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# SPIN NETWORKS AND $\mathrm{SL}(2,\mathbb{C})$-CHARACTER VARIETIES

SEAN LAWTON, ELISHA PETERSON

Abstract. Let $G = \mathrm{SL}(2,\mathbb{C})$, $\pi$ be a free group of rank $r$, and let $\mathcal{R} = \mathrm{Hom}(\pi, G)$ be the $G$-representation variety of $\pi$. $G$ acts on the coordinate ring $\mathbb{C}[\mathcal{R}]$ by simultaneous conjugation in the $r$ matrix variables of a regular function $f$, since $\mathcal{R} \cong G^{\times r}$. The subring $\mathbb{C}[\mathcal{R}]^G$ of invariants under this action is finitely generated, so its geometric points $\mathfrak{X} = \mathcal{R}/\!\!/G$ are an affine variety identified with the reducible characters of $\mathcal{R}$. $\mathfrak{X}$ is called the $G$-character variety of $\pi$. We present a constructive proof that the coordinate ring $\mathbb{C}[G]$ is isomorphic as a $G$-module to the invariant ring of matrix coefficients, giving a natural additive basis of $\mathbb{C}[\mathcal{R}]^G$ in terms of class functions on the irreducible sub-representations of $G$. These class functions may be explicitly described in terms of spin networks, a special type of graph useful in the study of $\mathrm{SL}(2,\mathbb{C})$ representations. We constructively determine the multiplicative structure of these graphs, giving a canonical "Taylor-like" series description of the regular functions on $\mathfrak{X}$. This allows us to describe $\mathfrak{X}$ and reprove a classical result of Fricke, Klein and Vogt.

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## 1. Introduction

The purpose of this work is to present a functional approach to the algebraic theory of $\mathrm{SL}(2,\mathbb{C})$ representations of free groups. Let $G = \mathrm{SL}(2,\mathbb{C})$, the $2\times 2$ complex matrices of determinant 1. For a rank $r$ free group $\pi = \{x_1, x_2, ..., x_r\}$, where $x_i$ are free letters, the representations $\mathrm{Hom}(\pi, G)$ form an affine variety denoted $\mathcal{R}$ and called the $\mathrm{SL}(2,\mathbb{C})$ representation variety of $\pi$. Indeed, the map $\rho \mapsto (\rho(x_1), \ldots, \rho(x_r))$ taking $\mathrm{Hom}(\pi, G) \longrightarrow G^{\times r}$ is a bijection and $G^{\times r}$ is the algebraic variety determined by the ideal

$$(x^i_{11}x^i_{22} - x^i_{12}x^i_{21} - 1 \mid 1 \leq i \leq r)$$

in the complex polynomial ring $\mathbb{C}[x^i_{11}, x^i_{12}, x^i_{21}, x^i_{22} \mid 1 \leq i \leq r]$.

We denote the coordinate ring of $\mathcal{R}$ by $\mathbb{C}[\mathcal{R}]$, and so

$$\mathbb{C}[\mathcal{R}] \cong \mathbb{C}[G]^{\otimes r}.$$

Let $\vec{\mathbf{x}} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_r)$ be the $r$-tuple consisting of *matrix variables* $\mathbf{x}_i = \begin{pmatrix} x^i_{11} & x^i_{12} \\ x^i_{21} & x^i_{22} \end{pmatrix}$. Then $G$ acts on $\mathbb{C}[\mathcal{R}]$ by diagonal conjugation, i.e., for $g \in G$ and $f(\vec{\mathbf{x}}) \in \mathbb{C}[\mathcal{R}]$,

$$g \cdot f(\vec{\mathbf{x}}) = f(g^{-1}\vec{\mathbf{x}}g) = f(g^{-1}\mathbf{x}_1 g, ..., g^{-1}\mathbf{x}_r g).$$

Hence, we may consider the subring of invariants $\mathbb{C}[\mathcal{R}]^G$. This ring is known to be finitely-generated as a $\mathbb{C}$-algebra, since $G$ is reductive [Do].

For $r = 2$, we decompose this ring into specific class functions associated with the finite-dimensional irreducible representations of $G$. These functions may be represented by a special type of graph called a *spin network*. This decomposition determines a canonical description of the invariants $\mathbb{C}[\mathcal{R}]^G$, and we subsequently determine, in these terms, the multiplicative structure on $\mathbb{C}[\mathcal{R}]^G$.

We are motivated not only by the interesting algebraic question of determining a complete description of the invariant ring and the subsequent knowledge of the character variety

$$\mathfrak{X} = \mathrm{Spec}_{max}(\mathbb{C}[\mathcal{R}]^G),$$

which encodes many geometric objects of interest [G3, G2], but also by the promise of a methodology and point of view that will allow for generalizations to other Lie groups and free groups of rank greater than two.

*The remainder of this paper is organized as follows.* In Section 2, we introduce spin networks after briefly reviewing background from invariant theory and representation theory. Spin networks are special types of graphs commonly used for angular momentum problems. They can

be identified with functions between tensor powers of $\mathbb{C}^2$, and offer a powerful means to work with regular functions on $\mathfrak{X}$ since they relate to each other by topological equivalences. We give this concept a full treatment, as our point of view leads us to a different definition for spin networks than is usually found in the literature.

In Section 3, we constructively prove

**Theorem 3.1.** *As $G$-modules*

$$\mathbb{C}[G] \cong \sum_{n\geq 0} V_n^* \otimes V_n \cong \sum_{n\geq 0} \mathrm{End}(V_n),$$

*where $\{V_n\}$ are the irreducible representations of $G$.*

This in turn shows, when $r = 2$, that

$$\mathbb{C}[\mathfrak{X}] \cong \sum_{\substack{a+b+c\in 2\mathbb{Z}\\ a+b\geq c\geq 0\\ a+c\geq b\geq 0\\ b+c\geq a\geq 0}} \mathrm{Span}_{\mathbb{C}}\left(\chi_c^{a,b}\right).$$

We refer to the invariant functions $\chi_c^{a,b}$ as *central functions*, and describe how they are represented as spin networks. This decomposition provides a canonical "Taylor-like" series description of the regular functions on $\mathfrak{X}$, with a highly nontrivial multiplicative structure.

In Section 4, we first compute a few simple central functions. Then, using spin networks, we demonstrate their surprising symmetries (Theorem 4.1), and prove our main theorem, which gives an explicit description of the multiplicative structure on $\mathbb{C}[\mathfrak{X}]$:

**Theorem 4.8** (Multiplication of Central Functions). *The product of two central functions $\chi_c^{a,b}$ and $\chi_{c'}^{a'b'}$ is given by*

$$\chi_c^{a,b}\chi_{c'}^{a'b'} = \sum_{j_1,j_2,k,l,m} \mathrm{C}_{j_1klm}\mathrm{C}_{j_2klm}\frac{\Theta(a,a',k)\Theta(b,b',l)\Theta(c,c',m)}{\Delta(k)\Delta(l)\Delta(m)}\chi_m^{kl},$$

*where the sum is taken over triples of integers*

$$\{a,a',k\},\ \{b,b',l\},\ \{c,c',m\},\ \{a',b,j_i\},\ \{c,k,j_i\},\ \{b,l,j_i\},\ \{k,l,m\},$$

*corresponding to the side lengths of Euclidean triangles with even perimeter, and the coefficients are written in terms of $6j$-symbols:*

$$\mathrm{C}_{j_iklm} = \frac{\Delta(j_i)}{\Theta(a',b,j_i)}\begin{bmatrix} a & a' & k\\ c & j_i & b\end{bmatrix}\begin{bmatrix} j_i & b & l\\ c' & b' & a'\end{bmatrix}\begin{bmatrix} k & l & m\\ c & c' & j_i\end{bmatrix}.$$

A special case of this, which is proven independently, is:

**Corollary 4.6** (Central Function Recurrence). *Provided $a > 0$ and $c > 0$, we can write*

$$\chi_c^{a,b} = x\cdot\chi_{c-1}^{a-1,b} - \frac{(a+b-c)^2}{4a(a-1)}\chi_c^{a-2,b} - \frac{(-a+b+c)^2}{4c(c-1)}\chi_{c-2}^{a,b} - \frac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)}\chi_{c-2}^{a-2,b}.$$

Finally, using this relation and the the symmetry given by Theorem 4.1, we give a constructive proof of the following classical theorem due to Fricke, Klein [FK], and Vogt [Vo]:

**Theorem 4.9** (Fricke-Klein-Vogt). *Let $G$ act on $G\times G$ by simultaneous conjugation. Then*

$$\mathbb{C}[G \times G]^G \cong \mathbb{C}[x, y, z],$$

*the complex polynomial ring in three indeterminants. Consequently, every function $f : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathbb{C}$ which is invariant under simultaneous conjugation by $\mathrm{SL}(2,\mathbb{C})$, i.e.,*

$$f(\mathbf{x}_1, \mathbf{x}_2) = f(g\mathbf{x}_1 g^{-1}, g\mathbf{x}_2 g^{-1}) \qquad \textit{for all } g \in \mathrm{SL}(2,\mathbb{C}),$$

*can be written as a polynomial in the three variables $x = \mathrm{tr}(\mathbf{x}_1)$, $y = \mathrm{tr}(\mathbf{x}_2)$, and $z = \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$.*

## 2. Preliminaries

2.1. **Algebraic Structure of $\mathfrak{X}$.** $G = \mathrm{SL}(2,\mathbb{C})$ has the structure of an irreducible affine variety since it is the zero set in $\mathbb{C}^4$ of the irreducible polynomial

$$x_{11}x_{22} - x_{12}x_{21} - 1 \in \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}].$$

Hence, the coordinate ring of $G$ is given by

$$\mathbb{C}[G] = \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}]/(x_{11}x_{22} - x_{12}x_{21} - 1).$$

This implies that the representation variety $\mathcal{R} \cong G^{\times r}$ is irreducible as well.

We use the following facts, which may be found in [Do]:

- The set of maximal ideals (or geometric points) of the ring of invariants is the categorical quotient $G^{\times r}//G$.
- If a reductive algebraic group $G$ acts rationally on the set of geometric points of a complex algebra $A$ without zero divisors, then the algebra of invariants $A^G$ is finitely generated.

Hence $\mathbb{C}[G^{\times r}]^G$ is finitely generated, and

$$\mathfrak{X} = \mathrm{Spec}_{max}(\mathbb{C}[\mathcal{R}]^G) = \mathcal{R}//G$$

is an irreducible affine variety referred to as the *$G$-character variety* of $\pi$. We identify the character variety $\mathfrak{X} = G^{\times r}//G$ with the usual (reducible) characters of the representation variety $\mathcal{R}$. This identification is natural since the points of $\mathfrak{X}$ correspond to conjugacy classes of reducible representations. Procesi [Pr] has shown that $\mathbb{C}[\mathcal{R}]^G$ is generated by traces of products of matrix variables.

2.2. **Representation Theory of $G$.** The coordinate ring $\mathbb{C}[G]$ decomposes into a direct sum of tensor products of the finite-dimensional irreducible representations of $G$. We will use this decomposition, given explicitly by Theorem 3.1, to understand $\mathbb{C}[\mathfrak{X}]$. To this end, we review the representation theory of $G$, which is described in [BtD], [Do], and [FH].

Denote by $K = \mathrm{SU}(2)$, the maximal compact subgroup of $G$. There is an equivalence of categories between the finite-dimensional $\mathbb{C}$-linear representations of $G$ and $K$, given by passing through the equivalence of categories of the associated representations of the Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$. Since $K$ is homeomorphic to the 3-sphere, it is compact and its finite-dimensional representations are completely reducible. The symmetric powers of the standard representation of $K$, and thus of $G$ as well, are all irreducible representations and moreover they comprise a complete list.

Let $V_0 = \mathbb{C} = V_0^*$ be the trivial representation of $G$ and denote the standard representation and its dual by

$$V = V_1 = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \quad V^* = V_1^* = \mathbb{C}e_1^* \oplus \mathbb{C}e_2^*,$$

where $e_1$ and $e_2$ are the basic vectors $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, respectively. Denote the symmetric powers of these representations by

$$V_n = \mathrm{Sym}_n(V) \text{ and } V_n^* = \mathrm{Sym}_n(V^*).$$

Since there exists an invariant non-degenerate $\mathbb{C}$-bilinear form on $V_n$, we have $V_n \cong (V_n)^*$.

Moreover, $V_n^*$ is naturally isomorphic to $(V_n)^*$, so elements in $V_n$ pair with elements in $V_n^*$. Let the element $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$ project to $v_1 \circ v_2 \circ \cdots \circ v_n \in V_n$, so that we can express the basis elements of $V_n$ and $V_n^*$ by

$$\mathsf{n}_{n-k} = e_1^{n-k} e_2^k = \underbrace{e_1 \circ e_1 \circ \cdots \circ e_1}_{n-k} \circ \underbrace{e_2 \circ e_2 \circ \cdots \circ e_2}_{k} \quad \text{and}$$

$$\mathsf{n}_{n-k}^* = (e_1^*)^{n-k}(e_2^*)^k = \underbrace{e_1^* \circ e_1^* \circ \cdots \circ e_1^*}_{n-k} \circ \underbrace{e_2^* \circ e_2^* \circ \cdots \circ e_2^*}_{k},$$

respectively, for $k = 0, \ldots, n$. Then, this pairing is described by

$$\mathsf{n}_{n-k}^*(v_1 \circ v_2 \circ \cdots \circ v_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (\mathsf{n}_{n-k})^*(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}),$$

where $\Sigma_n$ is the symmetric group of permutations on $n$ elements. In particular,

$$\mathsf{n}_{n-k}^*(\mathsf{n}_{n-l}) = \frac{(n-k)!k!}{n!}\delta_{kl} = \tfrac{\delta_{kl}}{\binom{n}{k}}.$$

The tensor product $V_a \otimes V_b$, where $a, b \in \mathbb{N}$, is also a representation of $G$. Since $G$ is completely reducible, it may be decomposed into irreducible representations by

**Proposition 2.1** (Clebsch-Gordan formula).

$$V_a \otimes V_b \cong \bigoplus_{j=0}^{\min(a,b)} V_{a+b-2j}.$$

Finally, we give several versions of Schur's lemma, which we will use frequently.

**Proposition 2.2** (Schur's Lemma). *Let $G$ be a group, $V$ and $W$ representations of $G$, and $f \in \mathrm{Hom}_G(V, W)$ with $f \neq 0$.*

(1) *If $V$ is irreducible, then $f$ is injective.*
(2) *If $W$ is irreducible, then $f$ is surjective.*
(3) *If $V = W$, then $f$ is a homothety (a multiple of the identity).*
(4) *Suppose $V, W$ are irreducible:*
*if $V \cong W$, then $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 1$;*
*if $V \ncong W$, then $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 0$.*

See [BtD] or [CSM] for proof of Propositions 2.1 and 2.2.

With respect to the basis described above, we explicitly formulate the $G$-action on $V_n$. For a given $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in G$, then

$$g \cdot \mathsf{n}_{n-k} = \sum_{\substack{0 \le j \le n-k \\ 0 \le i \le k}} \binom{n-k}{j}\binom{k}{i} \left( g_{11}^{n-k-j} g_{12}^{k-i} g_{21}^{j} g_{22}^{i} \right) \mathsf{n}_{n-(i+j)},$$

and $G$ acts on $V_n^*$ in the usual way:

$$(g \cdot \mathsf{n}^*_{n-k})(v) = \mathsf{n}^*_{n-k}(g^{-1} \cdot v) \text{ for } v \in V_n.$$

It is left to the reader to verify that these are in fact actions.

2.3. **Spin Networks in Representation Theory.** At its heart, a spin network is a graph which can be assigned a specific function between tensor powers of the standard $\mathrm{SL}(2, \mathbb{C})$ representation $V = \mathbb{C}^2$. They were first introduced by Roger Penrose to aid in angular momentum calculations, who interpreted $\mathbb{C}^2$ vectors as *spinors* and $\mathrm{SL}(2, \mathbb{C})$ representations as quantum angular momentum states [Pen].

Before defining spin networks, we note that our characterization of spin networks differs from that in the literature [Ka, Pen, St]. This is because the standard definition is not well-suited to studying the algebraic structure of the character variety $\mathfrak{X}$, and in particular trace

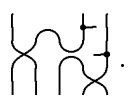
calculations. Also, our definition relates directly to the inherent structure of $G$ and $V$, and therefore easily extends to arbitrary matrix groups [Pet].

In this paper, we will fully develop our description of spin networks, due in part to the uniqueness of our presentation and in part to the uniqueness of diagrammatic methods.

**Definition 2.3.** A *spin network* $\mathcal{S}$ is a graph with vertex set $\mathcal{S}_i \sqcup \mathcal{S}_o \sqcup \mathcal{S}_v$ consisting of degree 1 'inputs' $\mathcal{S}_i$, degree 1 'outputs' $\mathcal{S}_o$ and degree 2 'ciliated vertices' $\mathcal{S}_v$. If there are $k_i = |\mathcal{S}_i|$ inputs and $k_o = |\mathcal{S}_o|$ outputs, then $\mathcal{S}$ is identified with a function $f_{\mathcal{S}} : V^{\otimes k_i} \to V^{\otimes k_o}$ where $V = \mathbb{C}^2$. If the spin network is *closed*, meaning $k_i = 0 = k_o$, it is identified with a complex scalar $f_{\mathcal{S}} \in \mathbb{C}$.

A *ciliation* is an ordering of edges incident to a vertex. When a graph is drawn in the plane, this may be represented by a mark drawn out from a vertex and labelling the edges in the order crossed when proceeding clockwise from this mark. Thus, in the degree 2 case, there are two possible ciliations: $\dashv^{1}_{2}$ and $\vdash^{2}_{1}$.

We normally draw spin networks in *general position* within a box (in the plane) with inputs at the bottom and outputs at the top. For example, the following spin network has two ciliated vertices and represents a function from $V^{\otimes 5} \to V^{\otimes 3}$:



This is convenient because the function composition $f_{\mathcal{S}'} \circ f_{\mathcal{S}}$ corresponds to the diagram composition $\mathcal{S}' \circ \mathcal{S}$ formed by placing $\mathcal{S}'$ on top of $\mathcal{S}$.

Since we are defining spin networks as ciliated graphs, it does not matter how we represent the graph in the plane. Thus, we can freely move strands about and "slide" ciliations along the strands without changing the underlying spin network.

Let $v, w \in V$ and let $\{e_1, e_2\}$ be the standard basis for $\mathbb{C}^2$. We compute the function $f_{\mathcal{S}}$ of a spin network $\mathcal{S}$ by decomposing it into the four *spin network component maps*:

- the *identity* $|\ : V \to V, \quad v \mapsto v$;
- the *cap* $\frown\ : V \otimes V \to \mathbb{C}, \quad v \otimes w \mapsto v^T w$ (inner product);
- the *cup* $\smile\ : \mathbb{C} \to V \otimes V, \quad 1 \mapsto e_1 \otimes e_1 + e_2 \otimes e_2$;
- the *cap vertex* $\curlywedge\ : V \otimes V \to \mathbb{C}, \quad v \otimes w \mapsto \det[v\ w]$.

In the most standard spin network definitions [CFS, Ka, Pen], $\frown$ and $\smile$ are omitted, while $\curlywedge$ is included with a multiplier of $i = \sqrt{-1}$. We will later see that omitting the $i$ factor gives an advantage with

trace calculations, while the $\frown$ and $\smile$ maps are included to simplify our next proof.

**Theorem 2.4.** *Any spin network $\mathcal{S}$ may be decomposed into the above component maps. Any such decomposition gives the same function, and so $f_{\mathcal{S}}$ is well-defined.*

*Proof.* It suffices to prove the statement for a connected spin network, which topologically must be either an arc or a circle. For each ciliated vertex in this network, there must be a $\curlywedge$. The only ambiguity in connecting the remainder of the diagram using cups, caps, and the identity, comes from 'kinks' of the form $\rotatebox{90}{$\sim$}$. However, the observation

$$| \frown \circ \smile | = | , \quad \text{or} \quad \rotatebox{90}{$\sim$} = | ,$$

makes this a moot point. For alternate proofs, see [CFS] or [Ka]. □

When computing a spin network's function, it is not convenient to limit ourselves to these four maps. We can make this process a little smoother by identifying some additional maps.

**Proposition 2.5.** *In the spin network sense,*

(1) *the swap,* $\times : V \otimes V \to V \otimes V$ *takes* $v \otimes w \mapsto w \otimes v$*;*
(2) *the vertex on a straight line,* $\dashv : V \to V$ *takes* $v \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) v$*;*
(3) *the vertex on a cup,* $\psi : \mathbb{C} \to V \otimes V$ *takes* $1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$*;*
(4) *with opposite ciliations,* $\curlyvee = -\curlywedge$, $\vdash = -\dashv$, $\curlyvee = -\psi$.

*Proof.* We have (1) since with respect to the overlying graph structure crossings are meaningless. We can compute (2) by recognizing that $\dashv = (\curlywedge \ |) \circ (| \ \smile)$, so $\dashv$ takes $v = \left(\begin{smallmatrix} v^1 \\ v^2 \end{smallmatrix}\right)$ to

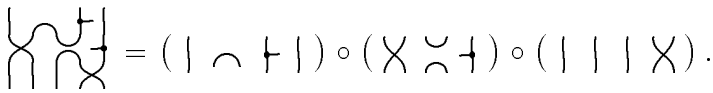
$$\begin{aligned} (\curlywedge \ |)\,(v \otimes e_1 \otimes e_1 + v \otimes e_2 \otimes e_2) &= \det[v\ e_1]e_1 + \det[v\ e_2]e_2 \\ &= -v^2 e_1 + v^1 e_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) v. \end{aligned}$$

Statement (3) is computed similarly, using the decomposition

$$\psi = (| \ \curlywedge \ |) \circ (\smile \ \smile).$$

Finally, (4) follows from the observation $\curlyvee = \curlywedge \circ \times$, which says that reversing the ciliation at a vertex corresponds to swapping the inputs, hence two columns of a determinant. □

As an example of these decompositions, we have:

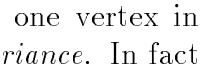


This could of course be further decomposed into just the four basic component maps.

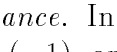
At this point, the cup and cap maps defined above are no longer necessary, and we make the following assumption:

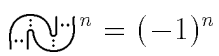
**Convention 2.6.** For the remainder of this paper, *we assume the set of ciliated vertices coincides exactly with the set of local extrema*, and we redefine the cap and cup functions:

$$= : 1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1;$$



$$= : v \otimes w \mapsto \det[v\ w].$$

This convention does preclude diagrams connecting vertices in $\mathcal{S}_i$ or $\mathcal{S}_o$ without at least one vertex in between, and comes at a price of *non-topological invariance*. In fact, each straightened kink  introduces a factor of $(-1)$, or more generally,



$$= (-1)^n$$

Thus, we have to be very careful with diagram manipulations which straighten such kinks. We avoid the common fix (multiplying and by $i$) for the following reason if the matrix map $\mathbf{x} : V \to V$ for $\mathbf{x} \in M_{2\times 2}$ is represented diagrammatically by , then we want

$$= \mathrm{tr}(\ \ ).$$

This works with our definition of spin networks, but one gets $-\mathrm{tr}(\ \ )$ for the definition commonly found in the literature.

2.3.1. *Reflections on Spin Networks.* There is a large amount of symmetry in spin networks which we can exploit to calculate more easily. First, since reflecting horizontally gives $= -$ , we see that:

**Proposition 2.7.** *If a spin network $\mathcal{S}$ corresponds to a function $f_{\mathcal{S}} : V^{\otimes k_i} \to V^{\otimes k_o}$, then its mirror image $\overleftrightarrow{\mathcal{S}}$ corresponds to the function $f_{\overleftrightarrow{\mathcal{S}}} = (-1)^{|\mathcal{S}_v|} \overleftrightarrow{f}_{\mathcal{S}} : V^{\otimes k_i} \to V^{\otimes k_o}$, where $|\mathcal{S}_v|$ is the number of local extrema in the diagram and $\overleftrightarrow{f}$ indicates that the ordering of inputs and outputs is reversed.*

There is also some vertical symmetry. The vertical flip of a diagram exchanges the inputs and outputs, and gives the *dual* of the original function, in the following sense:

**Proposition 2.8.** *If a spin network $\mathcal{S}$ corresponds to a function $f_{\mathcal{S}} : V^{\otimes k_i} \to V^{\otimes k_o}$, then its* dual diagram *$\mathcal{S}^{\ddagger}$, obtained by vertical reflection, corresponds to the function $f^*_{\mathcal{S}} : V^{\otimes k_o} \to V^{\otimes k_i}$ with*

$$f^*_{\mathcal{S}}(v_1 \otimes \cdots \otimes v_{k_i}) = \sum_{e_b \in \mathcal{B}(V^{\otimes k_i})} \left(f_{\mathcal{S}}(e_b) \cdot (v_1 \otimes \cdots \otimes v_{k_o})\right) e_b,$$

*where $\cdot$ is the inner product with respect to the standard basis for $V^{\otimes k_o}$, and $\mathcal{B}(V^{\otimes k_i})$ is the basis for $V^{\otimes k_i}$. Alternately stated, $f^*$ and $f$ are dual with respect to the standard inner product on $V$.*

*Proof.* Given $\mathcal{S} =$ and $v_i = \binom{v_i^1}{v_i^2}$, we have

$$f^*_{\mathcal{S}}(v_1 \otimes v_2) = \cup\,(1) \cdot (v_1 \otimes v_2) = (e_1 \otimes e_2 - e_2 \otimes e_1) \cdot (v_1 \otimes v_2)$$
$$= v_1^1 v_2^2 - v_1^2 v_2^1 = \det[v_1\ v_2] = \wedge\,(v_1 \otimes v_2).$$

In [Pet], it is shown that this, together with the computation for $\mathcal{S} =$ , is sufficient to show that the proposition holds in general. □

These symmetries give:

**Theorem 2.9** (Spin Network Reflection Theorem). *A relation*

$$\sum_m \alpha_m \mathcal{S}^m = 0$$

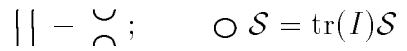
*among some collection of spin networks $\{\mathcal{S}^m\}$ is equivalent to the same relation for the vertically reflected spin networks $\{\mathcal{S}^{\updownarrow m}\}$ and (up to sign) for the horizontally reflected spin networks $\{\overleftrightarrow{\mathcal{S}}^m\}$, i.e.,*

$$\sum_m \alpha_m \mathcal{S}^{\updownarrow m} = 0 \qquad \textit{and} \qquad \sum_m \alpha_m (-1)^{|\mathcal{S}_v^m|} \overleftrightarrow{\mathcal{S}}^m = 0.$$

The proof is given in [Pet]. This fact allows us to freely use "reflected relations," simplifying many of the proofs in section 4.

2.3.2. *Working with Spin Networks.* Perhaps the most valuable asset of spin networks as a calculational tool is:

**Proposition 2.10.** *All crossings and loops may be removed from spin networks. In particular,*

(1) 

The proof is given in [Pet], although this is a good exercise.

The first of these relations is called the *Fundamental Binor Identity*, and represents a fundamental type of structure in mathematics; it is the core concept in defining both the *Kauffman Bracket Skein Module* in knot theory [Bu2] and the *Poisson bracket* on the set of loops on a surface, which Goldman describes in [G1]. It can also be identified with the *characteristic polynomial* for $2 \times 2$ matrices.

We now describe how spin networks interact with a given matrix $\mathbf{x} \in M_{2\times 2}$. Any such $\mathbf{x}$ acts naturally on $V^{\otimes k}$ by multiplication on each tensor factor:

$$\mathbf{x} \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \mathbf{x}v_1 \otimes \cdots \otimes \mathbf{x}v_n. \tag{2}$$

We represent the action $v \mapsto \mathbf{x} \cdot v$ by inserting a polygon on a strand, and thus identify  $\leftrightarrow \mathbf{x}$. Then, the action of $\mathbf{x}$ on $V^{\otimes k}$ is represented by . We are primarily interested in the case $\mathbf{x} \in \mathrm{SL}(2, \mathbb{C})$, for which spin networks are especially convenient.

**Proposition 2.11.** *The spin network component maps* , = , *and*  = , *and therefore all spin networks, are invariant under the natural action of* $\mathrm{SL}(2, \mathbb{C})$ *on* $V$ *described above.*

*Proof.* The case for the identity is clear, while

$(v \otimes w) = \det[\mathbf{x}v \ \mathbf{x}w] = \det(\mathbf{x} \cdot [v \ w])$

$= \det(\mathbf{x}) \cdot \det[v \ w] = 1 \cdot \det[v \ w] =$ $(v \otimes w)$

shows that $\circ \mathbf{x} = \mathbf{x} \circ$ .

The proof for follows by reflecting this relation. □

This means that matrices in such a diagram can "slide across" a vertex (local extremum) by simply inverting the matrix, so that

We can work with a general matrix $\mathbf{x} \in M_{2\times 2}$ almost as easily: the determinant frequently shows up since  $= \det($  $)$ and so $=$ $\det($  $)$ , provided $\mathbf{x}$ is invertible.

When a matrix  is placed somewhere within a *closed* spin network, we obtain a map $G \to \mathbb{C}$. For multiple matrices, we obtain a map $G \times \cdots \times G \to \mathbb{C}$. One of our primary motivations for this paper is the study of invariance properties of such maps. For example, we have the following interpretations of the simplest closed networks:

**Proposition 2.12.** *For* $\mathbf{x} \in M_{2\times 2}$ *and* $\mathbb{I} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, *we have:*

(3)  $= 2 = \mathrm{tr}(\mathbb{I})$;  $= \mathrm{tr}(\mathbf{x})$;  $= \det(\mathbf{x}) \cdot \mathrm{tr}(\mathbb{I})$.

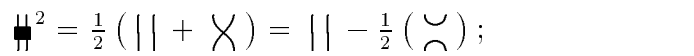
2.3.3. *Symmetrizers and Irreducible Representations.* We have one more $\mathrm{SL}(2, \mathbb{C})$-invariant map to introduce:

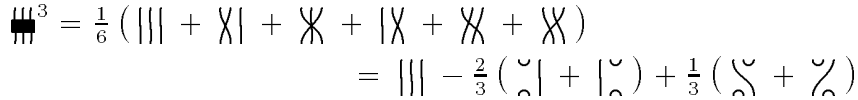
**Definition 2.13.** The *symmetrizer* $^n : V^{\otimes n} \to V^{\otimes n}$ is the map taking

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}, \tag{4}$$

where $v_i \in V$ and $\Sigma_n$ is the *symmetric group* of permutations on $n$ elements.

In the simplest examples, we have:

(5) 

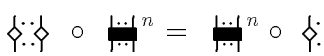
(6) 

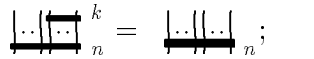
Note that the crossings are removed by applying the Fundamental Binor Identity.

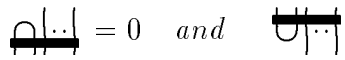
The defining equation (4) of should look familiar: its image is a subspace of $V^{\otimes n}$ isomorphic to the $n$th symmetric power $\mathrm{Sym}^n V$, and thus it can be thought of as either the projection $\pi : V^{\otimes n} \to \mathrm{Sym}^n V$ or as the inclusion $\iota : \mathrm{Sym}^n V \to V^{\otimes n}$ (see [FH], page 473).

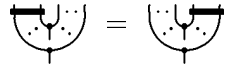
What does this mean for us? If a diagram from $V^{\otimes k_i}$ to $V^{\otimes k_o}$ has symmetrizers at its top and bottom, it can be thought of as a map between $V_{k_i}$ and $V_{k_o}$ instead. We freely interpret such spin networks as maps between tensor powers of these irreducible representations.

**Proposition 2.14** (Basic Symmetrizer Properties).

(7) *Invariance:* 

(8) *stacking relation:* 

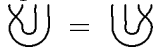
(9) *capping/cupping:*  $= 0$ *and*  $= 0$;

(10) *symmetrizer sliding:* 

*Proof.* The first relation (7) is evident if one expands the symmetrizer in terms of permutations, since permutations are $\mathrm{SL}(2,\mathbb{C})$-invariant.

The *stacking relation* is the statement that symmetrizing the last $k$ elements of a symmetric tensor has no effect, since they are already symmetric.

The *capping* and *cupping relations* follow from the equations $\circ$  $= 0$ and  $\circ$  $= 0$, and the stacking relation.

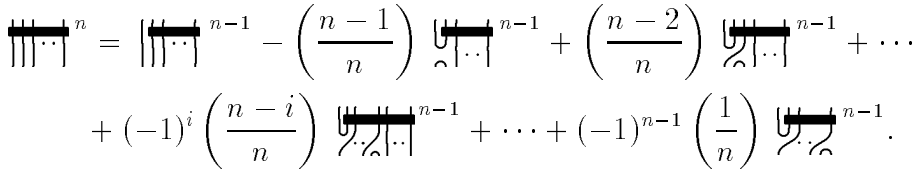
Two methods may be used to demonstrate (10). First, recall that the minimum corresponds to the map $=$ for $= g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$. Thus, this can be thought of as a special case of $\mathrm{SL}(2,\mathbb{C})$-invariance (7). Alternately, if we expand the symmetrizer, and write each permutation as a product of transpositions, then (10) follows from the simple relation  . □

We now move on to some more involved relations among symmetrizers. Although it is easy to write down an arbitrary in terms of permutations, it is usually rather difficult to write it down in terms of

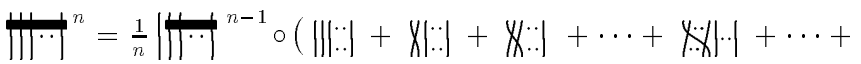
diagrams without crossings. The next two propositions give recurrence relations which can simplify this process.

**Proposition 2.15.** *The symmetrizer* *satisfies the following recurrence relation:*

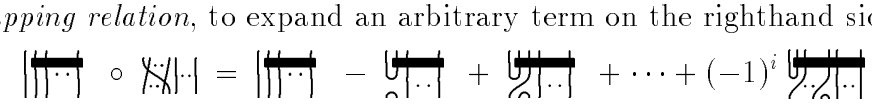
(11)

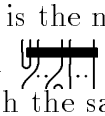


*Proof.* The proof begins with a basic combinatorial formula: a permutation on $n$ objects is determined uniquely by (a) the number mapping to 1; and (b) a permutation on the remaining $n-1$ numbers. In diagram form, this statement corresponds to the equation:

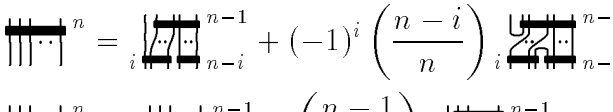


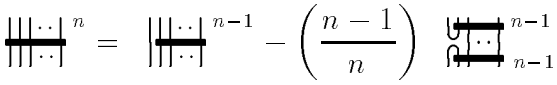
Now, use the binor identity and the key observation that any terms whose cups are not in the 'first position' on top will vanish, due to the *capping relation*, to expand an arbitrary term on the righthand side:



where $i$ is the number of 'kinks' in or 1 plus the number of kinks in  . Finally, group the number of terms on the righthand side with the same number of kinks together: there will be $n-i-1$ terms with $i$ kinks. □

**Proposition 2.16.** *also satisfies the recurrence relations:*

(12) 

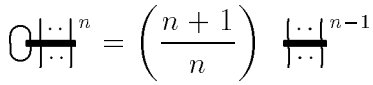
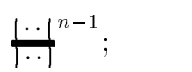
(13) 

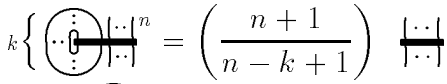
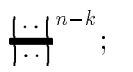
*Proof.* Compose relation (11) with $\otimes$ . This has no effect on the lefthand side, by the *stacking relation*. On the righthand side, all but one of the terms with a cap on the bottom vanish, due to the *capping relation*, since they will cap off either the or the . The one term which remains 'caps between' these two symmetrizers. The coefficient is $(-1)^i\left(\frac{n-i}{n}\right)$ since in recurrence (11), $i$ is equal to one more than the number of kinks in .

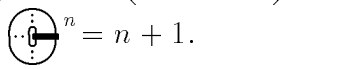
Relation (13) is clearly a special case of the first. □

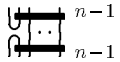
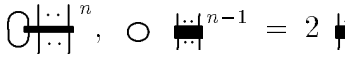
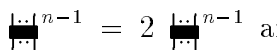
As a first application of the above, we have the *looping relation*:

**Proposition 2.17** (Looping Relations).

(14)  $= \left(\frac{n+1}{n}\right)$  ;

(15)  $= \left(\frac{n+1}{n-k+1}\right)$  ;

(16)  $= n+1.$

*Proof.* Close off the left strand in (13) above. Then, , , and  become  ,  $= 2$  and , respectively. Now collect terms to get (14), and proceed to (15) or (16) by applying the first relation $k$ or $n$ times. □

2.3.4. *Symmetrizers and Trivalent Spin Networks.* Recall the Clebsch-Gordan decomposition of Proposition 2.1,

$$V_a \otimes V_b \cong \bigoplus_{c \in \lceil a,b \rfloor} V_c, \qquad \lceil a,b \rfloor = \{a+b, a+b-2, \ldots, |a-b|\}.$$

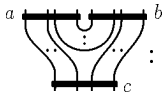
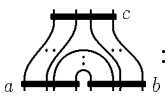
The condition $c \in \lceil a,b \rfloor$ is completely symmetric, since it is equivalent to the following:

**Convention 2.18.** We write $c \in \lceil a,b \rfloor$, or say that $\{a,b,c\}$ is an *admissible triple*, when $a,b,c$ are nonnegative integers satisfying:

(17) $$a+b+c \in 2\mathbb{Z}; \quad a \leq b+c; \quad b \leq a+c; \quad c \leq a+b.$$

This may alternately be interpreted as the integer side lengths of a Euclidean triangle with even perimeter.

Two maps arise from this decomposition: an injection $i_c^{a,b} : V_c \to V_a \otimes V_b$ and a projection $P_{a,b}^c : V_a \otimes V_b \to V_c$. Both have extremely simple diagrammatic representations (see [CFS]):

$i_c^{a,b} =$  $: V_c \to V_a \otimes V_b;$ $\qquad P_{a,b}^c =$  $: V_a \otimes V_b \to V_c.$

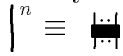
Note that the symmetrizers allow us to implicitly identify the tops and bottoms of these diagrams with tensor products of irreducible representations of $\mathrm{SL}(2,\mathbb{C})$.

We find it useful to introduce notation for the numbers of strands in these diagrams:

**Convention 2.19.** Given $a, b, c$, denote by $\alpha$, $\beta$, and $\gamma$ the total number of strands connecting $V_b$ to $V_c$, $V_a$ to $V_c$, and $V_a$ to $V_b$, respectively, and by $\delta$ the total number of strands in the diagram. Then:

$$\alpha = \tfrac{1}{2}(-a+b+c), \quad \beta = \tfrac{1}{2}(a-b+c), \quad \gamma = \tfrac{1}{2}(a+b-c); \quad \delta = \tfrac{1}{2}(a+b+c).$$

Note that $\{a, b, c\}$ is admissible if and only if $\alpha, \beta, \gamma \in \mathbb{N}$.

Since $i_c^{a,b}$ and $P_{a,b}^c$ will be so important for the remainder of this paper, we introduce a notation which simplifies their depiction. Let $n$ lines with a symmetrizer be represented by one **thick** line labelled $n$, so that  .

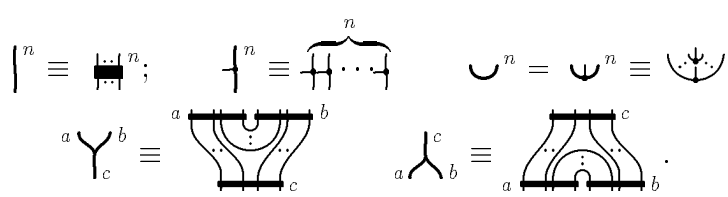
**Definition 2.20.** A *trivalent spin network* $\mathcal{S}$ is a graph drawn on the plane with vertices of degree $\leq 3$ and edges labelled by positive integers such that:

- 2-vertices are ciliated and coincide with local extrema;
- 3-vertices are drawn 'up' or 'down' ;
- any two edges meeting at a 2-vertex have the same label;
- the three labels adjacent to any vertex form an admissible triple.

If there are $m$ input edges with labels $l_i$ for $i = 1, \ldots, m$ and $n$ output edges with labels $l'_i$ for $i = 1, \ldots, n$, the network is identified with a map between tensor products of irreducible $\mathrm{SL}(2, \mathbb{C})$ representations,

$$f_{\mathcal{S}} : V_{l_1} \otimes \cdots \otimes V_{l_m} \to V_{l'_1} \otimes \cdots \otimes V_{l'_n},$$

which is computed by identifying $\mathcal{S}$ with a regular spin network using the following identifications:

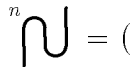


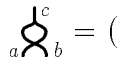

Note that ciliations are normally chosen to be on the local extrema, and degree-3 vertices, when expanded, also have a number of ciliated vertices. Thus, topological invariance becomes a more delicate operation, and the next section treats it as such.

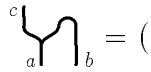
For the remainder of this paper, we assume that all sets of labels incident to a common vertex in a diagram are admissible. Moreover, whenever we sum over a label in a diagram, the sum is taken over all possible values of that label which make the requisite triples in the diagram admissible.

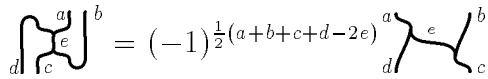
2.3.5. *Fun with Signs.* The identity gives rise to the following compendium of sign changes through diagram manipulations:

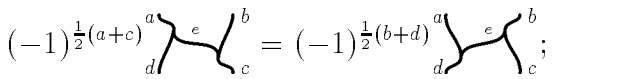
**Proposition 2.21.** *The following diagrams are equivalent up to sign:*

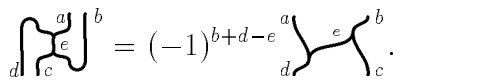
(18) 

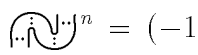
(19) 

(20) 

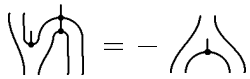
(21) 

(22) 

(23) 

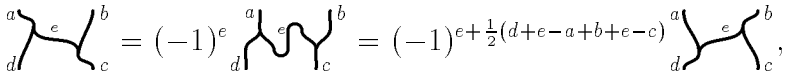
*Proof.* First, (18) is just a restatement of  , and (19) follows directly from the Reflection Theorem (2.22), since contains $\gamma = \frac{1}{2}(a+b-c)$ local extrema.

For (20), notice that in the simplest case



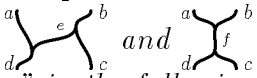
the negative sign comes from the strand on top of the diagram. Similarly, the general case for transforming into has a sign for each strand between $b$ and $c$, giving $(-1)^{\alpha} = (-1)^{\frac{1}{2}(-a+b+c)}$. This identity is used twice to give (21).

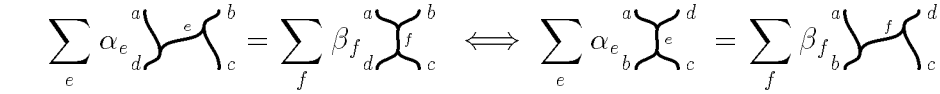
Finally, (22) follows from:



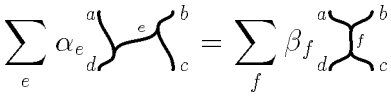
and (23) is given by combining (21) and (22). □

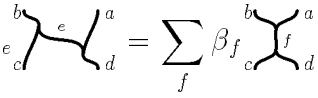
The above relations allow us to define a $\frac{\pi}{4}$ reflection on certain types of diagrams, which will be important later:

**Proposition 2.22.** *If a relation consists entirely of terms of the form*  *and* *, then one may "reflect about the line through a and c" in the following sense:*



*Proof.* By horizontally reflecting the first relation, using Theorem 2.9,

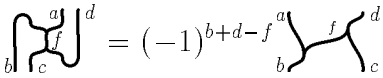


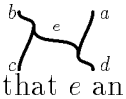
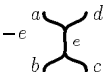
$$\iff \sum_e \alpha_e (-1)^{\frac{1}{2}(a+b+c+d-2e)}$$ 

$$\iff \sum_e \alpha_e$$ 

where the signs cancel due to the admissibility conditions.

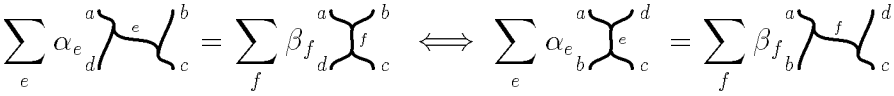
Now, add strands to both sides, so that the right side becomes



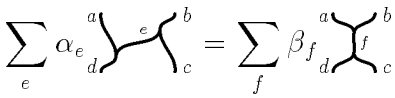
Likewise, on the left side,  becomes $(-1)^{b+d-e}$  . Once again, admissibility implies that $e$ and $f$ must have the same parity, so these signs cancel. □

We give two alternate versions of this proposition as a corollary, whose proof may be found in [Pet].

**Corollary 2.23.** *We have the additional equivalences:*

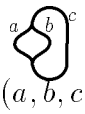


*and*



$$\iff \sum_e \alpha_e (-1)^{\frac{1}{2}(e-b)}$$ $$= \sum_f \beta_f (-1)^{\frac{1}{2}(d-f)}$$ .

2.3.6. *Properties of Trivalent Spin Networks.* As for regular spin networks, any closed trivalent spin network may be interpreted as a constant. The simplest such diagrams are given by

**Proposition 2.24.** *Let* $\Theta(a,b,c) =$  *and* $\Delta(c) =$ *(symmetrizer shown for clarity). Then* $\Theta(a,b,c)$ *is symmetric in* $\{a,b,c\}$

*and explicitly (recall the $\alpha, \beta, \gamma, \delta$ given in Convention 2.19):*

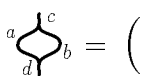
$$\Delta(c) = c + 1 = \dim(V_c); \tag{24}$$

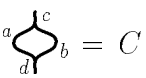
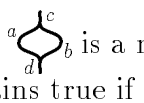
$$\Theta(a,b,c) = \frac{\left(\frac{-a+b+c}{2}\right)!\left(\frac{a-b+c}{2}\right)!\left(\frac{a+b-c}{2}\right)!\left(\frac{a+b+c+2}{2}\right)!}{a!b!c!} = \frac{\alpha!\beta!\gamma!(\delta+1)!}{a!b!c!}; \tag{25}$$

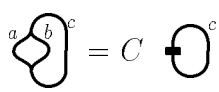
$$\Theta(1,a,a+1) = \Delta(a+1) = a+2. \tag{26}$$

*Proof.* The first equation (24) is a consequence of the *Looping Relation* (14). That $\Theta(1,a,a+1) = \Delta(a+1)$ is a consequence of the stacking relation, and demonstrates (26). We refer the reader to [CFS] for the $\Theta(a,b,c)$ formula. □

Ratios of $\Delta$ and $\Theta$ show up in the next two propositions, which tell us how to "pop bubbles" and how to "fuse together" two thick edges. The first demonstrates the usefulness of Schur's Lemma (Proposition 2.2) in diagrammatic techniques.

**Proposition 2.25** (Bubble Identity).  $= \left(\frac{\Theta(a,b,c)}{\Delta(c)}\right)\delta_{cd}$, *where $\delta_{cd}$ is the Kronecker delta.*

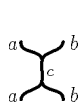
*Proof.* Schur's Lemma requires  $= C$ $\delta_{cd}$ for some constant $C$, since  is a map between irreducible representations. This equation remains true if we "close off" the diagrams, giving:

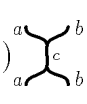
 $= C$  $\implies \quad C = \dfrac{\Theta(a,b,c)}{\Delta(c)}.$ □

**Proposition 2.26** (Fusion Identities).

$= \displaystyle\sum_{c\in\lceil a,b\rfloor}\left(\frac{\Delta(c)}{\Theta(a,b,c)}\right)$

$= \displaystyle\sum_{c\in\lceil a,b\rfloor}(-1)^{\frac{1}{2}(a-b+c)}\left(\frac{\Delta(c)}{\Theta(a,b,c)}\right)$ .

*Proof.* Maps of the form  for $c \in \lceil a,b\rfloor$ form a basis for the space of $\mathrm{SL}(2,\mathbb{C})$-invariant maps $V_a \otimes V_b \to V_a \otimes V_b$ (see [CFS]). Thus, we may express the first diagram as a linear combination:

$= \displaystyle\sum_{c\in\lceil a,b\rfloor} C(c)$  ,

Given a specific $d \in \lceil a, b \rfloor$, we may compute the constant $C(d)$ by composing this expression with ${}^{a}\!\Big\rceil^{b}_{d}$, giving:

$$
\begin{aligned}
{}^{a}\!\Big\rceil^{b}_{d} &= \sum_{c \in \lceil a,b \rfloor} C(c)\, {}^{a}\!\Big\rceil^{b}_{c} \circ {}^{a}\!\Big\rceil^{c}_{d}\!\Big\lceil^{b} = \sum_{c \in \lceil a,b \rfloor} C(c) \left( \frac{\Theta(a,b,c)}{\Delta(c)} \right) {}^{a}\!\Big\rceil^{b}_{c} \circ \Big\lceil^{d} \delta_{cd} \\
&= C(d) \left( \frac{\Theta(a,b,d)}{\Delta(d)} \right) {}^{a}\!\Big\rceil^{b}_{d} \qquad \Longrightarrow \qquad C(d) = \frac{\Delta(d)}{\Theta(a,b,d)}.
\end{aligned}
$$

For the second equation, we have

$$
\begin{aligned}
{}^{a}_{b}\!\times^{b}_{a} &= \sum_{c \in \lceil a,b \rfloor} (-1)^{\frac{1}{2}(-a+b+c)} \left( \frac{\Delta(c)}{\Theta(a,b,c)} \right) {}_{b}\!\Big\lceil\!\Big\rceil^{a}_{a}\!\times^{c}\!\Big\lceil\!\Big\rceil^{b} \\
&= \sum_{c \in \lceil a,b \rfloor} (-1)^{\frac{1}{2}(a-b+c)} \left( \frac{\Delta(c)}{\Theta(a,b,c)} \right) {}^{a}_{b}\!\Big\rceil\!\Big\lceil^{c}\!\Big\rceil\!\Big\lceil^{b}_{a}. \qquad \square
\end{aligned}
$$

## 3. Decomposition of $\mathbb{C}[G]$

The following theorem is a consequence of the "unitary trick"[Do], the Peter-Weyl Theorem [CSM], and the fact that the set of matrix coefficients of $G$ is exactly its coordinate ring [CSM]. We offer a self-contained constructive proof in section 3.2, since it gives us an explicit correspondence between regular functions and spin networks.

**Theorem 3.1.** *For $G = \mathrm{SL}(2, \mathbb{C})$, we have a $G$-module isomorphism:*

$$\mathbb{C}[G] \cong \sum_{n \geq 0} V_n^* \otimes V_n.$$

3.1. **Central Functions.** As a consequence of Theorem 3.1, we describe $\mathbb{C}[G \times G]^G$ in terms of an additive basis of class functions, which have an elegant realization as spin networks.

Indeed, as a consequence of Theorem 3.1 and the Clebsch-Gordan decomposition, we have the following decomposition:

$$\begin{aligned}
\mathbb{C}[G \times G] &\cong \mathbb{C}[G] \otimes \mathbb{C}[G] \\
&\cong \left( \sum_{a \geq 0} V_a^* \otimes V_a \right) \otimes \left( \sum_{b \geq 0} V_b^* \otimes V_b \right) \\
&\cong \sum_{a \geq 0} \sum_{b \geq 0} V_a^* \otimes V_a \otimes V_b^* \otimes V_b \\
&\cong \sum_{0 \leq a,b < \infty} (V_a^* \otimes V_b^*) \otimes (V_a \otimes V_b) \\
&\cong \sum_{0 \leq a,b < \infty} \left( \sum_{i=0}^{\min(a,b)} V_{a+b-2i}^* \right) \otimes \left( \sum_{j=0}^{\min(a,b)} V_{a+b-2j} \right) \\
&\cong \sum_{\substack{0 \leq a,b < \infty \\ 0 \leq i,j \leq \min(a,b)}} V_{a+b-2i}^* \otimes V_{a+b-2j}.
\end{aligned}$$

And hence, since all above maps are $G$-equivariant,

$$\mathbb{C}[G \times G]^G \cong \sum_{\substack{0 \leq a,b < \infty \\ 0 \leq i,j \leq \min(a,b)}} \left( V_{a+b-2i}^* \otimes V_{a+b-2j} \right)^G. \tag{27}$$

But by Schur's lemma (2.2),

$$\dim_{\mathbb{C}} \left( V_{a+b-2i}^* \otimes V_{a+b-2j} \right)^G = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

so

$$\mathbb{C}[G \times G]^G \cong \sum_{\substack{0 \leq a,b < \infty \\ 0 \leq j \leq \min(a,b)}} \mathrm{End}(V_{a+b-2j})^G.$$

**Definition 3.2.** Given the above isomorphism, for each $c \in \lceil a, b \rfloor$, there exists a class function $\chi_c^{a,b} \in \mathbb{C}[G \times G]^G$ which corresponds to a generating homothety (unique up to scalar) in $\mathrm{End}(V_c)^G$. We refer to the functions $\chi_c^{a,b}$ as *central functions*.

Denote by $\mathrm{Span}_{\mathbb{C}} \left( \chi_{a+b-2j}^{a,b} \right) \subset \mathbb{C}[G \times G]^G$ the linear span over $\mathbb{C}$ of $\chi_{a+b-2j}^{a,b}$. Then we can rewrite (27) as

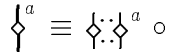
$$\mathbb{C}[G \times G]^G \cong \sum_{\substack{0 \leq a,b < \infty \\ c \in \lceil a,b \rfloor}} \mathrm{Span}_{\mathbb{C}} \left( \chi_c^{a,b} \right).$$

In terms of the functions $\chi_c^{a,b}$ we understand the additive structure of $\mathbb{C}[G \times G]^G$, and so have a canonical "Taylor-like" series description of the regular functions on $\mathfrak{X}$. In section 4, we will describe the multiplicative structure of $\mathbb{C}[G \times G]^G$ in terms of this additive basis of central functions.

This decomposition allows us to explicitly express these class functions as

$$\chi_c^{a,b}(\mathbf{x}_1, \mathbf{x}_2) = \operatorname{tr}\Big(\iota(\mathsf{c}_i^*)\Big((\mathbf{x}_1, \mathbf{x}_2) \cdot \iota(\mathsf{c}_j)\Big)\Big)_{ij},$$

where $\iota$ is the "Clebsch-Gordan" injection $V_c \hookrightarrow V_a \otimes V_b$, and $\{\mathsf{c}_j\}$ is a basis for $V_c$.

The functions $\chi_c^{a,b}$ take a natural diagrammatic form. If the matrix $\mathbf{x}_1$ is represented diagrammatically by , then its action on $V_a$ can be represented by  . If there are $r$ different matrices in a closed spin network, we can interpret it as a function $G^{\times r} \to \mathbb{C}$. In particular, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are depicted by and , respectively, then

$$\chi_c^{a,b}(\mathbf{x}_1, \mathbf{x}_2) = \text{[diagram]} = \text{[diagram]}.$$

As a special case, setting $\mathbf{x}_1 = \mathbf{x}_2 = \mathbb{I}$, where $\mathbb{I}$ is the identity matrix in $G$, gives $\chi_c^{a,b}(\mathbb{I}, \mathbb{I}) = \Theta(a, b, c)$.

Before explicitly computing the multiplicative structure of $\mathbb{C}[\mathfrak{X}]$, we prove the decomposition theorem.

3.2. **Proof of $\mathbb{C}[G]$ Decomposition Theorem.** Define

$$\Upsilon : \sum_{n \geq 0} V_n^* \otimes V_n \longrightarrow \mathbb{C}[G]$$

by linear extension of the mapping

$$\mathsf{n}_{n-k}^* \otimes \mathsf{n}_{n-l} \mapsto \mathsf{n}_{n-k}^*(\mathbf{x} \cdot \mathsf{n}_{n-l}),$$

where $\mathbf{x} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is a matrix variable. We will show this is an isomorphism in the following steps:

(1) Show that $\Upsilon$ is well-defined.
(2) Construct isomorphisms
  (i) $\bigoplus_{n \geq 0}(\operatorname{Hom}_G(V_n, \mathbb{C}[G]_R) \otimes V_n) \xrightarrow{\Phi} \mathbb{C}[G]$
  (ii) $\operatorname{Hom}_G(V_n, \mathbb{C}[G]_R) \xrightarrow{\Psi_n} V_n^*$.
(3) Show that $\Upsilon = \Phi \circ (\oplus \Psi_n \otimes \mathrm{id})$.

We first verify that $\Upsilon$ is a $G$-equivariant with the calculation:

$$\begin{aligned}\Upsilon(g\cdot(\mathsf{n}^*_{n-k}\otimes\mathsf{n}_{n-l})) &= \Upsilon\left((g\cdot\mathsf{n}^*_{n-k})\otimes(g\cdot\mathsf{n}_{n-l})\right)\\ &= (g\cdot\mathsf{n}^*_{n-k})(\mathbf{x}\cdot(g\cdot\mathsf{n}_{n-l})) = \mathsf{n}^*_{n-k}((g^{-1}\mathbf{x}g)\cdot\mathsf{n}_{n-l})\\ &= g\cdot\mathsf{n}^*_{n-k}(\mathbf{x}\cdot\mathsf{n}_{n-l}) = g\cdot\Upsilon(\mathsf{n}^*_{n-k}\otimes\mathsf{n}_{n-l}).\end{aligned}$$

Its image consists of regular functions since

$$\begin{aligned}\mathsf{n}^*_{n-k}(\mathbf{x}\cdot\mathsf{n}_{n-l}) &= \mathsf{n}^*_{n-k}\left((x_{11}e_1+x_{21}e_2)^{n-l}(x_{12}e_1+x_{22}e_2)^l\right)\\ &= \sum_{\substack{i+j=k\\0\le i\le n-l\\0\le j\le l}}\tbinom{n}{k}^{-1}\tbinom{n-l}{i}\tbinom{l}{j}x_{11}^{n-l-i}x_{12}^{l-j}x_{21}^{i}x_{22}^{j},\end{aligned}$$

which is clear a polynomial. We have now shown $\Upsilon$ is well-defined.

$G$ acts on the right of $\mathbb{C}[G]$ by $(f,g)\mapsto f\cdot g$, where

$$f\cdot g(\mathbf{x}) = f(\mathbf{x}g).$$

We let $\mathbb{C}[G]_R$ be the ring $\mathbb{C}[G]$ with this right action, to distinguish it from the diagonal conjugation action already imposed on $\mathbb{C}[G]$. Additionally, $G$ acts on the left of $\mathrm{Hom}_G(V_n,\mathbb{C}[G]_R)$ by $(g,\gamma)\mapsto g\cdot\gamma$, where

$$(g\cdot\gamma)(v)(\mathbf{x}) = \gamma_v(g^{-1}\mathbf{x}),$$

and $\gamma_v=\gamma(v)\in\mathbb{C}[G]$. This action is well-defined since

$$(g\cdot\gamma)(g'\cdot v)(\mathbf{x}) = \gamma_{g'\cdot v}(g^{-1}\mathbf{x}) = \gamma_v(g^{-1}\mathbf{x}\cdot g') = (g\cdot\gamma)(v)(\mathbf{x}\cdot g').$$

**Lemma 3.3.** *Define*

$$\Phi : \bigoplus_{n\ge 0}(\mathrm{Hom}_G(V_n,\mathbb{C}[G]_R)\otimes_{\mathbb{C}}V_n)\longrightarrow\mathbb{C}[G],$$

*by linear extention of the mappings $\gamma\otimes v\mapsto\gamma(v)$. $G$ acts on $\bigoplus(\mathrm{Hom}_G(V_n,\mathbb{C}[G]_R)\otimes_{\mathbb{C}}V_n)$ by $g\cdot(\sum\gamma\otimes v)=\sum(g\cdot\gamma)\otimes(g\cdot v)$. With respect to this action, $\Phi$ is an isomorphism of $G$-modules.*

Before we prove Lemma 3.3, we require some preliminary technical results.

**Lemma 3.4.** *Every regular function is contained in a finite-dimensional sub-representation of $\mathbb{C}[G]$.*

*Proof of Lemma 3.4.* Since we are considering two actions of $G$, namely the diagonal and right $G$-actions, we consider the following $G\times G$-action which encompasses both by restriction. Let this action

$$\alpha : G\times G\times G\longrightarrow G$$

be defined by $(g_1, g_2, \mathbf{x}) \mapsto g_1 \mathbf{x} g_2^{-1}$, and further let

$$\alpha^* : \mathbb{C}[G] \longrightarrow \mathbb{C}[G \times G \times G] \cong \mathbb{C}[G]^{\otimes 3} \tag{28}$$

defined by $f \mapsto f \circ \alpha$ be the *pull-back* of regular functions on $G$ to regular functions on $G^{\times 3}$. For $f \in \mathbb{C}[G]$, (28) implies that there exists $n_f \in \mathbb{N}$ and regular functions $f_i, f_i', f_i''$ for $1 \leq i \leq n_f$ such that

$$\alpha^*(f) = \sum_{i=1}^{n_f} f_i \otimes f_i' \otimes f_i''.$$

Therefore

$$\alpha^*(f)(g_1^{-1}, g_2^{-1}, \mathbf{x}) = \sum_{i=1}^{n_f} f_i(g_1^{-1}) f_i'(g_2^{-1}) f_i''(\mathbf{x}).$$

On the other hand,

$$\alpha^*(f)(g_1^{-1}, g_2^{-1}, \mathbf{x}) = f(\alpha(g_1^{-1}, g_2^{-1}, \mathbf{x})) = f(g_1^{-1} \mathbf{x} g_2) = ((g_1, g_2) \cdot f)(\mathbf{x}),$$

which implies

$$(g_1, g_2) \cdot f = \sum_{i=1}^{n_f} f_i(g_1^{-1}) f_i'(g_2^{-1}) f_i''. \tag{29}$$

Let $(G \times G)f = \{(g_1, g_2) \cdot f : f \in G\}$ be the $G \times G$-orbit of $f$, and let $V_f$ be the linear subspace spanned over $\mathbb{C}$ by $(G \times G)f$ in $\mathbb{C}[G]$. $V_f$ is finite-dimensional by (29), and so $\{f_i''\}$ is a finite spanning set. Clearly $V_f$ is $G \times G$-invariant, and so invariant with respect to the diagonal and right $G$-actions. Thus, it is a finite-dimensional sub-representation containing $f$. □

**Lemma 3.5.** $\mathbb{C}[G]$ *is completely* $G \times G$*-reducible.*

*Proof of Lemma 3.5.* Let $\mathcal{I}$ be the set of direct sums of irreducible finite-dimensional sub-representations of $\mathbb{C}[G]$. $\mathcal{I}$ is partially ordered by set inclusion and is non-empty since $\mathbb{C}[G] \neq \{\phi\}$ and any $V_f$ is completely reducible since it is a finite-dimensional $G$-module. Thus, by Zorn's lemma there exists a maximal element $M \in \mathcal{I}$. If $M \neq \mathbb{C}[G]$, then consider any $f \notin M$. There exists $V_f$, a finite-dimensional sub-representation that contains $f$, by Lemma 3.4. Recall that $K = \mathrm{SU}(2)$ is the maximal compact subgroup of $G$. Restricting the action of $G \times G$ to $K \times K$, we find an orthogonal complement to $V_f$ in $M \cup V_f$, which we denote by $M^\perp$. But then $M^\perp \oplus V_f \in \mathcal{I}$, since $K \times K$ representations extend to $G \times G$ representations. Hence we contradict the maximality of $M$. Therefore $\mathbb{C}[G]$ is completely reducible with respect to the

$G \times G$-action, and so with respect to the diagonal and right $G$-actions. In particular,

$$\mathbb{C}[G] \cong \sum_{j \geq 0} c_j V_j,$$

where $c_j \in \mathbb{N}$ is the multiplicity of $V_j$ in $\mathbb{C}[G]$. This decomposition holds for both $\mathbb{C}[G]$ and $\mathbb{C}[G]_R$ since they both are restrictions of the same $G \times G$-action. □

*Proof of Lemma 3.3.* By Lemma 3.5,

$$\Phi : \bigoplus_{n \geq 0} (\mathrm{Hom}_G(V_n, \mathbb{C}[G]_R) \otimes_{\mathbb{C}} V_n) \longrightarrow \mathbb{C}[G]$$

is an isomorphism if and only if

$$\bigoplus_{n \geq 0} \left( \sum_{j \geq 0} c_j \mathrm{Hom}_G(V_n, V_j) \otimes_{\mathbb{C}} V_n \right) \longrightarrow \sum_{j \geq 0} c_j V_j$$

is an isomorphism. By Schur's Lemma, this reduces to

$$\bigoplus_{n \geq 0} (c_n \mathrm{Hom}_G(V_n, V_n) \otimes_{\mathbb{C}} V_n) \longrightarrow \sum_{n \geq 0} c_n V_n,$$

which is in turn equivalent to

$$\bigoplus_{n \geq 0} (c_n \mathbb{C} \otimes_{\mathbb{C}} V_n) \longrightarrow \sum_{n \geq 0} c_n V_n.$$

However, this is the map sending $\sum \lambda \otimes v \mapsto \sum \lambda v$ for $\lambda \in \mathbb{C}$ and $v \in V_n$, which is canonically an isomorphism. □

We can now finish the proof of the theorem. Define

$$\Psi_n : V_n^* \longrightarrow \mathrm{Hom}_G(V_n, \mathbb{C}[G]_R)$$

by $w^* \mapsto \mathbf{F}_{w^*}$, where $\mathbf{F}_{w^*}(v)(\mathbf{x}) = w^*(\mathbf{x} \cdot v)$. $\Psi_n$ is well-defined since

$$\begin{aligned} \mathbf{F}_{w^*}(g \cdot v)(\mathbf{x}) &= w^*(\mathbf{x} \cdot (g \cdot v)) = w^*((\mathbf{x}g) \cdot v) \\ &= g \cdot w^*(\mathbf{x} \cdot v) = g \cdot (\mathbf{F}_{w^*}(v))(\mathbf{x}), \end{aligned}$$

and is $G$-equivariant because

$$\begin{aligned} \Psi_n(g \cdot w^*)(v)(\mathbf{x}) &= \mathbf{F}_{g \cdot w^*}(v)(\mathbf{x}) = (g \cdot w^*)(\mathbf{x} \cdot v) = w^*((g^{-1}\mathbf{x}) \cdot v) \\ &= \mathbf{F}_{w^*}(v)(g^{-1}\mathbf{x}) = (g \cdot \mathbf{F}_{w^*})(v)(\mathbf{x}) = g \cdot \Psi_n(w^*)(v)(\mathbf{x}). \end{aligned}$$

Since $V_n^*$ is irreducible, Schur's Lemma implies $\Psi_n$ is injective. We now show surjectivity. Consider $\gamma \in \mathrm{Hom}_G(V_n, \mathbb{C}[G]_R)$. For $\mathbb{I} \in G$, the function $\gamma(v)(\mathbb{I})$ is linear in $v \in V_n$. Hence there exists $w^* \in V_n^*$ such

that $w^*(v) = \gamma(v)(\mathbb{I})$ for all $v \in V_n$. We now show $\mathbf{F}_{w^*} = \gamma$ which proves that $\Psi_n$ is surjective.

$$\begin{aligned}\mathbf{F}_{w^*}(v)(\mathbf{x}) &= w^*(\mathbf{x}\cdot v) = \gamma(\mathbf{x}\cdot v)(\mathbb{I})\\ &= (\mathbf{x}\cdot\gamma)(v)(\mathbb{I}) = \gamma(v)(\mathbb{I}\cdot\mathbf{x}) = \gamma(v)(\mathbf{x}).\end{aligned}$$

Therefore,

$$\sum_{n\geq 0} V_n^* \otimes V_n \cong \bigoplus_{n\geq 0} \left(\mathrm{Hom}_G(V_n, \mathbb{C}[G]_R) \otimes V_n\right),$$

given by the map $\Psi = \oplus(\Psi_n \otimes \mathrm{id})$.

Finally, we verify $\Upsilon = \Phi \circ \Psi$:

$$\begin{aligned}\Phi\circ\Psi(w^*\otimes v)(\mathbf{x}) &= \Phi(\mathbf{F}_{w^*}\otimes v)(\mathbf{x}) = \mathbf{F}_{w^*}(v)(\mathbf{x})\\ &= w^*(\mathbf{x}\cdot v) = \Upsilon(w^*\otimes v)(\mathbf{x}). \quad \square\end{aligned}$$

3.3. **Ring Structure of $\mathbb{C}[G]^G$.** We have established

$$\mathbb{C}[G] \cong \sum_{n\geq 0} V_n^* \otimes V_n.$$

Since $V_n^* \cong (V_n)^*$ , $V_n^* \otimes V_n \cong \mathrm{End}(V_n)$, for all $n \in \mathbb{N}$. Hence, by Schur's Lemma,

$$\mathbb{C}[G]^G \cong \sum_{n\geq 0} (V_n^* \otimes V_n)^G \cong \sum_{n\geq 0} \mathrm{Span}_{\mathbb{C}}(\chi_n),$$

where $\chi_n \in \mathrm{End}(V_n)^G$ is a homothety.

Therefore, using the isomorphism $V_n^* \otimes V_n \to \mathrm{End}(V_n)$ given by

$$\mathsf{n}_{n-k}^* \otimes \mathsf{n}_{n-l} \mapsto (\mathsf{n}_{n-k})^*(\mathbf{x})\mathsf{n}_{n-l},$$

we compute $\chi_0 = \mathsf{n}_0^* \otimes \mathsf{n}_0 = 1$, and $\chi_1 = \mathsf{n}_0^* \otimes \mathsf{n}_0 + \mathsf{n}_1^* \otimes \mathsf{n}_1$.

Since $V_n$ is irreducible, Burnside's Theorem [La] implies $\mathrm{End}(V_n)$ is algebraically generated by $G < \mathrm{Aut}(V_n)$. Hence, $\mathsf{n}_{n-k}^* \otimes \mathsf{n}_{n-l}$ is the matrix coefficient, column $n-k+1$ and row $n-l+1$, of $V_n$.

For example, consider $\chi_1$. $V_1$ is the standard representation and its diagonal matrix coefficients, for

$$\mathbf{x} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

are $x_{11}$ and $x_{22}$. Hence

$$\chi_1 = x_{11} + x_{22} = \mathrm{tr}(\mathbf{x}).$$

Recall that a triple of non-negative integers $\{a, b, c\}$ is *admissible* if

$$a+b+c \in 2\mathbb{Z}; \quad a \leq b+c; \quad b \leq a+c; \quad c \leq a+b,$$

and that the set of integers $c$ for which $\{a, b, c\}$ is admissible is denoted by $\lceil a, b \rfloor$ (Convention 2.18).

Given the functions $\chi_0$ and $\chi_1$, we can determine a recursive formula for a general $\chi_n$ using the following multiplication formula.

**Theorem 3.6** (Product Formula).

$$\chi_a \chi_b = \sum_{c \in \lceil a,b \rfloor} \chi_c \tag{30}$$

*Proof.* From the Clebsch-Gordan decomposition,

$$(V_a \otimes V_b)^* \otimes (V_a \otimes V_b) \cong \sum_{0 \le j,k \le \min(a,b)} V^*_{a+b-2j} \otimes V_{a+b-2k},$$

and so from Schur's Lemma

$$\mathrm{End}(V_a \otimes V_b)^G \cong \sum_{0 \le j \le \min(a,b)} \mathrm{End}(V_{a+b-2j})^G.$$

Hence the characters satisfy

$$\chi_a \chi_b \quad = \quad \chi_{(V_a \otimes V_b)} = \chi_{(\oplus_j V_{a+b-2j})} = \sum_{c \in \lceil a,b \rfloor} \chi_c. \quad \square$$

Using the product formula (30) and the initial calculations of $\chi_0$ and $\chi_1$, we prove

**Theorem 3.7.** $\mathbb{C}[G]^G \cong \mathbb{C}[t]$

*Proof.* Consider the ring homomorphism $\Phi : \mathbb{C}[t] \to \mathbb{C}[G]^G$ defined by $f \mapsto f \circ \mathrm{tr}$. Suppose $f(\mathrm{tr}(g)) = 0$ for all $g \in G$. If $f \neq 0$, then since $f$ has a finite number of zeros, $\mathrm{tr}(g)$ must have a finite number of values. However,

$$\begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \in G$$

for all values of $t$. Hence, $f = 0$ and $\Phi$ is injective. It remains to establish surjectivity. We have already shown $t \mapsto \chi_1$ and $1 \mapsto \chi_0$. Suppose $a \geq 2$ and $\chi_b$ is in the image of $\Phi$ for all $b < a$. Equation (30) implies $\chi_1 \chi_{a-1} = \chi_a + \chi_{a-2}$. Thus, by induction,

$$t\Phi^{-1}(\chi_{a-1}) - \Phi^{-1}(\chi_{a-2}) \mapsto \chi_a. \quad \square$$

We can also express $\chi_n$ as a function of eigenvalues. Since $\chi_n$ is determined by its values on normal forms,

$$\begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{pmatrix} \in G,$$

computing the matrix representations of such forms give formulas for the functions $\chi_n$. Explicitly, $\left(\begin{smallmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ acts on $V_n$ by the matrix

$$\begin{pmatrix} \lambda^n & * & * & \cdots & * \\ 0 & \lambda^{n-2} & * & \cdots & * \\ \vdots & 0 & \ddots & * & * \\ 0 & \vdots & 0 & \lambda^{2-n} & * \\ 0 & 0 & \cdots & 0 & \lambda^{-n} \end{pmatrix}.$$

Hence,

$$\chi_n = \lambda^n + \lambda^{n-2} + \cdots + \lambda^{2-n} + \lambda^{-n} = \frac{\lambda^{n+1} - \lambda^{-n-1}}{\lambda - \lambda^{-1}},$$

which are the Chebyshev polynomials. These rational functions defined in terms of eigenvalues, correspond to elements in $\mathbb{C}[t]$ by $t \mapsto \lambda + \lambda^{-1}$.

## 4. Structure of $\mathbb{C}[G \times G]^G$

We now consider $\mathbb{C}[G \times G]^G$ and in like manner begin with some basic computations. As before, $\chi_0^{0,0} = 1$.

Let $\mathbf{x}_1 = \begin{pmatrix} x_{11}^1 & x_{12}^1 \\ x_{21}^1 & x_{22}^1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} x_{11}^2 & x_{12}^2 \\ x_{21}^2 & x_{22}^2 \end{pmatrix}$ be matrix variables, and let

$$\begin{aligned} x &= \operatorname{tr}(\mathbf{x}_1) = x_{11}^1 + x_{22}^1, \\ y &= \operatorname{tr}(\mathbf{x}_2) = x_{11}^2 + x_{22}^2, \\ z &= \operatorname{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}) = (x_{11}^1 x_{22}^2 + x_{22}^1 x_{11}^2) - (x_{12}^1 x_{21}^2 + x_{21}^1 x_{12}^2). \end{aligned}$$

From the Clebsch-Gordan decomposition and Schur's lemma, for $c \in \lceil a, b \rfloor$ there exists a unique (up to a scalar) injection

$$\iota : V_c \hookrightarrow V_a \otimes V_b.$$

Similarly, we have an inclusion

$$V_c^* \hookrightarrow V_a^* \otimes V_b^* \cong (V_a \otimes V_b)^*.$$

Recall that we have shown

$$\mathbb{C}[G \times G]^G \cong \sum_{\substack{0 \le a,b < \infty \\ c \in \lceil a,b \rfloor}} \operatorname{Span}_{\mathbb{C}}(\chi_c^{a,b}),$$

where $\chi_c^{a,b}$ corresponds to

$$\sum_{0 \le k \le c} \mathsf{c}_k^* \otimes \mathsf{c}_k \in V_c^* \otimes V_c \hookrightarrow V_a^* \otimes V_b^* \otimes V_a \otimes V_b,$$

and $\{\mathsf{c}_k^* \otimes \mathsf{c}_k\}$ is a basis for $V_c$.

Let $\{\mathsf{a}_k^* \otimes \mathsf{a}_k\}$ be a basis for $V_a$, and $\{\mathsf{b}_k^* \otimes \mathsf{b}_k\}$ be a basis for $V_b$. We will describe $\iota$. To motivate its construction, we begin with the invariance of the inclusion $V_0 \hookrightarrow V_1 \otimes V_1$ given by

$$\mathsf{c}_0 \mapsto \mathsf{a}_0 \otimes \mathsf{b}_1 - \mathsf{a}_1 \otimes \mathsf{b}_0.$$

The invariance follows from a straightforward calculation or the observation that it is the exterior product, which is unimodularly invariant. When $a = b$, the mapping $V_0 \hookrightarrow V_a \otimes V_b$ is given by projection of the map

$$\mathsf{c}_0 \mapsto (\mathsf{a}_0 \otimes \mathsf{b}_1 - \mathsf{a}_1 \otimes \mathsf{b}_0)^{\otimes \frac{a+b}{2}}, \tag{31}$$

and therefore is also invariant.

Moreover when $c = a + b$, the map

$$V^{\otimes c} \xrightarrow{s} V^{\otimes a} \otimes V^{\otimes b},$$

which independently symmetrizes the first $a$ and last $b$ factors of a basic element of $V^{\otimes c}$, is equivariant, since it preserves tensor degree. The following diagram

$$\begin{array}{ccc} V^{\otimes c} & \xrightarrow{s} & V^{\otimes a} \otimes V^{\otimes b} \\ \downarrow & & \downarrow \\ V_c & \xrightarrow{\iota} & V_a \otimes V_b, \end{array}$$

given by projection, is commutative, and so $\iota$ is given by

$$\tbinom{c}{k}\mathsf{c}_k \mapsto \sum_{\substack{0 \le i \le a \\ 0 \le j \le b \\ i+j=k}} \tbinom{a}{i}\mathsf{a}_i \otimes \tbinom{b}{j}\mathsf{b}_j, \tag{32}$$

and is also equivariant.

Before we write down the general form of $\iota$ we do some calculations. For $\chi_0^{1,1}$ we consider $V_0 \hookrightarrow V_1 \otimes V_1$. In this case

$$\mathsf{c}_0 \mapsto \mathsf{a}_0 \otimes \mathsf{b}_1 - \mathsf{a}_1 \otimes \mathsf{b}_0, \qquad \mathsf{c}_0^* \mapsto \mathsf{a}_0^* \otimes \mathsf{b}_1^* - \mathsf{a}_1^* \otimes \mathsf{b}_0^*.$$

So

$$\begin{aligned} \mathsf{c}_0^* \otimes \mathsf{c}_0 \mapsto \chi_0^{1,1} &= (\mathsf{a}_0^* \otimes \mathsf{b}_1^* - \mathsf{a}_1^* \otimes \mathsf{b}_0^*) \otimes (\mathsf{a}_0 \otimes \mathsf{b}_1 - \mathsf{a}_1 \otimes \mathsf{b}_0) \\ &= (\mathsf{a}_0^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_1^* \otimes \mathsf{b}_1) - (\mathsf{a}_1^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_1) \\ &\quad -(\mathsf{a}_0^* \otimes \mathsf{a}_1) \otimes (\mathsf{b}_1^* \otimes \mathsf{b}_0) + (\mathsf{a}_1^* \otimes \mathsf{a}_1) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_0) \\ &= x_{11}^1 \otimes x_{22}^2 - x_{12}^1 \otimes x_{21}^2 - x_{21}^1 \otimes x_{12}^2 + x_{22}^1 \otimes x_{11}^2 \\ &= (x_{11}^1 x_{22}^2 + x_{22}^1 x_{11}^2) - (x_{12}^1 x_{21}^2 + x_{21}^1 x_{12}^2) \\ &= z \end{aligned}$$

We next calculate $\chi_1^{1,0}$ and $\chi_1^{0,1}$, corresponding to the inclusions $V_1 \hookrightarrow V_0 \otimes V_1$ and $V_1 \hookrightarrow V_1 \otimes V_0$, respectively. With respect to the former,

$$\mathsf{c}_0 \mapsto \mathsf{a}_0 \otimes \mathsf{b}_0, \qquad \mathsf{c}_0^* \mapsto \mathsf{a}_0^* \otimes \mathsf{b}_0^*,$$
$$\mathsf{c}_1 \mapsto \mathsf{a}_1 \otimes \mathsf{b}_0, \qquad \mathsf{c}_1^* \mapsto \mathsf{a}_1^* \otimes \mathsf{b}_0^*,$$

and with respect to the latter,

$$\mathsf{c}_0 \mapsto \mathsf{a}_0 \otimes \mathsf{b}_0, \qquad \mathsf{c}_0^* \mapsto \mathsf{a}_0^* \otimes \mathsf{b}_0^*,$$
$$\mathsf{c}_1 \mapsto \mathsf{a}_0 \otimes \mathsf{b}_1, \qquad \mathsf{c}_1^* \mapsto \mathsf{a}_0^* \otimes \mathsf{b}_1^*.$$

Hence

$$\begin{aligned}\mathsf{c}_0^* \otimes \mathsf{c}_0 + \mathsf{c}_1^* \otimes \mathsf{c}_1 \mapsto \chi_1^{1,0} &= (\mathsf{a}_0^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_0) + (\mathsf{a}_1^* \otimes \mathsf{a}_1) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_0) \\ &= x_{11}^1 \otimes 1 + x_{22}^1 \otimes 1 = x_{11}^1 + x_{22}^1 = x\,.\end{aligned}$$

and

$$\begin{aligned}\mathsf{c}_0^* \otimes \mathsf{c}_0 + \mathsf{c}_1^* \otimes \mathsf{c}_1 \mapsto \chi_1^{0,1} &= (\mathsf{a}_0^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_0) + (\mathsf{a}_0^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_1^* \otimes \mathsf{b}_1) \\ &= 1 \otimes x_{11}^2 + 1 \otimes x_{22}^2 = x_{11}^2 + x_{22}^2 = y\,.\end{aligned}$$

Note that $V_{a-\gamma} \otimes (V \otimes V)^{\otimes\gamma} \otimes V_{b-\gamma}$ projects naturally to $V_a \otimes V_b$ via multiplication in the graded tensor ring

$$\sum_{0 \le a,b < \infty} V_a \otimes V_b.$$

With this in mind, we may combine (31) and (32) to give the general form of $\iota$, which is determined by mapping $\binom{c}{k}\mathsf{c}_k$, for $0 \le k \le c$ and $\gamma = \frac{a+b-c}{2}$, to the projection of

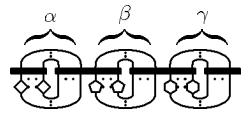
$$\sum_{\substack{0 \le i \le a-\gamma \\ 0 \le j \le b-\gamma \\ i+j=k}} \tbinom{a-\gamma}{i} \mathsf{a}_i \otimes (\mathsf{a}_0 \otimes \mathsf{b}_1 - \mathsf{a}_1 \otimes \mathsf{b}_0)^{\otimes\gamma} \otimes \tbinom{b-\gamma}{j} \mathsf{b}_j\,.$$

4.1. **Symmetry of Central Functions.** Our next result is not at all obvious via the algebraic definition of central functions, but essentially trivial in diagram form. In the theorem, we will use $\sigma(\diamondsuit_1, \diamondsuit_2, \diamondsuit_3)$ to denote the ordered triple $(\diamondsuit_{\sigma(1)}, \diamondsuit_{\sigma(2)}, \diamondsuit_{\sigma(3)})$ obtained by applying a given permutation $\sigma \in \Sigma_3$ to the triple $(\diamondsuit_1, \diamondsuit_2, \diamondsuit_3)$.

**Theorem 4.1** (Symmetry of Central Functions). *Suppose a central function is expressed as a polynomial $P$ in the variables $x = \mathrm{tr}(\mathbf{x}_1)$, $y = \mathrm{tr}(\mathbf{x}_2)$, and $z = \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$, so that $P_{a,b,c}(x,y,z) = \chi_c^{a,b}(\mathbf{x}_1, \mathbf{x}_2)$ for some admissible triple $\{a,b,c\}$. These polynomials are symmetric with respect to $(x,y,z)$ in the following sense:*

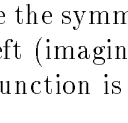
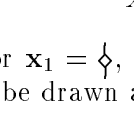
$$P_{\sigma(a,b,c)}(x,y,z) = P_{a,b,c}(\sigma^{-1}(y,x,z)).$$

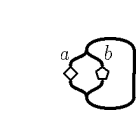
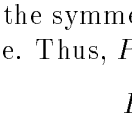
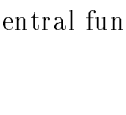
*Proof.* Define the following function $G \times G \times G \to \mathbb{C}$:

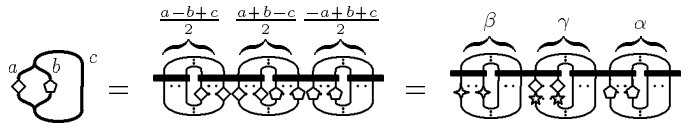
$X_{\alpha,\beta,\gamma}($  $) =$ 

,

where the symmetrizer on the right is assumed to 'wrap around' to the one on the left (imagine this diagram being drawn on a cylinder). By construction this function is symmetric, in the sense that:

$X_{\sigma(\alpha,\beta,\gamma)}\left(\sigma\left(\right.\right.$  $\left.\left.\right)\right) = X_{\alpha,\beta,\gamma}\left(\right.$  $\left.\right)$ .

For $\mathbf{x}_1 =$  , $\mathbf{x}_1^{-1} =$  , $\mathbf{x}_2 =$ , $\mathbf{x}_2^{-1} =$  , a central function $\chi_c^{a,b}(\mathbf{x}_1, \mathbf{x}_2)$ may be drawn as:



,

with the symmetrizers in the last two diagrams assumed to wrap around as before. Thus, $P_{a,b,c}(x,y,z) = X_{\alpha,\beta,\gamma}(\mathbf{x}_2, \mathbf{x}_1^{-1}, \mathbf{x}_1\mathbf{x}_2^{-1})$ and so:

$$\begin{aligned} P_{\sigma(a,b,c)}(x,y,z) &= X_{\sigma(\alpha,\beta,\gamma)}(\mathbf{x}_2, \mathbf{x}_1^{-1}, \mathbf{x}_1\mathbf{x}_2^{-1}) \\ &= X_{\alpha,\beta,\gamma}(\sigma^{-1}(\mathbf{x}_2, \mathbf{x}_1^{-1}, \mathbf{x}_1\mathbf{x}_2^{-1})) \\ &= P_{a,b,c}(\sigma^{-1}(y,x,z)). \end{aligned}$$

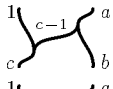
□

4.2. **A Recurrence Relation for Central Functions.** Define the *rank* of a central function to be:

$$\delta = \mathrm{rank}(\chi_c^{a,b}) = \tfrac{1}{2}(a+b+c).$$

We will obtain a recurrence relation for an arbitrary central function $\chi_c^{a,b}$ by manipulating diagrams to express the product $\mathrm{tr}(\mathbf{x}_1) \cdot \chi_c^{a,b}(\mathbf{x}_1, \mathbf{x}_2)$ as a sum of central functions. This formula can be rearranged to write $\chi_c^{a,b}$ as a linear combination of central functions *with lower rank*. There are three main ingredients to the diagram manipulations: the *bubble identity* and the *fusion identity* from section 2.3.6, and two *recoupling formulae* which we prove in the following lemma.

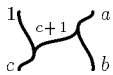
**Lemma 4.2.** *For $i = \frac{1}{2}(a+1-b+c)$ and appropriate triples admissible,*

(33) 

$=$

$-(-1)^i \left(\frac{a+b-c+1}{2(a+1)}\right)$

;

(34) 

$=(-1)^i \left(\frac{-a+b+c+1}{2(c+1)}\right)$

$+ \left(\frac{(a+b+c+3)(a-b+c+1)}{4(a+1)(c+1)}\right)$

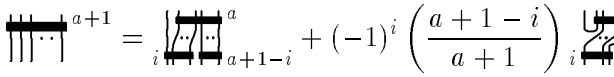
.

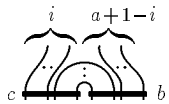
*Proof.* Note that $i$ is just the number of strands connecting $^{a+1}$ to $^{c}$ in

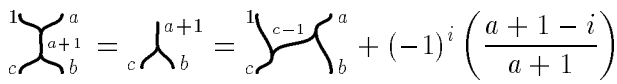
$=$

. For (33), use $n = a+1$ and $i$ in recurrence relation (12)

to get:

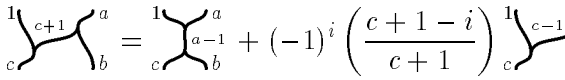


Compose this equation with  to get, via the *stacking relation*:

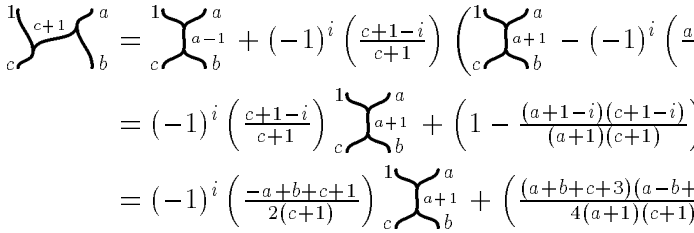


which is the desired result.

To prove (34), notice that if we switch $a$ and $c$ in the previous relation, and apply a $\frac{\pi}{4}$ reflection to the relation about the $1 \leftrightarrow b$ axis as in Proposition 2.22, then $i$ is unchanged and the equation becomes:



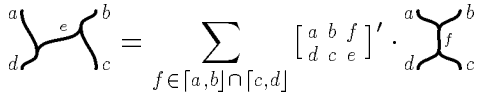
Rearrange this equation, and use (33) in its exact form to get:



To show the last computation, note that $a+1-i = \frac{1}{2}(a+b-c+1)$ and $c+1-i = \frac{1}{2}(-a+b+c+1)$, so the numerator of the last term is:

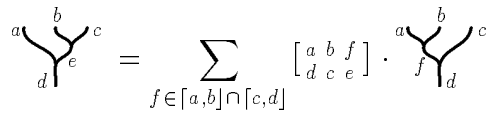
$$\begin{aligned}
4((a+1)(c+1)-(a+1-i)(c+1-i)) &= 4(a+1)(c+1)-((b+1)+(c-a))((b+1)-(c-a)) \\
&= 4(a+1)(c+1)-(b+1)^2+(a-c)^2 \\
&= ((a+1)-(c+1))^2+4(a+1)(c+1)-(b+1)^2 \\
&= ((a+1)+(c+1))^2-(b+1)^2 \\
&= (a+1+c+1+b+1)(a+1+c+1-b-1) \\
&= (a+b+c+3)(a-b+c+1). \qquad \square
\end{aligned}$$

The coefficients we have computed are examples of *6j-symbols*, most easily defined to be the coefficients $\left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix}\right]'$ in the following *change of basis* equation:



We use a prime because we will need an alternate version later:

**Definition 4.3.** The *6j-symbols* $\left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix}\right]$ are the coefficients given by




Both versions given here differ from those in the literature [CFS, Ka]. It is not hard to show, using Corollary 2.23, that

$$\left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix}\right]' = (-1)^{\frac{1}{2}(b+d-e-f)} \left[\begin{smallmatrix} a & b & f \\ d & c & e \end{smallmatrix}\right].$$

Thus, as a corollary to the above lemma we have the following $6j$-symbols, given by replacing $c$ with $c+1$ or $c-1$, which we will need in our next theorem.

**Corollary 4.4.**

$\left[\begin{smallmatrix} 1 & a & a+1 \\ c+1 & b & c \end{smallmatrix}\right] = 1;$ $\qquad \left[\begin{smallmatrix} 1 & a & a-1 \\ c+1 & b & c \end{smallmatrix}\right] = (-1)^{\frac{1}{2}(a-b+c+2)} \frac{(a+b-c)}{2(a+1)};$

$\left[\begin{smallmatrix} 1 & a & a+1 \\ c-1 & b & c \end{smallmatrix}\right] = (-1)^{\frac{1}{2}(a-b+c+2)} \frac{(-a+b+c)}{2c};$ $\qquad \left[\begin{smallmatrix} 1 & a & a-1 \\ c-1 & b & c \end{smallmatrix}\right] = \frac{(a+b+c+2)(a-b+c)}{4(a+1)c}.$

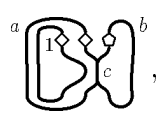
We can now prove the "multiplication by $x$" formula.

**Theorem 4.5.** *The product $x \cdot \chi_c^{a,b}(x,y,z)$ can be expressed by:*

$$x \cdot \chi_c^{a,b} = \chi_{c+1}^{a+1,b} + \frac{(a+b-c)^2}{4a(a+1)}\chi_{c+1}^{a-1,b} + \frac{(-a+b+c)^2}{4c(c+1)}\chi_{c-1}^{a+1,b} + \frac{(a+b+c+2)^2(a-b+c)^2}{16a(a+1)c(c+1)}\chi_{c-1}^{a-1,b}. \tag{35}$$

*This equation still holds for $a=0$ or $c=0$, provided we exclude the terms with $a$ or $c$ in the denominator.*

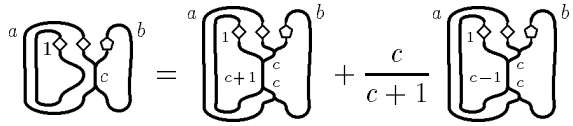
*Proof.* Diagrammatically, $x \cdot \chi_c^{a,b}(x,y,z)$ is represented by




,

since $x = \mathrm{tr}(\mathbf{x}_1) =$ and multiplication is automatic on disjoint diagrams. Now manipulate the diagram to obtain a sum over $\chi$'s with the following three steps.

First, we can apply the fusion identity to connect the lone strand to the $\chi_c^{a,b}$:



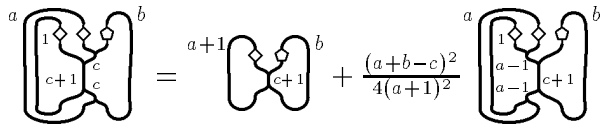

(36)

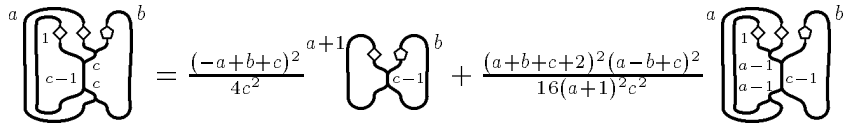
,

where the coefficients are evaluated from

$$\frac{\Delta(c\pm 1)}{\theta(1,c,c\pm 1)} = \frac{c\pm 1+1}{c+\frac{3}{2}\pm\frac{1}{2}}.$$

Second, use the $6j$-symbols computed in Corollary 4.4 above to move the $a$ strand from one side of the diagram to the other:

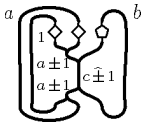
(37) 


(38) 

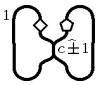
.

In each case, we are recoupling twice: once for the top piece and once for the corresponding bottom piece. In doing this, we would actually get four terms, but since the $a \pm 1$ labels *must be the same* on both the top and the bottom (a consequence of Schur's Lemma or the bubble identity), two of the terms vanish.

In the final step, use the bubble identity to collapse the final pieces:




$$= \frac{\Theta(1, a, a \pm 1)}{\Delta(a \pm 1)}$$




$$= \chi^{a+1,b}_{c\hat{\pm}1} \quad \text{or} \quad \left(\tfrac{a+1}{a}\right) \chi^{a-1,b}_{c\hat{\pm}1},$$

where $\hat{\pm}$ represents a sign which may differ from that in $a \pm 1$. At this point, obtaining (35) is simply a matter of multiplying the coefficients obtained in the previous formulae.

Now consider the special cases. For $a = 0$, since $b = c$ and consequently $\frac{c}{c+1} = \frac{(-a+b+c)^2}{4c(c+1)}$, the desired formula is exactly (36). Similarly, for $c = 0$, the desired formula is (37). □

We find it interesting that, for all our discussion of signs introduced by non-topological invariance, all signs introduced are eventually squared and thus do not show up in this result.

We can rearrange the terms in (35) and re-index to get:

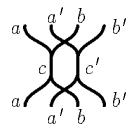
**Corollary 4.6** (Central Function Recurrence). *Provided $a > 0$ and $c > 0$, we can write*

$$\chi^{a,b}_c = x \cdot \chi^{a-1,b}_{c-1} - \tfrac{(a+b-c)^2}{4a(a-1)} \chi^{a-2,b}_c - \tfrac{(-a+b+c)^2}{4c(c-1)} \chi^{a,b}_{c-2} - \tfrac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)} \chi^{a-2,b}_{c-2}.$$

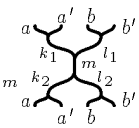
The condition $a > 0$, $c > 0$ arises because decrementing $a$ and $c$ in (35) means $\{a-1, b, c-1\}$ must now be admissible. Also, note that by applying the symmetry relation of Theorem 4.1, we could easily write down formulae for multiplication by $y$ and $z$ and two more recurrence relations. This fact is indispensable in our proof of Theorem 4.9.

4.3. **Multiplication of Central Functions.** It is not difficult to write down the formula for the product of two central functions, although the formula is by no means simple. We begin with a lemma which encapsulates the most tedious diagram manipulations:

**Lemma 4.7.**




$$= \sum_{i,j,k,l,m} \mathrm{C}^{abc,a'b'c'}_{j_1k_1l_1,j_2k_2l_2,m}$$




,

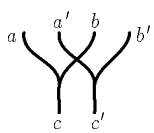
*where the coefficients are given by the formula*

$$\mathrm{C}^{abca'b'c'}_{j_1k_1l_1,j_2k_2l_2,m} = \frac{\Theta(c,c',m)}{\Delta(m)} \prod_{i=1,2} \frac{\Delta(j_i)}{\Theta(a',b,j_i)} \cdot \begin{bmatrix} a & a' & k_i \\ c & j_i & b \end{bmatrix} \begin{bmatrix} j_i & b & l_i \\ c' & b' & a' \end{bmatrix} \begin{bmatrix} k_i & l_i & m \\ c & c' & j_i \end{bmatrix},$$

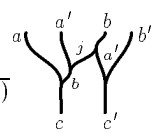
*and the following 13 triples are assumed to be admissible:*

$$\{a,a',k_i\},\ \{b,b',l_i\},\ \{c,c',m\},\ \{a',b,j_i\},\ \{c,k_i,j_i\},\ \{b,l_i,j_i\},\ \{k_i,l_i,m\}.$$

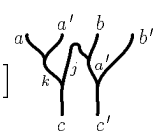
*Proof.* We will just demonstrate the diagram manipulation for the top half of the diagram, which by symmetry must be the same as for the bottom half. Combining these two manipulations and applying a bubble identity will give the desired result. We will save enumeration of admissible triples until after the manipulation, but keep a close eye on signs in the meantime.



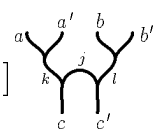

$$= \sum_j (-1)^{\frac{1}{2}(a'-b+j)} \frac{\Delta(j)}{\Theta(a',b,j)}$$



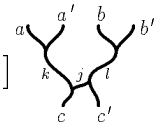

$$= \sum_{j,k} (-1)^{\frac{1}{2}(a'-b+j)+j} \frac{\Delta(j)}{\Theta(a',b,j)} \begin{bmatrix} a & a' & k \\ c & j & b \end{bmatrix}$$



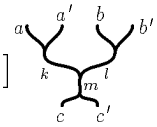

$$= \sum_{j,k,l} (-1)^{\frac{1}{2}(a'-b-j)} \frac{\Delta(j)}{\Theta(a',b,j)} \begin{bmatrix} a & a' & k \\ c & j & b \end{bmatrix} \begin{bmatrix} j & b & l \\ c' & b' & a' \end{bmatrix}$$




$$= \sum_{j,k,l} (-1)^{\frac{1}{2}(a'-b-j)+\frac{1}{2}(j+l-c')} \frac{\Delta(j)}{\Theta(a',b,j)} \begin{bmatrix} a & a' & k \\ c & j & b \end{bmatrix} \begin{bmatrix} j & b & l \\ c' & b' & a' \end{bmatrix}$$




$$= \sum_{j,k,l,m} (-1)^{\frac{1}{2}(a'-b+c-c'-j-m)+l} \frac{\Delta(j)}{\Theta(a',b,j)} \begin{bmatrix} a & a' & k \\ c & j & b \end{bmatrix} \begin{bmatrix} j & b & l \\ c' & b' & a' \end{bmatrix} \begin{bmatrix} k & l & m \\ c & c' & j \end{bmatrix}$$



The $(-1)$ terms all cancel in the end, a consequence of the fact that the following triples must be admissible:

$$\{a,a',k\},\ \{b,b',l\},\ \{c,c',m\},\ \{a',b,j\},\ \{c,k,j\},\ \{b,l,j\},\ \{k,l,m\}$$

One computes the 13-parameter coefficients $\mathrm{C}^{abc,a'b'c'}_{j_1k_1l_1,j_2k_2l_2,m}$ above by reflecting this result vertically, taking two sets of indices for the variables $j,k,l,m$ on the two halves, and noting that the resulting bubble in the middle collapses with a factor of $\frac{\Theta(c,c',m)}{\Delta(m)}$ for $m=m_1=m_2$. $\square$

With that out of the way, we can describe the central function multiplication table explicitly. Note the symmetry with respect to $k,l,m$, which is guaranteed by Theorem 4.1.

**Theorem 4.8** (Multiplication of Central Functions). *The product of two central functions $\chi^{a,b}_c$ and $\chi^{a'b'}_{c'}$ is given by:*

$$\chi^{a,b}_c\chi^{a'b'}_{c'} = \sum_{j_1,j_2,k,l,m} \mathrm{C}_{j_1klm}\mathrm{C}_{j_2klm}\frac{\Theta(a,a',k)\Theta(b,b',l)\Theta(c,c',m)}{\Delta(k)\Delta(l)\Delta(m)}\chi^{kl}_m,$$

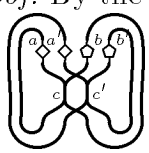
*where the sum is taken over admissible triples*

$$\{a,a',k\},\ \{b,b',l\},\ \{c,c',m\},\ \{a',b,j_i\},\ \{c,k,j_i\},\ \{b,l,j_i\},\ \{k,l,m\}$$

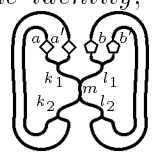
*and the coefficients are given by:*

$$\mathrm{C}_{j_iklm} = \frac{\Delta(j_i)}{\Theta(a',b,j_i)}\begin{bmatrix} a & a' & k \\ c & j_i & b\end{bmatrix}\begin{bmatrix} j_i & b & l \\ c' & b' & a'\end{bmatrix}\begin{bmatrix} k & l & m \\ c & c' & j_i\end{bmatrix}.$$

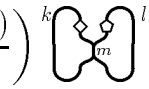
*Proof.* By the previous lemma and the *bubble identity*, we have:



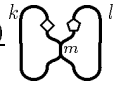

$$= \sum_{j_1,k_1,l_1,j_2,k_2,l_2,m} \mathrm{C}^{abc,a'b'c'}_{j_1k_1l_1,j_2k_2l_2,m}$$




$$= \sum_{j_1,j_2,k,l,m} \mathrm{C}^{abc,a'b'c'}_{j_1kl,j_2kl,m}\left(\frac{\Theta(a,a',k)\Theta(b,b',l)}{\Delta(k)\Delta(l)}\right)$$




$$= \sum_{i,j,k,l} \mathrm{C}_{j_1klm}\mathrm{C}_{j_2klm}\frac{\Theta(a,a',k)\Theta(b,b',l)\Theta(c,c',m)}{\Delta(k)\Delta(l)\Delta(m)}$$




. $\square$

4.4. **Applications.** Spin networks offer a novel approach to a classical theorem of Fricke, Klein, and Vogt [FK, Vo]. We give both a nonconstructive proof and a new *constructive* proof which depends on the symmetry, recurrence, and multiplication formulae for central functions.

**Theorem 4.9** (Fricke-Klein-Vogt Theorem). *Let $G$ act on $G\times G$ by simultaneous conjugation. Then*

$$\mathbb{C}[G\times G]^G \cong \mathbb{C}[x,y,z],$$

*the complex polynomial ring in three indeterminants. Consequently, every function* $f : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathbb{C}$ *which is invariant under simultaneous conjugation by* $\mathrm{SL}(2,\mathbb{C})$*, i.e.,*

$$f(\mathbf{x}_1, \mathbf{x}_2) = f(g\mathbf{x}_1 g^{-1}, g\mathbf{x}_2 g^{-1}) \qquad \textit{for all } g \in \mathrm{SL}(2,\mathbb{C}),$$

*can be written as a polynomial in the three variables* $x = \mathrm{tr}(\mathbf{x}_1)$*,* $y = \mathrm{tr}(\mathbf{x}_2)$*, and* $z = \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$*.*

*Proof.* Define the ring homomorphism

$$\Gamma : \mathbb{C}[x,y,z] \to \mathbb{C}[G \times G]^G$$

by $f(x,y,z) \mapsto f(\mathrm{tr}(\mathbf{x}_1), \mathrm{tr}(\mathbf{x}_2), \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}))$.

We first show that $\Gamma$ is injective. Suppose $f(\mathrm{tr}(\mathbf{x}_1), \mathrm{tr}(\mathbf{x}_2), \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})) = 0$ for all pairs $(\mathbf{x}_1, \mathbf{x}_2) \in G \times G$. Let $(x,y,z) \in \mathbb{C}^3$, $\epsilon_x = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$, and $\eta_y(\zeta)^{-1} = \begin{pmatrix} 0 & \frac{-1}{\zeta} \\ \zeta & y \end{pmatrix}$, where $\zeta = \frac{z+\sqrt{z^2-4}}{2}$. Then

$$(x,y,z) = (\mathrm{tr}(\epsilon_x), \mathrm{tr}(\eta_y(\zeta)), \mathrm{tr}(\epsilon_x \eta_y(\zeta)^{-1})).$$

Hence $f = 0$ on $\mathbb{C}^3$, $\mathrm{Ker}(\Gamma) = \{0\}$, and $\Gamma$ is injective. This is the "Fricke slice" given by Goldman in [G3].

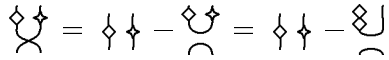
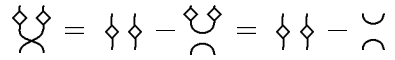
It remains to show that $\Gamma$ is surjective.

*Nonconstructive diagrammatic proof of surjectivity.* First, Theorem 3.1 implies that the central functions form a basis for such functions, so it suffices to prove the theorem for all $\chi_c^{a,b}$. But expanding the symmetrizers in $\chi_c^{a,b}$ gives a collection of circles with matrix elements, each of which correspond to a product of traces of words in $\mathbf{x}_1, \mathbf{x}_2$, so it suffices to express the trace of any word in $\mathbf{x}_1, \mathbf{x}_2$ as a polynomial in $x, y, z$.

This reduction depends entirely on the binor identity, which when composed with $\mathbf{x}_1 \otimes \mathbf{x}_2 =$ gives:

(39)

As special case we have, with  $= \mathbf{x}_1^{-1}$:

 and 

The first relation allows us to assume no loop has both $\mathbf{x}_1$ and $\mathbf{x}_1^{-1}$, while the second allows us to assume no word has more than one of any matrix. This leaves us with just the traces $\mathrm{tr}(\mathbf{x}_1), \mathrm{tr}(\mathbf{x}_2), \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2)$, and $\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$, using $\mathrm{SL}(2,\mathbb{C})$ trace identities. Finally, closing off (39) gives:

$$\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2) = \mathrm{tr}(\mathbf{x}_1)\mathrm{tr}(\mathbf{x}_2) - \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}),$$

which allows us to remove $\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2)$.

*Constructive diagrammatic proof of surjectivity.* We show that an arbitrary central function $\chi_c^{a,b}$ may be written as a polynomial in $x, y, z$, using an induction argument on its rank $\delta = \frac{1}{2}(a+b+c)$. For the base case $\delta = 0$ recall our earlier computations demonstrating

$$\chi_0^{0,0} = 1,\ \chi_1^{1,0} = x,\ \chi_1^{0,1} = y,\ \chi_0^{1,1} = z.$$

For $\delta > 0$, we may inductively assume that all central functions with rank less than $\delta$ are in $\mathbb{C}[x, y, z]$. The admissibility conditions imply that at least two out of the triple $\{a, b, c\}$ are positive, which we can assume to be $a$ and $c$, without loss of generality, by the Symmetry Theorem (4.1). In this case, the recurrence given by Corollary 4.6,

$$\chi_c^{a,b} = x \cdot \chi_{c-1}^{a-1,b} - \frac{(a+b-c)^2}{4a(a-1)}\chi_c^{a-2,b} - \frac{(-a+b+c)^2}{4c(c-1)}\chi_{c-2}^{a,b} - \frac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)}\chi_{c-2}^{a-2,b},$$

allows us to write $\chi_c^{a,b}$ in terms of central functions of lower rank, which by induction must be in $\mathbb{C}[x, y, z]$. Thus, $\chi_c^{a,b} \in \mathbb{C}[x, y, z]$, and we have established surjectivity. $\square$

Using this constructive approach with Mathematica, we computed the following table of low-rank central functions. Note the three-fold symmetry guaranteed by Theorem 4.1.

| **a =** | **b = 0** | **b = 1** | **b = 2** |
|---|---|---|---|
| **0** | $\chi_0^{0,0} = 1$ | $\chi_1^{0,1} = y$ | $\chi_2^{0,2} = y^2 - 1$ |
| **1** | $\chi_1^{1,0} = x$ | $\chi_0^{1,1} = z$<br>$\chi_2^{1,1} = xy - \frac{1}{2}z$ | $\chi_1^{1,2} = yz - \frac{1}{2}x$<br>$\chi_3^{1,2} = xy^2 - \frac{2}{3}(yz + x)$ |
| **2** | $\chi_2^{2,0} = x^2 - 1$ | $\chi_1^{2,1} = xz - \frac{1}{2}y$<br>$\chi_3^{2,1} = x^2y - \frac{2}{3}(xz + y)$ | $\chi_0^{2,2} = z^2 - 1$<br>$\chi_2^{2,2} = xyz - \frac{1}{2}(x^2 + y^2 + z^2) + 1$<br>$\chi_4^{2,2} = x^2y^2 - xyz + \frac{1}{6}z^2 - \frac{1}{2}(x^2 + y^2) + \frac{1}{3}$ |
| **3** | $\chi_3^{0,0} = x^3 - 2x$ | $\chi_2^{3,1} = x^2z - \frac{2}{3}(xy + z)$<br>$\chi_4^{3,1} = x^3y - \frac{3}{4}x^2z - \frac{1}{2}(3xy - z)$ | $\chi_1^{3,2} = xz^2 - \frac{2}{3}(yz + x)$<br>$\chi_3^{3,2} = x^2yz - \frac{2}{3}(xz^2 + xy^2) - \frac{1}{2}x^3 - \frac{1}{9}(2yz - 13x)$<br>$\chi_5^{3,2} = x^3y^2 - \frac{6}{5}(x^2yz + xy^2) + \frac{3}{10}xz^2 - \frac{2}{5}x^3 + \frac{3}{5}(yz + x)$ |

TABLE 1. Central Functions $\chi_c^{a,b}$ with $a \leq 3, b \leq 2$.

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Mathematics Department University of Maryland, College Park, MD 20742

*E-mail addresses*: **slawton@math.umd.edu, peterson@math.umd.edu**