor Identity
- Generalizations

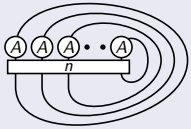
3 Applications
- Linear Algebra
- Surfaces and Character Varieties

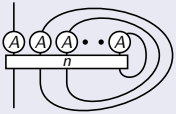
4 Concluding Remarks

A Not-So-Characteristic Equation

Proposition

Diagrammatically, the characteristic equation for an $n \times n$ matrix is:

$\det(A) * I =$ 

$= n! *$ 

Example

$\det(A) * I = 2$

$=$

$-$

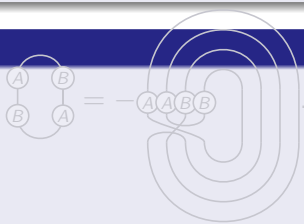
$= A * \mathrm{tr}(A) - A^2.$

Trace Relations: the Commutator

Proposition

$$\mathrm{tr}(AB\bar{A}\bar{B}) = \mathrm{tr}(A)^2 + \mathrm{tr}(B)^2 + \mathrm{tr}(AB)^2 - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB) - 2.$$

Proof.

Represent the trace diagrammatically as 

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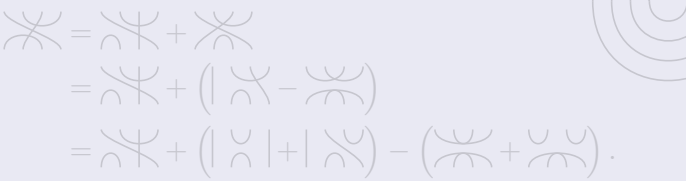

Re-insert matrix and keep track of signs:

$$\mathrm{tr}(AB\bar{A}\bar{B}) = \mathrm{tr}(B)^2 - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A\bar{B}) + \mathrm{tr}(A)^2 + \mathrm{tr}(A\bar{B})^2 - 2.$$

Use $\mathrm{tr}(A\bar{B}) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB)$ to obtain the desired relation. □

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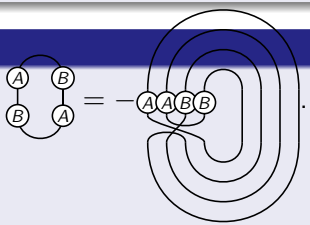
Represent the trace diagrammatically as 

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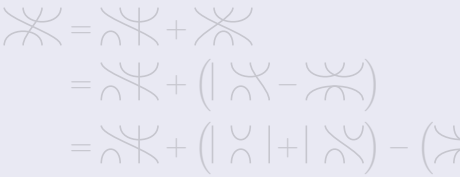

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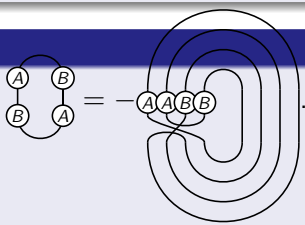
Represent the trace diagrammatically as 

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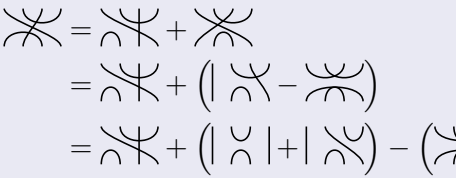

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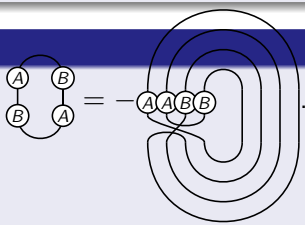
Represent the trace diagrammatically as 

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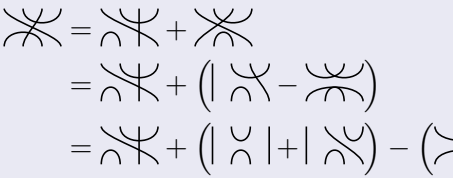

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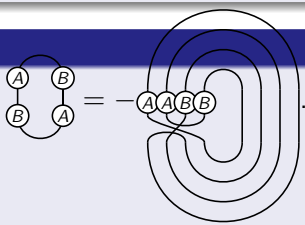
Represent the trace diagrammatically as 

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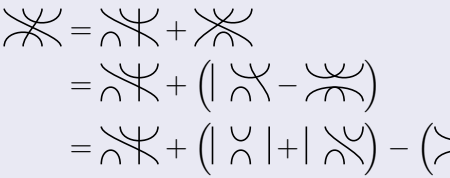

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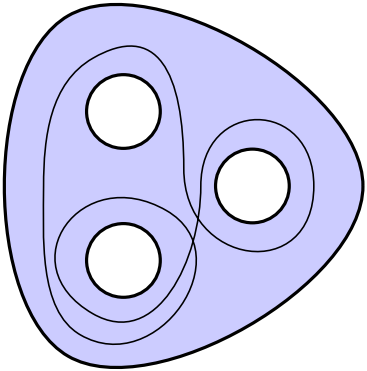
Surface Group Representations

Suppose we are given a representation $\pi_1 \to G$ for some matrix group G. What happens when a closed trace diagram is drawn on a surface?

- Assign 'surface cuts' to elements of the fundamental group.
- Mark loops at the cuts using the representation $\pi_1 \to G$.
- Assigns a trace polynomial to each surface loop [here, $\mathrm{tr}(\bar{A}BBC)$].



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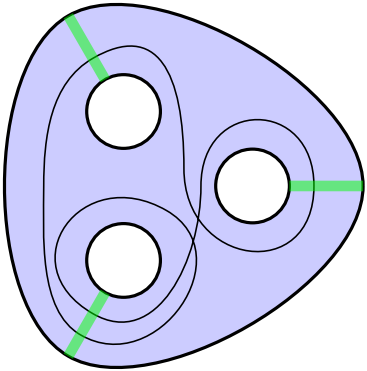

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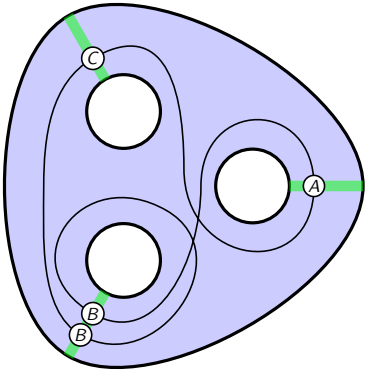

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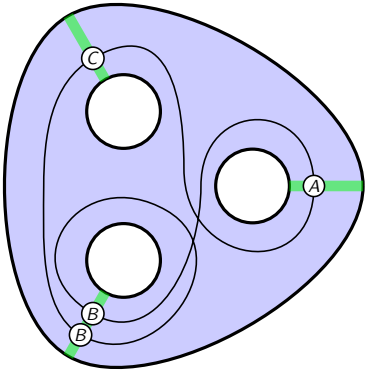

The Character Variety

The ring of trace polynomials may be used to construct the following:

Definition

The *G-character variety* is the algebraic variety whose coordinate ring is the trace ring generated by representations.

Example

For surfaces with free group of rank two, the coordinate ring of the $SL(2, \mathbb{C})$-character variety is a polynomial ring in three undeterminants.

This can be proved by recognizing that (via the binor identity) all trace loops on the three-holed sphere can be reduced to three basic loops.

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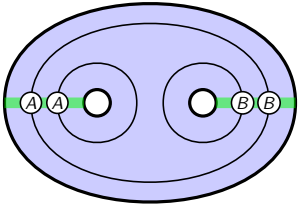
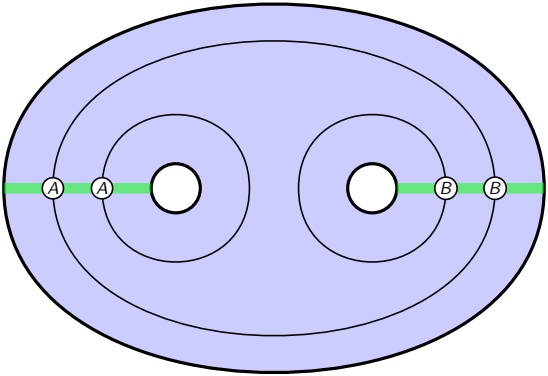


Basis for the Trace Ring I

Diagrams can be used to construct a basis for the trace ring.



Expand symmetrizers and remove crossings to obtain a trace polynomial

$$\chi_5^{7,6}(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(A\bar{B}))$$

Theorem

The polynomials $\chi_c^{a,b}$ comprise a basis for the coordinate ring of the $\mathrm{SL}(2,\mathbb{C})$-character variety of the three-holed sphere.

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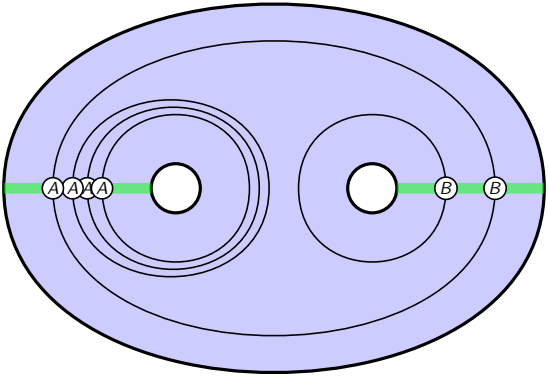

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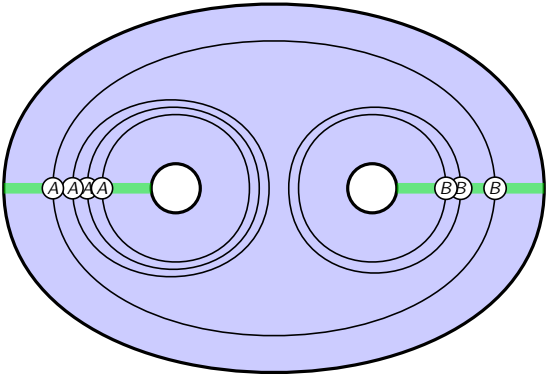

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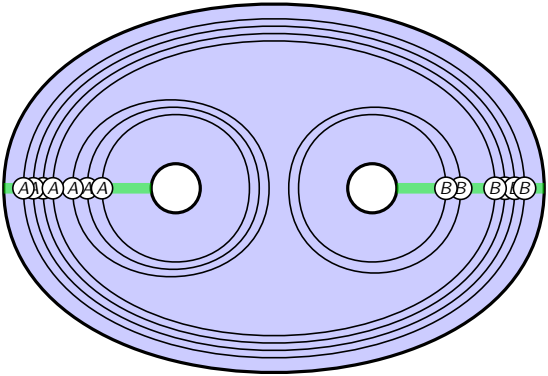

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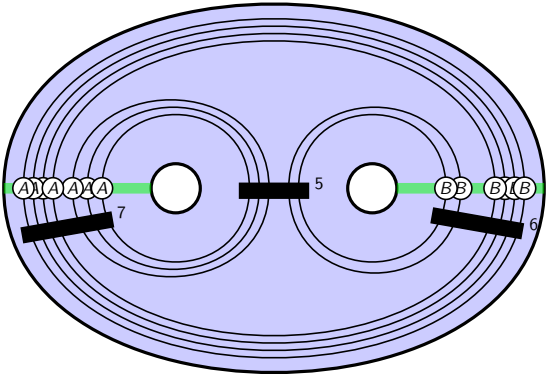
Diagrams can be used to construct a basis for the trace ring.



Theorem

The polynomials $\chi_c^{a,b}$ comprise a basis for the coordinate ring of the $\mathrm{SL}(2,\mathbb{C})$-character variety of the three-holed sphere.

- Proof uses the unitary trick and the Peter-Weyl Theorem.

Expand symmetrizers and remove crossings to obtain a trace polynomial

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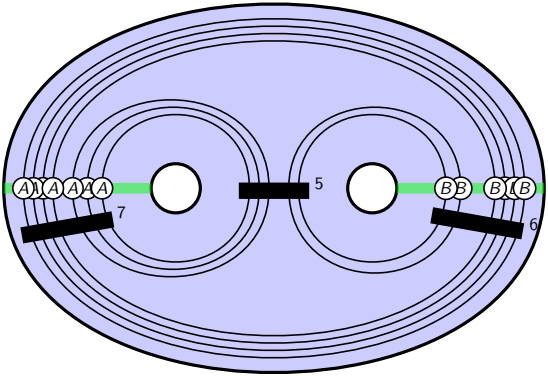

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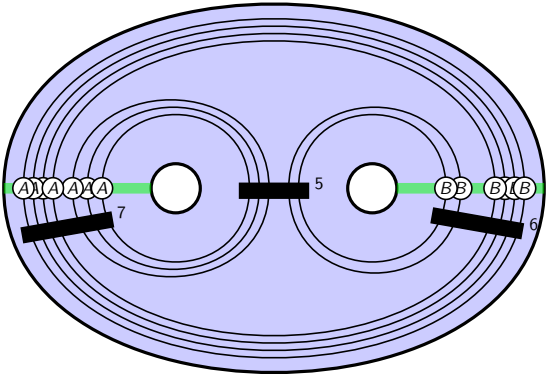

Expand symmetrizers and remove crossings to obtain a trace polynomial

$$\chi_5^{7,6}(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(A\bar{B}))$$

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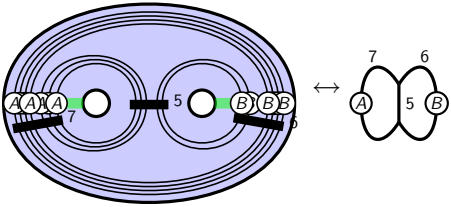
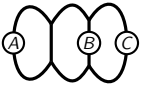
Basis for the Trace Ring II

Shorthand:



Remarks:

- The basis exhibits considerable symmetry.
- The basis depends only on fundamental group of the surface.
- To generalize for other surfaces, add more loops:



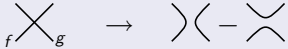
- For most surfaces, the trace ring is not free.
- Edges are labeled by *representations*.

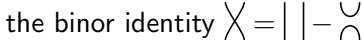
Poisson Structure

To capture the boundary structure of a surface, one can introduce a locally-defined Poisson structure on this ring:

Definition

The Goldman bracket $\{f, g\}$ of two loops on a surface is the sum over all essential intersections of the following:



- This bracket satisfies the Jacobi and Leibniz identities, and so gives the ring a Poisson structure.
- The bracket is simply the application of the binor identity  to the essential crossings in a diagram.

1 Introduction
- What is a Trace Diagram?

2 Working with Trace Diagrams
- Interpreting Trace Diagrams
- The Fundamental Binor Identity
- Generalizations

3 Applications
- Linear Algebra
- Surfaces and Character Varieties

4 Concluding Remarks

References

Trace Diagrams for *any* Lie group:

- Predrag Cvitanovic, *Group Theory*, http://chaosbook.org/GroupTheory/