# Why the sine has a simple derivative

S^r Isaac Newton,
speaking of M^r Cotes, said
"If He had lived
we might have known something."[1]

Many advocate the use of writing in the teaching of mathematics and there is much to recommend it. David G. Hartz, now of the College of Saint Benedict in Minnesota, has long had the habit of requiring his students to write a short summary of the most important mathematical points that had arisen in class that week and their reaction to them. He wants them to explain, for another student in the class, what they did the previous week, and to do it in their own words, keeping formulas to a minimum. The purpose of this assignment was to encourage students to review what they had done the previous week and to reflect about it. Here is what one student wrote:

> The derivatives of the trigonometric functions are rather amazing when one thinks about it. Of all the possible outcomes, $D_x \sin x = \cos x$. Simply $\cos x$; *not*
>
> $$\frac{1}{542}\cos x\left(\frac{1}{\pi}\right)\cdot 2x.$$
>
> But simply $\cos x$. Is it just *luck* on the part of the mathematicians who derived trig and calculus? I assume trig was developed before calculus, why or how could the solution prove to be so simple? Luck.[2]
>
> A. Student
> Fl. 1988

Now it would be easy to dismiss this rather exuberant student as one in a creative writing class. But what we realize from these comments is that the student has not come to grips with at least one important mathematical idea, thinking instead that the formula $D_x \sin x = \cos x$ is a matter of luck, "just *luck*." Now we mathematicians know that this is not the case, but we must explain it to our students, or, better yet, provide exercises that lead them to discover this for themselves.

We measure angles in radians, but this student does not understand why we do. One way to rectify this view is to ask for $D_x \sin x^\circ$, where $x$ is measured in degrees. Since $2\pi$ radians $= 360$ degrees we have $1^\circ = 2\pi/360$. Consequently,

$$\begin{aligned} D_x \sin x^\circ &= D_x \sin\left(\frac{2\pi}{360}x\right) \\ &= \frac{2\pi}{360}\cos\left(\frac{2\pi}{360}x\right) \\ &= \frac{2\pi}{360}\cos x^\circ. \end{aligned}$$

No one wants to write the factor $2\pi/360$ each time they take the derivative of $\sin x$. This is the mathematical reason why radians are used.

What I like best about the above quotation is that it cries out for some historical comments. The student reasonably assumes that trigonometry was developed before calculus since it is such an integral part of our calculus courses. But it wasn't that simple. Technically the student is correct, but not in spirit. Trigonometry was developed by the Greeks, but there were no trigonometric *functions* until Euler introduced them in 1739 in order to solve differential equations, specifically linear differential equations with constant coefficients.[3]

## A Discovery of Cotes

In his *Harmonia mensurarum* of 1722, Roger Cotes (1682–1716) states and proves the following lemma:

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1 So wrote Robert Smith, Cotes's cousin and successor as Plumian Professor of Astronomy at Trinity College, Cambridge, in his copy of Cotes's *Harmonia mensuarum*. See Ronald Gowing, *Roger Cotes. Natural Philosopher*, Cambridge University Press, 1983, pp. 141–142, for a nice illustration of the original.

2 David G. Hartz, "Writing abstracts as a means of review," pp. 101–103 in *Using Writing to Teach Mathematics*, MAA Notes Number 16, edited by Andrew Sterrett.

3 Victor J. Katz, "The calculus of the trigonometric functions," *Historia Mathematica*, 14(1987), 311-324. Much of the historical material in this note derives from this paper.

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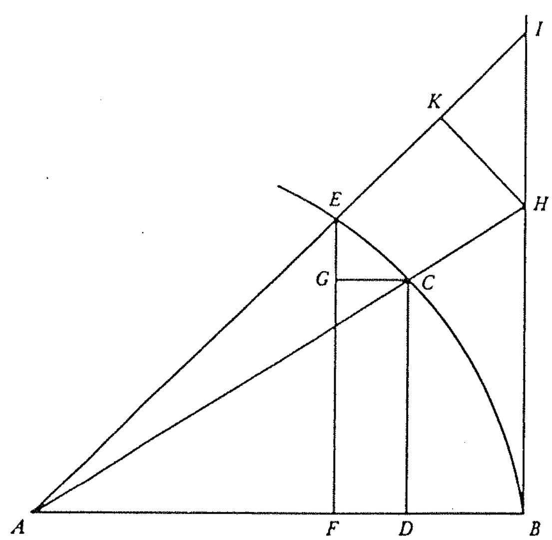
The small variation of any arc of a circle is to the small variation of the sine of that arc, as the radius to the sine of the complement.

Proof: With centre $A$, radius $AB$, arc $BC$ is drawn; $CG$ is parallel to the radius $AB$; $CE$ is the variation of arc $BC$; $EG$ is the variation in the sine of that arc. I say $CE : CG :: AC : AD$, when the variations are very small.[4]




Reformulating this in modern notation, the "small variation of any arc of a circle" becomes $d\theta$ while "the small variation of the sine of that arc" is $d(\sin\theta)$. Hence the formula is

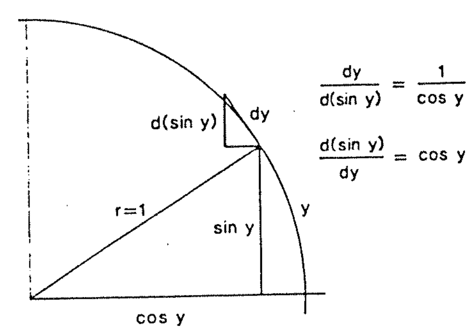
$$\frac{d(r\theta)}{d(\sin\theta)} = \frac{r}{\cos\theta}.$$

In 1748, Euler simplified this by using the unit circle. Thus:

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta,$$

where $r$ is the radius of the circle under consideration.

To see why this is obviously true, note that if the arc $BC$ has measure $\theta$ and $EC$ is a small increment of that arc, then it is $d\theta$. The line segment $DC$ is $\sin\theta$ and so $GE$ is $d(\sin\theta)$. In the triangle $\triangle ADC$, radius $AC$ has, by convention, length 1 and side $AD$ is $\cos\theta$. Since $\triangle ADC$ is similar to $\triangle EGC$, we have the desired result.




Those who object to the lack of rigor in this differential argument should realize that theorems have to be discovered before they are proved. What we are really after here is an intuitive explanation of why the derivative of the sine is the cosine.

## The Word 'radian'

Curiously, the word "radian" entered mathematics rather late — not until 1871, and then only in the private papers of James Thomson, brother of Lord Kelvin (nee John Thomson). More interestingly, it first occurred in an examination paper, dated June 5, 1873, at Queen's College, Belfast. The next year John Muir adopted the term after consultation with Thomson.[5]

[4] Quoted from Gowing, op. cit., p. 93.

[5] Florian Cajori, *A History of Mathematics*, second edition, 1924, p. 484. He cites *Nature*, vol, 83, pp. 156, 217, 459, 460.

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It was not a matter of luck. The choice of radians as a measure for angles was the result of careful thought.

### Exercises

1. "See how many first year mathematics graduate students understand that the derivative of $\sin\theta$ is equal to $\cos\theta$ only if $\theta$ is measured in radians. (Ask them to plot $\sin\theta$ vs. $\theta$ from $0^\circ$ to $90^\circ$ on a graph, and then measure the slope at $30^\circ$.)"[6]
2. In the diagram, show that $KH = \sec\theta \cdot d\theta$, $KI = d(\sec\theta)$ and $HI = d(\tan\theta)$. Then, using similar triangles, find the formulas for the derivatives of $\tan\theta$ and $\sec\theta$.

Bust of Roger Cotes, in the Wren Library at Trinity College.

[6] Clifford E. Swartz, "What do physics teachers want?," *The UMAP Journal*, vol. 1, no. 1 (1980), pp. 115–125, esp. p. 116. This physics professor at Stony Brook reports that students are coming to college with as much mathematics as ever, some even with calculus, but that "They just can't use mathematics." He outlines the rather minimal amount of mathematics that physics students need to know — and *use* with some sophistication.

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