The Integral Sign

One of the greatest treasures for the historian of mathematics is the notes that Leibniz made as he was discovering the calculus. We can follow the details of his invention in an English translation in *The Early Mathematical Manuscripts of Leibniz* (1920), edited by J. M. Child. In the manuscript dated October 29, 1675 Leibniz wrote:

> Utile erit scribi $\int$ pro omn. ut $\int l$ pro omn. l, id est summa ipsorum l. [It will be useful to write $\int$ for omn., as $\int l$ for omn. l, that is, the sum of these l's.][1]

A little later in the same manuscript he indicated that he was aware that he was creating a new field of mathematics when he wrote:

> Satis haec nova et nobilia, cum novum genus calculi inducant. [These are sufficiently new and notable, since they will lead to a new calculus.]

The integral symbol itself was the long form of the letter S which was frequently used by Leibniz at the time in his ordinary writing. Previously he had used the notation of Cavalieri: *omn.l*, for *omnes linea*, that is, for the sum of the lines.[2] Cavalieri's notation had also been used by Mengoli and Fabri.

In the same manuscript Leibniz wrote $\int x^2 = x^3/3$.[3] Leibniz also used the characteristic triangle in this manuscript, though both Pascal and Barrow had used it before him.

The integal sign first appeared in print eleven years later in 1686 in Leibniz's first paper on the integral calculus, "De geometria recondita et analysi indivisibilium atque infinitorum" [On a deeply hidden geometry and the analysis of indivisibles and infinities], which appeared in *Acta eruditorum*[4], a journal which Leibniz was instrumental in founding. In this paper the bottom half of the symbol was amputated so that it looked more like our letter f. It is interesting that one of the things that Leibniz did in this paper was to give an equation of the cycloid involving the integral sign.

The paper dealt with John Craig's little book *Methodus figurarum lineis rectis et curvis comprehensarum quadraturas determinandi*, [Method of Determining the Quadratures of figures Bounded by Curves and Straight Lines], London 1685. A Scottish theologian who loved to apply mathematics to matters divine, Craig (1660?–1731) was around Cambridge about 1680 and so knew of Newton's work on the calculus. The book refers to Leibniz's work on the calculus and uses his differential notation (being the first work in England to do so), so this explains why Leibniz was reviewing it. The stimulus for Leibniz writing this paper was that Craig had attributed to Leibniz a paper that was actually written by Tschirnhaus. Leibniz wanted to correct any false impressions concerning his work and so he sent this paper on the integral calculus to *Acta eruditorum*. In it he showed that quadratures were a special case of the inverse tangent problem. It is curious that Leibniz's work on the calculus was referred to in a book published in England before Newton published on the calculus.[5]

Leibniz originally spoke of the integral calculus as the *calculus summatorius*, a name obviously connected with summation [the sign Σ was introduced by Euler in 1755]. He first saw the word "integral" in a paper of Jakob Bernoulli in the May 1690 *Acta eruditorum* = *Opera*, 421–426, wherein Bernoulli solved the isochrone problem which Leibniz had posed to the public in 1687:

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[1] Cajori, *A History of Mathematical Notations*, vol. 2, §570, p. 203.

[2] J. E. Hofmann, *Leibniz in Paris*, p. 188.

[3] A photo of this is in Cajori (*Notations*, vol 2, p. 243; This facsimile is reproduced from C. I. Gerhardt's *Briefwechsel von G. W. Leibniz mit Mathematikern* (1899). See §§544, 570, and especially §620 of Cajori.

[4] 5 (1686), 292–299. The integral sign appeared first on p. 297. The original is reprinted in his *Mathematische Schriften*, Abth. 2, Band III, 226–235. A German translation is in Ostwald's *Klassiker*, No. 162, and the relevant passage, in English translation, is in Struik's *Source Book* (where the original date of the manuscript is misprinted). See Aiton 1985a, pp. 119 and 168.

[5] See Aiton, *Leibniz. A Biography* (1985), p. 119.

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Find the curve such that when a point falls along this curve the vertical component of its velocity is constant. Later the younger Bernoulli brother, Johann, reduced the problem to the simple differential equation $dy \cdot \sqrt{y} = dx \cdot \sqrt{a}$, and then "integrated" to obtain $\frac{2}{3}y\sqrt{y} = x\sqrt{a}$, which is a semicubical parabola. To see the details of how Johann Bernoulli presented this problem in lectures to his pupil L'Hospital in 1691-1692 see *Die erste Integralrechnung*, Ostwald's *Klassiker*, no. 194 (1914), pp. 142–143.

Leibniz tried, in the year 1695, to persuade Johann Bernoulli, Jakob's younger brother, to use his terminology of "sums":

> I leave it to your deliberation if it would not be better in the future, for the sake of uniformity and harmony, not only between ourselves but in the whole field of study, to adopt the terminology of summation instead of your integrals. Then for instance $\int y\,dx$ would signify the sum of all y multiplied by the corresponding dx, or the sum of all such rectangles. I ask this primarily because in that way the geometrical summations, or quadratures, correspond best with the arithmetical sums or sums of sequences. ... I do confess that I found this whole method by considering the reciprocity of sums and differences, and that my considerations proceeded from sequences of numbers to sequences of lines or ordinates.[6]

This request of Leibniz provided Bernoulli with the opportunity to explain the origin of the term integral:

> Further, as regards the terminology of the sum of differentials I shall gladly use in the future your terminology of summations instead of our integrals. I would have done so already much earlier if the term integral were not so much appreciated by certain geometers [a reference to French mathematicians, especially l'Hôpital, who had studied Bernoulli's *Integral Calculus*] who acknowledge me as the inventor of the term. It would therefore be thought that I rather obscured matters if I indicated the same thing now with one term and now with another. I confess that indeed the terminology does not aptly agree with the thing itself (the term suggested itself to me as I considered the differential as the infinitesimal part of a whole or *integral*; I did not think further about it).[7]

Leibniz reluctantly accepted Bernoulli's suggestion to call his new invention the integral calculus, but everyone else quickly accepted it.

Limits of integration were at first indicated only in words. Euler was the first to use symbols for limits of integration. He did this in his three volume work on the integral calculus, *Institutiones calculi integralis* (1768–1770). Fourier in his 1822 *La Théorie analytique de la chaleur* used $\int\limits_a^b$, although he had used it previously in print in 1819. In 1823 F. Sarrus first used the signs $\left|F(x)\right|_a^x$ and $\Big|_a^x F(x)$ to indicate the process of substituting the limits a and x in the general integral $F(x)$. Cauchy and Moingo also later used this notation, but several others were suggested before the mathematical community settled on the variant of Sarrus's notation that we use today.[8]

Perhaps it will be amusing to mention that there are several facetious[9] origins of the integral sign. Franklin M. Turrell, in a paper entitled "The definite integral symbol,"[10] notes that "if an apple is peeled by hand with a knife, beginning at the stem end and circling about the central axis without breaking the peel until the opposite end is reached, a regular spiral is obtained which forms an elongated S when placed on a flat surface with the inside of the peel up." With illustrations and scholarly references to the apple literature, Turrell conjectures that the apple peel suggested the integral symbol to Leibniz. In a later paper[11] with the same title, Martin G. Beumer relates that around the turn of the century when the calculus "began to penetrate the circle of the adepts of physical chemistry" the 1901 Nobel prize winner in Chemistry, Jacobus Henricus van't Hoff (1852–1911) gave the following derivation of the word integral: "The word is derived from *Integer* (whole, entire) and *Aal* (German word for: eel)."

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[6] Leibniz to Johann Bernoulli, February 28, 1695; quoted from Bos, *AHES*, 14 (1974), 21.

[7] Johann Bernoulli to Leibniz April 30, 1695; Bos, ibid, 21–22.

[8] See Cajori's *A History of Mathematical Notations*, §625–629 for details.

[9] "Abstemious" and "caesious" are the only other English words I know that contains all five vowels in alphabetical order.

[10] *American Mathematical Monthly*, 67 (1960), 656–658.

[11] *Monthly*, 81 (1974), 1095–1096.

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