Torricelli's Trumpet

An Infinite Solid With Finite Volume

I do not remember this of Torricellio ...
to understand this for sense, it is not required
that a man should be a geometrician or a logician,
but that he should be mad.
Thomas Hobbes[1]

Evangelista Torricelli (1608–1647) showed such promise as a youth that he was sent to study with the Camaldolese monk, Benedetto Castelli (1578–1643), a mathematician, hydraulic engineer, and student of Galileo (1564–1642). A treatise Torricelli wrote on projectile motion attracted Galileo's attention, so he was able to live with Galileo in Florence during the last few months of Galileo's life. Bonaventura Cavalieri (1598–1647) and Vincenzo Viviani (1622–1703) were also part of the circle of Galileo at this time, so they were all friends. After Galileo died[2], Torricelli was appointed to Galileo's position of mathematician and philosopher to the Duke of Tuscany.

In 1641 Torricelli showed that an infinite solid could have finite volume. He thought he was the first to discover this, but he may have been anticipated by his contemporaries Fermat and Roberval, and certainly by was Oresme in the fourteenth century. The result was published in 1644 in *Opera geometrica*, the only work Torricelli published in his lifetime. Publication caused a sensation. In addition, Marin Mersenne (1588–1648), who had met Torricelli while visiting Italy in 1645, quickly spread the word. The best mathematicians of the seventeenth century were amazed and perplexed by this result, just as our students should be. Mathematical intuition is not always inherent; it takes time to develop.

Consider the rectangular hyperbola $y = 1/x$, which has been truncated at some point, say $(1/2, 2)$ to be specific. Then add a horizontal line segment from $(0, 2)$ to $(1/2, 2)$ to form a new curve. This curve is to be rotated around the x-axis and we wish to find the volume of the resulting solid.

Consider an arbitrary point (x_0, y_0) on the hyperbola, and draw a horizontal line from it to $(0, y_0)$. When this line segment is rotated around the x-axis a cylinder of radius y_0 and height x_0 is generated. This cylinder has lateral surface area $(2\pi y_0)x_0$. Since the point (x_0, y_0) lies on the hyperbola, the surface area of the cylinder is 2π, which is constant, and so independent of the choice of the point on the hyperbola.

Torricelli now forms a circle of area 2π, the surface area of each of the cylinders, and places it in the horizontal plane $y = y_0$. This is repeated for each point on the hyperbola. The result is a stack of circles which form a cylinder with base area 2π and height 2. The volume of this cylinder is 4π. By Cavalieri's Principle, this is the volume of the Trumpet of Torricelli[3].

In this proof, Torricelli has extended Cavalieri's Principle to allow curved indivisibles (not infinitesimals). To see that Cavalieri's original statement applies, think of the solid as made up of concentric horizontal cylinders. Then cut these cylinders up through the x-y-plane to the x-axis. Then unroll the cylinders so that they form rectangles that live in the same horizontal plane as the top of each cylinder. It is in this horizontal plane that Torricelli places a circle of equal area.

This result of Torricelli is incredible. It is the most interesting and ingenious application of Cavalieri's Principle. He has found a solid, that is infinitely long, and yet has finite volume. Moreover, his construction is remarkably simple. There seems to be no end to what clever people can do. Yet, this technique is *ad hoc*; it is crucial that the area of each horizontal cylinder be the same, for otherwise he could not have constructed the vertical cylinder.

In 1658 Christiaan Huygens (1629–1695) and René-François de Sluse (1622–1685) found an even

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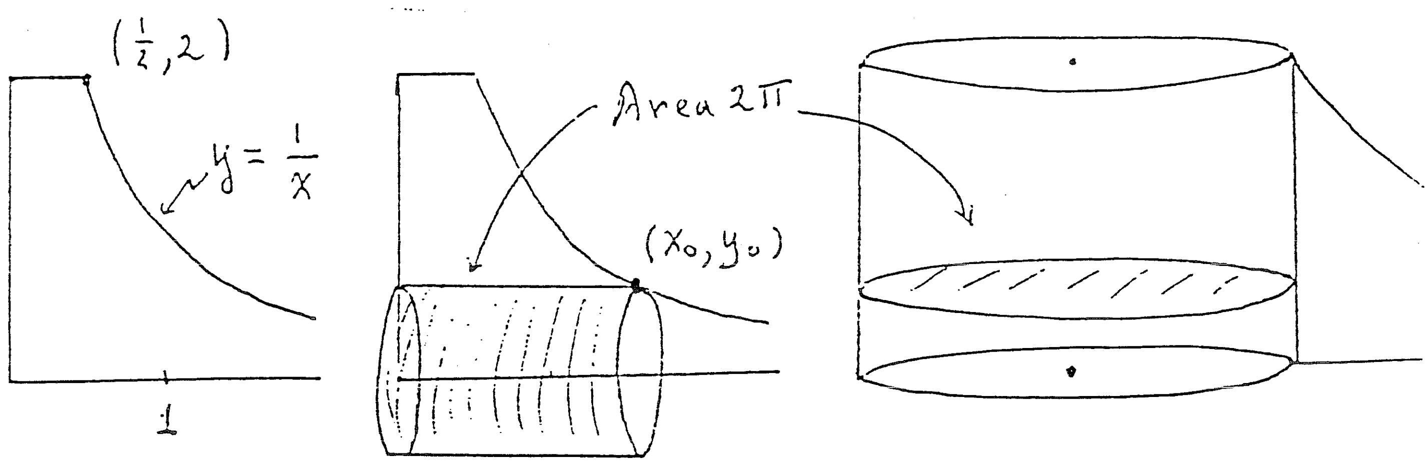
1 *The English Works of Thomas Hobbes*, vol. 7, p. 445.

2 Galileo is buried in a magnificent tomb to the left of the narthex in the church of Santa Croce in Florence. Across the nave, the tomb of Michelangelo (1475–1564) is of similar grandure. This indicates that scientists and artists held equal social positions in Florence. Would that it were the same today.

3 A neologism of Jan van Maanen.

This file, `torr.tex`, was typeset using TeX on May 29, 1996 at 3:51 P.M.



more remarkable solid, one with finite surface area and infinite volume. Sluse, in a letter to Huygens, proudly described it as a drinking glass, that had a small weight, but that even the hardiest drinker could not empty ("levi opera deducitur mensura vasculi, pondere non magni, quod interim helluo nullus ebibat."[4]).

Exercises:

1. Evaluate the integral $\int_{1/2}^{\infty} \pi(\frac{1}{x})^2\, dx$ and compare the value with the volume of Torricelli's Trumpet. Explain why the answers disagree.
2. Find the surface area of Torricelli's Trumpet.
3. The first time I presented this in class, I commented that the result was *ad hoc* and probably would not generalize. My students observed that if $y = 1/x^n$, $n > 1$, is rotated about the x-axis, then the volume is finite because it fits into the cylinder. This is a physical realization of the comparison test for integrals that I had never seen before. Are there any cases where you can obtain an exact result, say by making the solid a cone? [This is a serious question; I don't know the answer.]
4. Show that when the portion of $y = x^{-3/4}$ that lies above the ray $[1, \infty)$ is rotated aroung the x-axis, the resulting solid has finite volume and infinite surface area. The integral for surface area is not elementary, so you will need to use the comparison test to show that it is infinite.
5. View the film "Infinite Acres."
6. If the region that lies above the inteval $(0, 1]$ and below the graph of $y = \sqrt{x}$ is revolved about the x-axis, a solid called "Thor's anvil" is generated. Show that it has invinite volume and that the region generating it has finite area. Examples of this type seem not to be in our calculus books. [Dawson]

Sources:

This note is based on: (1) The *Dictionary of Scientific Biography*, (2) Carl Boyer, *The History of the Calculus and Its Conceptual Development*, pp. 125–126, (4) John W. Dawson, Jr., "Contrasting examples in improper integration," *The Mathematics Teacher*, March 1990, 201–202, and (4) Jan A. van Maanen, "Alluvial deposits, conic sections, and improper glasses, or history of mathematics applied in the classroom," pp. 73–91 in *Learn From the Masters*, edited by Frank Swetz, et alia, MAA, 1995. Boyer cites "De solido hyperbolico acuto" (Concerning pointed hyperbolic solids), which appears in *Opera di Evangelista Torricelli* (1919), vol. 1 (of 3), part I, pp. 173–221. Van Maanen cites Christiaan Huygens, *Œuvres Complètes*, vol. 2, pp. 164, 168, 212 and vol. 14, pp. 199, 200, 306–312, and also *Correspondance du P. Marin Mersenne*, vols., 12–14. these original sources. Margaret E. Baron, *The Origins of the Infinitesimal Calculus* (1969), which has been reprinted by Dover, gives a good treatment of Torricelli's work on pp. 182–194.

After this note was written a detailed discussion of the philosophical importance of this result was published by Paolo Mancosu and Ezio Vailanti as "Torricelli's infinitely long solid and its philosophical reception in the seventeenth century," *Isis*, 82 (March 1991), 50–70.

[4] Huygens, *Œuvres Complètes*, vol. 2, p. 168.