The Clepsydra

I never lost a football game.
Once in a while time ran out.
Vince Lombardi

The clepsydra, which measures time by the flow of water, was the chronometer of the Egyptians, Greeks, and Romans. The word derives from the Greek $\kappa\lambda\epsilon'\pi\tau\epsilon\iota\nu$, to steal, and $\nu\delta\omega\rho$, water. Ancient water clocks consisted of an earthenware bowl with one or more small holes in the bottom from which water escaped, or "stole away." The oldest extant clepsydra, which today is in the Science Museum in London, comes from Karak in Upper Egypt and was made about 1400 B.C.E.

As you can see from the picture, it was already a refined object, and so was hardly a new invention.[1]

The clepsydra was used to determine the maximum length of speeches in the courts of justice and in the Roman senate. For the verbose among these orators, time literally 'ran out,' and this must be the origin of our phrase today.[2] When used in this way a clepsydra simply measured a fixed unit of time.

But there was also a need for a device to tell time. In the ancient world this was complicated by their custom of having twelve hours in each day and twelve in each night. Thus the length of the ancient hour, both daytime and nightime hours, varied according to the seasons. Consequently, various arrangements, of which we have no clear account, were needed to obvitiate this irregularity.[3] Thankfully, this custom has been abandoned today.

More importantly, the rate of flow of water out of a punctured vessel is not uniform. The rate becomes less as the vessel empties itself. This is due to the decreasing force caused by the weight of the water above the hole. That the Egyptians had some empirical knowledge of this is demonstrated by the fact that their waterclock has sides which slant in at the bottom, so that there is less water to flow out where the rate of flow is less. A simpler way of overcoming this difficulty was by keeping the level of water in the clepsydra constant, say by having water flow into the vessel from a stream while the excess flows out at the top. Then all that one must do is note the volume of water discharged into a right circular cylinder. A uniform measuring scale for the depth of water in the cylinder thus gives an accurate measure of time. Such a device

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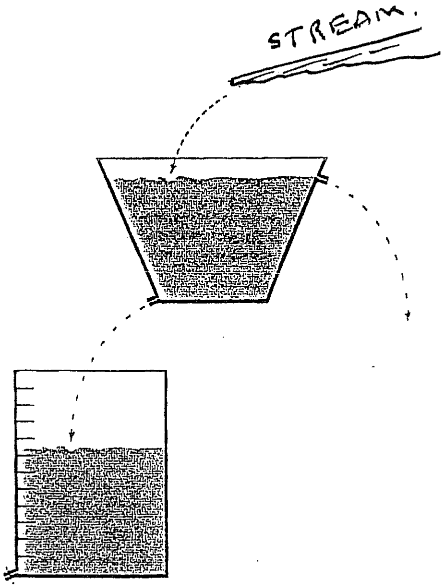
[1] The best single volume dealing with water clocks is *The Time Museum: Catalogue of the Collection*, Vol. 1, part 3, *Water-Clocks/Sand-Glases/ Fire-Clocks*, by Anthony J. Turner, Rockford, IL: Time Museum, 1984. The general editor of this catalog is historian of mathematics Bruce Chandler, professor of mathematics at the City University of New York. This lavishly illustrated volume has a detailed history of water clocks and an extensive bibliogrphy. The volume has been reviewed by Silvio A. Bedini, *Technology and Culture*, 28 (1987), 159–161.

[2] Unfortunately, the etymological dictionaries do not confirm this claim, but it must be true.

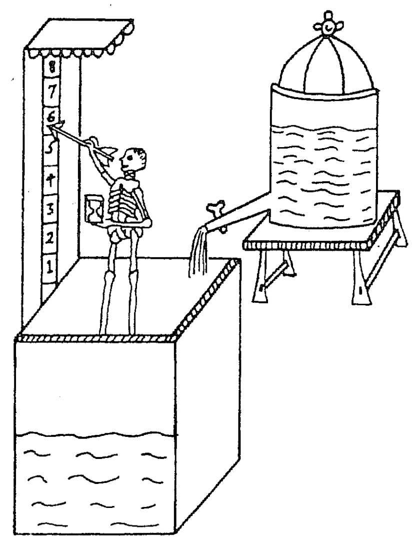
[3] *Encyclopedia Britannica,* 11^{th} edition, 1910, vol. 6, pp. 495-496.



is called an outflow clepsydra.

It is interesting that the young Isaac Newton built several waterclocks at his home at Woolsthorp Manor. Sadly they do not survive.[4] He most likely got the idea for these clocks from a book by John Bate entitled *The Mysteryes of Nature and Art.*[5] This was first published in 1634, but Newton, who was twelve when he read it, probably used the third edition of 1654. The two illustrations on which Newton based his clocks are from this edition and are reproduced below.

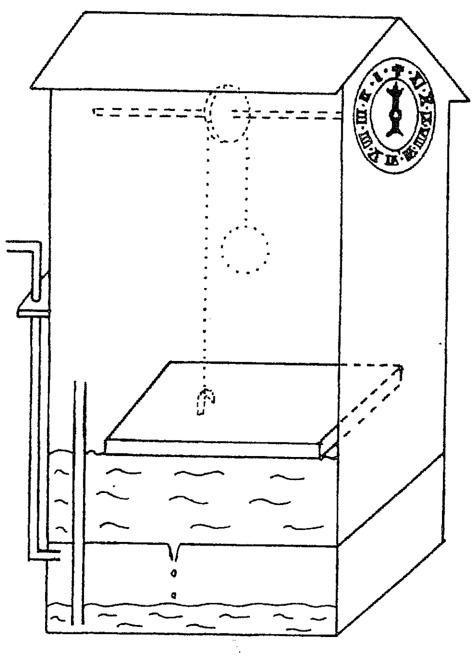
In the first of these clocks the water flows from the clepsydra into a reservoir that has a floating skeleton which points out the hour on the scale. Although the scale is uniform, the graduations between the hours at the top of the scale should be closer together.



The second is more interesting technologically. Water flows out of an upper chamber which contains a float. As the float descends, a string attached to the float turns the dial in a counterclockwise (!) direction. At the end of the day when the upper reservoir is empty the owner simply blows on the tube at the left. This forces the water back up into the upper reservoir. Thus it is not at all

[4] A. A. Mills, "Newton's water clocks and the fluid mechanics of clepsydrae," *Notes and Records of the Royal Society of London*, 37 (1982–83), 35–61. Errata, 38 (1983), 146. This article has an extensive bibliography. I would like to thank Julio Gonzales Cabillon for bringing this very interesting article to my attention.

[5] One would think it would be very difficult to obtain a copy of a work such as this, but, as a matter of fact, virtually everything published in England (regardless of language) and its colonies before 1700 is available on microfilm. See the catalog of Alfred W. Pollard and G. R. Redgrave, *A Short-Title Catalogue of Books Printed in England, Scotland, & Ireland, and of English Books Printed Abroad, 1475-1640.* For the next period consult Donald G. Wing, *Short-Title Catalogue of Books Printed in England, Scotland, Ireland, Wales, and British America, and of English Books Printed in Other Countries, 1641-1700.* These works, together with their indices of microfilms, are somewhat hard to use, so you should consult your reference librarian for assistance. The 1654 edition of Bate is on reel 1415, item 15 (Wing number B1092). Even handier, the 1634 edition was reprinted in 1977 as volume 845 of *The English Experience, its Record in Early Printed Books Published in Facsimile.* This series contains a number of other works of interest to the historian of mathematics. In particular Robert Record's *The Whetstone of Witte* (1570) is volume 142. Reporting on this book, which contains the first use of the equals sign, makes a good student project.



messy to reset the clock.

Note that in neither of Newton's clocks are the hours of uniform length. This is because the rate of flow decreases as the head of water — the distance between the surface the water and the oriface — diminishes. This led to a mathematical question. What shape should the vessel be so that the level of water decreases at a constant rate. While the Egyptians had some awareness of this problem, its first known clear statement is by Evangelista Torricelli[6], the founder of hydraulics. The first printed solution of the problem was in 1686 in a posthumous book by the French physicist Edme Mariotte.[7]

"In the absence of the calculus Mariotte's solution is verbally tortuous,"[8] so we relegate it to the appendix and proceed with a modern solution.

[6] See the appendix.

[7] For information about Mariotte, see B. Davies, "Edme Mariotte, 1620-1684," *Physics Education*, 9 (1974), 275-278 and Christian Licoppe, "The crystallization of a new narrative form in experimental reports (1660–1690): The experimental evidence as a transaction between philosophical knowledge and aristocratic power," *Science in Context*, 7 (1994), 205-244, which uses Mariotte's discovery of the blind spot in the eye as a case study of discovery.

Let $y = f(x)$ be a monotonically increasing function defined for positive x. When it revolves around the y-axis it will form the bowl of a clepsydra. The feature we want this bowl to have is that, when a small hole is put in the bottom, the water drains out in such a way that the water level drops equal vertical intervals in equal time spans. If this can be arranged, then time can be measured by a linear scale placed in the center of the bowl. This means that if y is the depth of the water and t is time, then

$$\frac{dy}{dt} = c,$$

where c is some constant.

Since it is the water that is draining from the bowl, we need to know its volume. But that is an easy computation using infinitesimally thin (dy) disks of radius x:

$$V = \int_0^y \pi x^2 \, dy,$$

where the radius x is the inverse of the function f, i.e., $x = f^{-1}(y)$. Thus,

$$V = \int_0^y \pi [f^{-1}(y)]^2 \, dy.$$

Since we want to find the function f, we shall apply the Fundamental Theorem of Calculus, to obtain

$$\frac{dV}{dy} = \pi [f^{-1}(y)]^2.$$

We now have a connection between V and y, but we want a connection between y and t. These three variables can be connected via the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt}.$$

The needed connection between V and y is provided by Torricelli's law, namely that the velocity of the outflow is proportional to the square root of the head, i.e., the distance between the surface of the water and the hole.[9]:

$$\frac{dV}{dt} = \sqrt{2gy}.$$

[8] This is how A. A. Mills describes it on p. 43 of his paper cited in note 4.

[9] Torricelli's law is sometimes explained by noting that water falling freely from a height y attains

Now substituting these derivatives into the chain rule we have

$$\sqrt{2gy} = \pi[f^{-1}(y)]^2 \cdot c.$$

Solving for f^{-1} we obtain

$$f^{-1}(y) = \frac{(2gy)^{1/4}}{\sqrt{c\pi}}$$

or

$$f(x) = ax^4$$

where a is the constant $(c\pi)^2/2g$. Thus we see that the shape of the inside of the bowl should be a fourth degree monomial.[10]

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speed $\sqrt{2gy}$ at height 0, but water in a clepsydra is not 'falling freely.' However, equating the potential energy of a mass m of water at height y with the kinetic energy of an equal mass flowing out of the bottom of the vessel with velocity v, we have

$$mgy = mv^2/2$$

or

$$v = \sqrt{2gy}.$$

This argument is paraphrased from R. D. Driver, "Torricelli's law — an ideal example of an elementary ODE," *The American Mathematical Monthly*, 105 (1998), 453–455. Driver continues by assuming the surface of the water has area A, the hole area a, and that Δy and Δt are small increments of depth and time. Then

$$A\Delta y \approx -av\Delta t \approx -a\sqrt{2gyt}.$$

Thus,

$$\frac{dy}{dt} = -\frac{a}{A}\sqrt{2gy}.$$

[10] I would like to thank my Bowling Green colleague Steven Seubert for bringing the beautiful mathematics of the clepsydra problem to my attention and Calvin Jongsma for pointing out that it has appeared in several calculus books, including C. H. Edwards, Jr. and David E. Penny, *Calculus with Analytic Geometry*, fourth edition (1994), p. 383, and George F. Simmons, *Calculus with Analytic Geometry*, second edition (1996), p. 230. It is also in Deborah Hughes-Hallet et al., *Calculus* (1994), p. 475. None of these books include the history of the problem.

Curiously, Mariotte may not have been the first to solve the clepsydra problem. Vincenzo Viviani (1622–1703), a student of Galileo and Toricelli, wrote a long manuscript in Italian entitled *Treatise of Clepsydrae* which was never published but has been described by Sylvio A. Bedini.[11] The date of this manuscript is uncertain. It may have been written prior to 1670 or as late as 1684, and if the later date is correct then the discovery was simultaneous with that of Mariotte.

A. A. Mills after citing one modern author who misread Mariotte as talking about a paraboloid of revolution notes that "the parabola is indeed mentioned by a few authors as the theoretically correct shape, although no sound justification has been traced in the existing literature. The earliest of these appears to be Viviani".[12] Since this casts doubt on the correctness of Viviani's result, we shall check it.

Viviani's manuscript illustrates a parabolic trough (as Mills notes and as can be seen in the illustration) and describes the construction of his clepsydra very clearly, beginning with the words "Let us make four exactly identical parabolas of clear crystal glass.[13] While it might seem natural that the solution to the clepsydra problem should be a solid of revolution, there is nothing in the problem that dictates that it should be so. The problem was stated so that a "vessel" was sought, not a "bowl."

Suppose $y = f(x)$ is the equation of the curve that is to form the end of the cylindrical trough. If the depth of the water in the trough is y then its volume is

$$V = l\int_0^y x\,dy = l\int_0^y f^{-1}(y)\,dy,$$

where l is the length of the trough. By the Fundamental Theorem of Calculus, we have

$$\frac{dV}{dy} = lf^{-1}(y) = lx.$$

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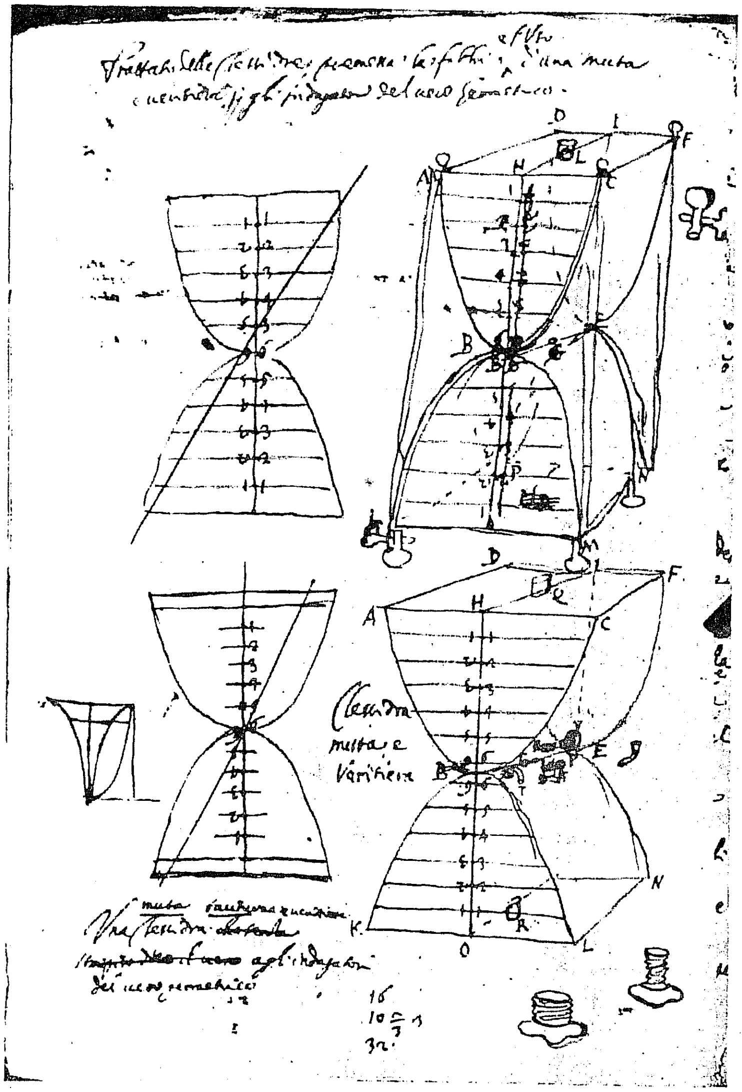
[11] "The 17th century table clepsydrae," *Physis. Rivista Internazionale di Storia della Scienza*, 10 (1968), 25-52. The illustration below was reproduced from p. 46.

[12] P. 43 of the Mills article cited in note 4.

[13] See pp. 44-45 of the paper of Bedini cited in note 11.



Using the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt}.$$

and Toricelli's law,

$$\frac{dV}{dy} = \sqrt{2gy}.$$

we obtain

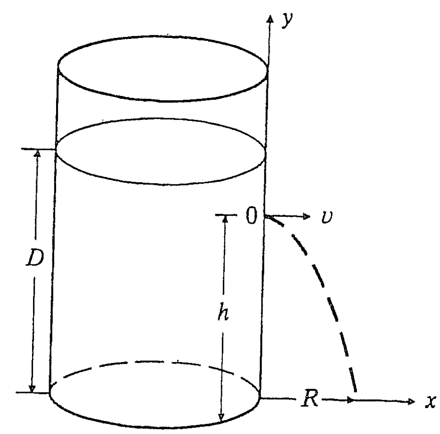
$$f(x) = kx^2$$

where k is a constant. Thus Viviani is correct. A parabolic trough will produce an accurate clepsydra.

Three More Things to Do

1. For an "impressive and memorable" classroom discussion one can take a two liter plastic bottle, puncture it on the side near the bottom, and then record the height of the water at time $t = 0, 10, 20, \cdots$. You will get results very close to what Torricelli's law predicts. Your students will be impressed that mathematics works.[14]

2. A common calculus problem is to ask for the range R of the initial spurt of water out of a tank when h is the height of the hole and D is the depth of the water. Even more interesting is to ask the inverse problem: For a given range R at what height h should one puncture the vessel so that the initial spurt of water has range R? This simple differential equation sometimes has two solutions, so provides a wonderful learning experince for students. Of course, you should put another hole, properly placed, in your plastic bottle and then do a demonstration to illustrate the two solutions.[15]



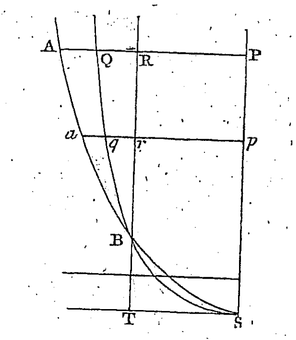
[14] Careful instructions on how to do this are nicely written up in Tom Farmer and Fred Gass, "Physical demonstrations in the calculus classroom," *The College Mathematics Journal*, 23 (1992), 146-148.

[15] C. W. Groetsch, "Inverse problems and Torricelli's law," *The College Mathematics Journal*, 24 (1993), 210-217. This problem and the clepsydra problem are also treated in his *Inverse Problems: Activities for Undergraduates*, The Mathematical Association of America, 1999.

3. Charles Hutton (1737-1823) published *A Mathematical and Philosophical Dictionary*[16] in 1796 and 1795 (the second volume appeared first) and while the article entitled "Clepsydra" does not mention Mariotte, Hutton was aware of his result.

> But such a figure might be given to the containing vessel as would require the dividing marks to be equi-distant, which Dr. Hutton, in his recent edition of "Ozanam's Recreations," has asserted to be a paraboloid, or vessel, formed by the circumvolution of a parabola of the fourth degree, the method of describing which, he has given thus:
>
> Let $ABS, \ldots$ be a common parabola, the axis of which is PS, and the summit S. Draw, in any manner, the line, RrT, parallel to that axis, and then draw any ordinate of the parabola AP, intersecting RT in R; make PQ a mean proportional between PR and PA, and let pq be a mean proportional also between pr and pa; and so on; The curve passing through all the points Q, q, &c. will be the one required, which, being made the mould for a vessel to be cast by, will produce an instrument, which, when perforated at the apex, will have the singular property of equalizing the scale, so as to correspond to equal times while the water is running out.[17]

Now the challenge here is to see if Hutton's description of the required quartic is accurate.[18]



The Advantages of History

It is worthwhile to reflect on the advantages of introducing this particular example into the calculus classroom.

1. Time measurement is an important question, one which people cared about (and still care about), so it makes sense to introduce it in the classroom. Curiously, the precise mathematical solution to the clepsydra problem came only after the clepsydra had been replaced by the mechanical clock, and thus the problem ceased to be a real world problem. This is in marked contrast to what usually happens in mathematics: Abstract problems are solved and then they lead to applications (e.g., Kepler's ellipses) or physical problems give rise to mathematical ones which are solved and applied to the physical situation.
2. This problem is easy to understand, both in its statement and in its solution. This is a *sine qua non* of classroom examples. This is also a non-standard problem, both in its statement and solution. It provides a beautiful example of the power of combining a variety of mathematical ideas.
3. Several famous people are involved in the solution and it is always good to introduce our students to them. Also, a relatively unknown individual played a crucial part in the solu-

[16] This was reprinted in 2000 with a new introduction by Richard Gregory, Bristol, England and Sterling, VA: Thoemmes Press.

[17] Available on the web at
www.ubr.com/clocks/pub/clep/clep.html
where it is reprinted from *Ree's Clocks, Watches and Chronometers* (1819). This, in turn, is from *The Cyclopdia, or, Universal Dictionary of Arts, Scieences, and Literature* (1819) by Abraham Rees. The passage continues: "Mr. Varignon has given a geometrical and general method of determining the scale for a clepsydra, whatever may be the shape and magnitude of the vessel. (See "Memoires de I'Academi Royale des Sciences" p. 78, 1699.)" but I have not seen this work.

[18] In interpreting this passage one needs to realize that "in any manner, the line, RrT" means that ST should have length 1. Also m is a mean proportional between a and b iff $a/m = m/b$.

tion. This allows us to make the point that mathematics is not just a field for geniuses.

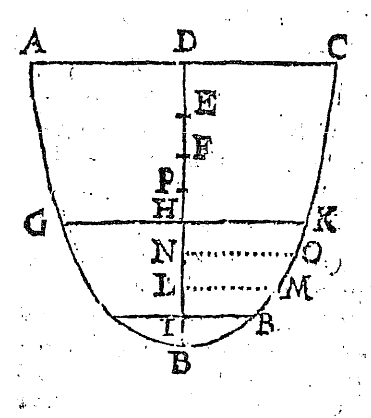
4. Nice illustrations are available to capture the interest of the students. It is far more interesting to show the pictures of Newton's waterclocks rather than just talk about them.
5. Some nice mathematics is involved. Students must see examples where a variety of techniques are combined in powerful ways. Here they get to see solids of revolution, inverse functions, and the chain rule all used together. Most importantly, the first Fundamental Theorem of Calculus was used in both solutions of this practical problem. This is a rare event, for usually it is only used in theoretical contexts. To say it a different way, the solutions of the clepsydra problem involved ideas not computation and this makes it particularly suited for the classroom.
6. Physical principles, in particular the law of Torricelli, are involved and so students see an example of how mathematics interacts with physical science.
7. The problem has two distinct solutions, a solid of revolution and a trough. This offers the opportunity to point out to students that we should not make assumptions that are not part of the statement of the problem and we should not get psychologically trapped into assuming the problem has a certain form.

Appendix

The following is the original text of Mariotte's solution of the clepsydra problem in the English translation of Desaguliers.[19] It is included here so that you may contrast it with the solution given above and thereby come to appreciate what a wonderful contribution the calculus of Newton and Leibniz was. This should help you appreciate the value of symbolic algebra and the algorithmic nature of the calculus. It is only by reading original sources that one comes to understand the history of mathematics and the magnificence of its methods.

[19] Edmé Mariotte, *The Motion of Water, and Other Fluids, Being a Treatise of Hydrostatics*, tanslated by J. T. Desaguliers, London, 1743, pp. 185–187.

"It is proper enough in this Place to resolve a pretty curious Problem which *Toricelly* has not undertaken to resolve, tho' he proposed it; this Problem is to find a Vessel of such a Figure that being pierc'd at the Bottom with a Small Hole the Water should go out, its upper Surface descending from equal Heights in equal Times. If in the Conoidal figure (*Fig. 75*[20]) BL is to BN, as the



Square squared of LM is to the Squared square of NO; and BN to BH, as the Squared squar'd of NO, to the Square squar'd of HK, an[d so] on; the Water will descend from ADC unif[orml]y, till it comes to the Hole at B; for let BP [be t]he mean Proportional betwixt BD and BH; since the Square squared of KH and of DC, are to each other as the Heights BH, BD; the Squares of HK, DC, will be in a subduplicate *Ratio* of BH to BD, or as the Heights BP, BD; but the Velocity of the Wate^r that goes out at B by reason of the Pressure of the Height BD, is to the Velocity of that that goes out by reason of the Pressure of the Height BH, in a Subduplicate *Ratio* of BD

[20] The digaram is from Edmé Mariotte, *Traité du mouvement des eaux et des autres corps fluides*, Nouv. éd. corr., Paris, 1700. The diagram agrees with that in the English edition, except that the 'B' at five O'Clock in the French is a K or R in the English.

to BH, that is to say as BP to BD, therefore the Velocity of the Water descending from H is to the Velocity of the Water descending from D, as the Square of HK to the Square of DC; but the Circular Surface of the Water at H is to the circular Surface of the Water at D, as the square of HK to the Square of DC, therefore they will descend and run out one as fast as the other; and if the Surface ADC, runs out in a Second, the Surface GHK will run out likewise in a Second, since the Quantities are as the Velocities. The same Thing will happen to the other Surfaces at E and F, &c. But the Hole at B must be very small, that no considerable Acceleration may be made, and that the Water may not go out thro' the Hole sensibly, but in proportion to its Weight. Such a Vessel may serve for a *Clepsydra* or Water Clock.

"The Explanation of it in Numbers.

"Let BD be 16 and BI 1; the Square square'd of IR will be one, if the Squar squared of DC be 16, and consequently DC will be 2 if IR be one. Let BH be the mean Proportional betwixt BI, and BD, which consequently will be 4. The Velocity by [rea]son of the Weight IB will be 4, if the Veloc[ity] by reason of the Weight DB be 16; but the C[ircle] or Surface IR will be 1, and the Circle DC be 4; therefore these Quantities will be as their Velocities, and consequently the Circles or Surfaces DC, and IR will run out in the same Time; and if the Surface IR requires a Second of Time to run out; Four times as much will run out in the same Time, by the Quadruple Velocity, namely, the Surface DC, since that is the Quadruple of the other. The same Proportion will be found in all the other Surfaces that compose the whole of the Water, or in the Solids, whose Thickness is indefinitely little. You suppose in all these Experiments, that there is no Turning, or Circular Motion in the Water; nor any little Pit, as there is in Funnels, when they empty themselves."

No doubt few will take up pencil and devote the time to read this with understanding, but they will undoubtedly have a richer understanding of this problem and the calculus. Such are the benefits of reading original sources. We should always remember the comment that Niels Henrik Abel (1902–1828) scribbled in one of his notebooks: "It appears to me that if one wants to make progress in mathematics one should study the masters and not their pupils.[21]

[21] This reluctance to read original sources reminds me of a letter from Baltazar Mathias Keilhau to Christian Peter Bianco Boeck (both friends of Abel since their student years) in the Spring of 1830: "Yet here our friend has such a touching resting place. It will rarely be visited by anyone capable of understanding his worth, but once in a while through the years it will happen, and then only for his own sake. Then let such a wanderer find a secure and imperishable mark on this place of his pilgrimage, a sumptuous one would not be in harmony with his feelings." The passage can be found in the last chapter of *Niels Henrik Abel, mathematician extraordinary* written by Oystein Ore, Minneapolis: University of Minnesota Press, 1957, page 267, reprinted New York: Chelsea Publishing Co., 1974. These words were quoted by Man Keung Siu in the summer of 1988 when a group of historians visited Abel's grave, and I thank him for the reference and a touching moment.