Mathematics of the Gregorian Calendar

V. Frederick Rickey

October 15, 1982 was the beginning of the second 400 year cycle of the Gregorian calendar, an event insufficiently celebrated by mathematicians. Although a number of recent articles deal nicely with the history of the calendar, one has to go back to early editions of the *Encyclopaedia Britannica* [12]—a place few would think to look—to find a good treatment of the mathematical issues involved. While most of these are elementary, the calendar does give rise to many curious facts, some of which are presented in italics below as exercises (with references for the lazy).

By 45 B.C., the old Roman calendar was in such a chaotic state, due mostly to political manipulations, that Julius Caesar, with the advice of the Graeco-Egyptian astronomer, Sosigenes, declared that every year divisible by four was to be a leap year with 366 days; the other years were to be common years of 365 days. The extra day was inserted before the sixth day before the Calends of March, hence that month contained two sixths, or was, as we still say, bissextile. The new year then began in March, thus explaining why the months September through December are etymologically the seventh through tenth.

In the Julian calendar, the years had an average length of 365.25 days while the tropical year was then 11 minutes 4 seconds shorter and so every 131 years the calender retrogressed one day with respect to the seasons. This defect was known at the time of reform, but it took several centuries for the calendar to become out of whack enough to matter.

Scores of scholars devised schemes for revising the Julian calendar. Regiomontanus (1436–1476), the father of trigonometry as a science independent of astronomy, became involved at papal behest, but before he could do anything he was poisoned by the sons of a man that he had written a polemic against. This was doubly unfortunate: the Protestant reformation was yet to come and change would have been easier. In 1514 Pope Leo X asked Copernicus for assistance, but he declined saying that the length of the year was not known precisely enough; this was one of his reasons for writing *De Revolutionibus*.

Pope Gregory VIII appointed a commission of mathematicians, astronomers, and clerics to revise the calendar. It was led by the mathematician, Christopher Clavius (1537–1612) who is generally given most of the credit for the reform, primarily since he defended it so staunchly. The plan that was used was the work of the physician, Luigi Lilio, who died shortly before. (For additional historical details, see Moyer [7] or Philip [8].) Gregory's reform, which was promulgated in the papal bull *Inter gravissimas* (Of the gravest concern) of February 24, 1582, was intended to solve several problems:

1. How could the year be brought back in accord with the seasons?
2. How could the Julian leap year rule be corrected so as to stop further dislocation?
3. Where should the beginning of the year be fixed?
4. How should the date of Easter be determined?

Although the new leap year rule now seems like the important point, it was actually the Easter issue that motivated papal intervention. The determination of the date of Easter has always been of mathematical interest, but it is complicated enough to deserve a note of its own.

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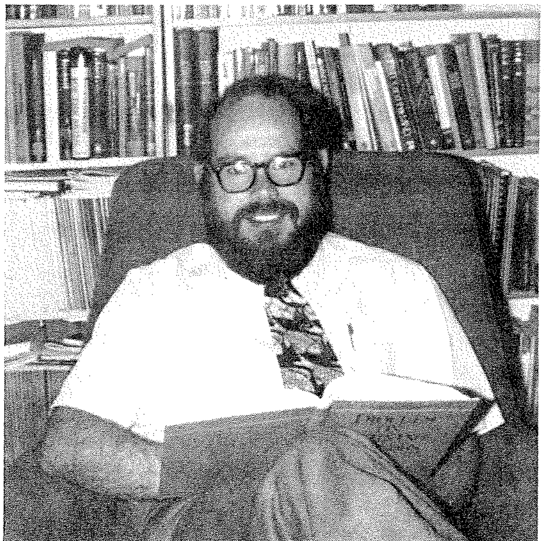

In A.D. 325 the Council of Nicaea "fixed" the vernal equinox at March 21, but by the sixteenth century it had slipped ten days behind. To convince Pope Gregory XIII of this the Tower of Winds was built at the Vatican containing a Meridian Room with an inlaid marble meridional sundial line across the floor. At noon on March 11, 1582, not March 21 as it should have, the sun shone in through the mouth of the South Wind, a mural on one wall, and crossed the meridional line.

This room had another connection with mathematics. In 1655, when Christina converted to Catholicism and abdicated as Queen of Sweden, she lived in this tower. She was, you will remember, the spartan queen who contributed to Descartes' death by insisting that he tutor her in an unheated library early in the morning, contrary to his custom of staying abed until noon. For years there were rumors that parts of the frescoes had been overpainted before she arrived, perhaps by adding drapery to the figures so as not to offend her. Later when the paintings were restored it was learned that what was overpainted was a paraphrase of the Biblical motto Jeremiah 1:14, "All bad things come from the North." No doubt this would have offended her.

Bringing the calendar back into accord with the seasons was accomplished by declaring that the day after Tuesday, October 4, should be Wednesday, October 15, in the year 1582. These dates were chosen so as to minimize the importance of the feast days omitted. Dropping these ten dates brought the equinox back to the time of the Council of Nicaea (A.D. 325), not to the time of Caesar's reform, thereby giving the whole thing an ecclesiastical touch. Nonetheless, this retrospective aspect was a mistake; it has confused historians ever since.

The reform was adopted quickly by Catholic Europe, but it was resisted elsewhere both for religious and academic reasons. For example, Françoise Viète (1540–1603) condemned the reform as a corruption of the Julian calendar. Certainly the two bitterest critics were Michael Maestlin, Kepler's teacher, and Joseph Justus Scaliger. (Scaliger's father made his reputation by attacking Cardano. When Cardano didn't reply immediately Scaliger thought he had died; Scaliger had a change of heart and wrote a laudatory funeral oration. Then Cardano replied.) Although "the calendar reform literature is on the whole 'interesting to few and entertaining to none' and a scholar of sense and taste will readily turn to other labours rather than cultivate this barren field" [11], the history of this dispute has been interestingly presented by Moyer [7].

The calendar was not adopted quickly. It was still possible in 1908 for a University of Michigan professor to state that he traveled for 43 days in August in Northern Europe, but that his September had only eighteen days. The explanation here is the same as why the "October Revolution" is celebrated in November—Russia did not adopt the Gregorian calendar until 1918.

Shakespeare and Cervantes died on the same date. Which one died first?

The New Leap Year Rule

The Gregorian reform also introduced a new leap year rule which was designed to keep the equinox near March 21. In the Gregorian calendar every year divisible by four is a leap year, except that years divisible by 100 are common, except years divisible by 400 are leap. Thus the Gregorian calendar has 97 leap years every 400 years, three fewer than the Julian. For example, the year 1500 was a common year under the new Gregorian calendar, but not under the Julian. Thus England, which did not adopt the reform until 1752, fell one day further behind. Judging by past performance, the year 2000 will confuse many. It will be a leap year, but it will not be the first year of the 21st century, rather the last of 20th.

The adoption of the Gregorian calendar was more traumatic in England than elsewhere. John Wallis, the theologian appointed as the first Savilian Professor with no more mathematical reputation than breaking a few coded messages for the king, strongly opposed the new calendar as a popish plot.

The delay in England led to additional historical confusion. George Washington's birth was recorded as 11 February 1731/32 O.S. This is an old style or Julian date. It became 22 February in our calendar. The "slash date" indicates that the year was 1731 in England, but 1732 on the continent. This is because the beginning of the year was then celebrated on March 25 in England. It was also the day taxes were due. When the Gregorian change occurred in 1752, the beginning of the year was changed to January 1 (this was also part of the Gregorian reform), but the tax date was delayed eleven days to April 5, where it remains. Apparently the tax collectors couldn't do fractions.

The 400 year cycle of our calendar leads to a number of peculiar results. My favorite—for it never fails to perplex—is that the thirteenth of the month is more likely to occur on Friday than on any other day of the week. B. H. Brown [1] posed this as a *Monthly* problem in 1933 and it continues to appear [5]. Since there are $400 \times 12 = 4800$ thirteenths of the month in one cycle and 7 does not divide 4800, it is clear that the 13th does not occur equally often on the different days of the week. (This is an example of a non-constructive existence proof that was not done by contradiction, something many feel can't happen.) To show that Friday is the most likely day for the 13th to occur on requires a brute force compilation with a perpetual calendar (I would be happy to see an elegant solution). The distribution of the thirteenth of the month in the 4800 months of the cycle is as follows:

Sunday	687
Monday	685
Tuesday	685
Wednesday	687
Thursday	684
Friday	688
Saturday	684.

So Friday wins, but not by much.

January 20 is inauguration day in the U.S. Although 20 ≡ 13 (mod 7), it is more likely to fall on Sunday than any other day. (Skolnik, [10])

It is an interesting exercise to prove that a calendar year has at least one and at most three Friday-the-thirteenths. (The solution of Bush and Heuer [2], which uses equivalence relations, is a nice classroom example.) In this respect 1983 was a good year (May was the sole offending month) but 1984 may be as bad as George Orwell predicted. It's the last leap year of the century with three Friday-the-thirteenths. They occur in a seven month period (January, April, and July), the most densely packed that three can be. The phrase "calendar year" is important in this exercise, for there are thirteen month periods (but no longer ones) that are free of Friday-the-thirteenths (the next being August 1984 to August 1985). The triskaidekaphobics among you will be encouraged that this century will end with a good sign: The 37-month period from September 1999 to October 2002 will contain only three Friday-the-thirteenths.

Is This the Best Possible Leap Year Rule?

There are a number of difficulties in designing an ideal solar calendar. Practicality dictates that only integers be used, yet the year does not contain an integral number of days, so something must be done. For the moment we shall leave the moon out of this, even though it gives another natural (as opposed to hours and weeks which are conventional) division of time. We shall deal with a purely solar calendar.

The tropical year is the time it takes the earth to return to the equinox point of its orbit. This consists of 365.24219879 days (in 1895 according to Simon Newcomb. In 1582 the value was 365.24222—24 seconds different from Clavius'). To simplify computation, we take 365.2422.

If .2422 is converted to a continued fraction (this is easy on a calculator: take the reciprocal, record the integral part, subtract it, then repeat), one obtains:

$$.2422 = \cfrac{1}{4 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}}}}}}$$

$$= (0; 4, 7, 1, 3, 4, 1, 1, 1, 2).$$

The convergents, obtained by throwing away the fractional denominators after a certain point, are

$$\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{31}{128}, \frac{132}{545}, \frac{163}{673}, \frac{295}{1218}, \frac{458}{1891}, \frac{1211}{5000}.$$

The first, third . . . of these are over approximations of .2422 and the others are too small. These represent the best rational approximations to our (rational) number that can be made without using a larger denominator.

The first of these means that one leap day should be added every four years—that's the Julian calendar. The next—add seven leap days every 29 years—isn't much good. Around 1079, Al-Khayyāmī, the poet and mathematician Omar Khayyam, suggested a 33 year calendar cycle with years 4, 8, 12, 16, 20, 24, 28, and 33 designated as leap years. Ironically, 8/33 = (0; 4, 7, 1) = .2424 is a better correction than the Gregorian 97/400 = (0; 4, 8, 12) = .2425, which is not even an intermediate fraction. None of the other convergents leads to practical calendars except the last "exact" value, 1211/5000, which means that 1211 leap days should be added every 5000 years. This is easy to achieve if the Gregorian rule is altered so that every fifth (not fourth) century is leap and there is a double leap year every 5000 years. We add 1250 days every 5000 years because every fourth year is leap, but subtract 50 days for the century year, and add ten days for the fifth centuries and add one at the end. Thus we added 1250 − 50 + 10 + 1 = 1211 days every 5000 years.

This rule has been suggested by Rasof [9], but the one that gets the most press is that suggested by Herschel: Every 4000th year should be ordinary. From the mathematical point of view this isn't so wonderful (perhaps numerology keeps all those 4s in): while 969/4000 lies between 8/33 and 31/128, it is not an intermediate fraction.

This continued fraction argument goes back at least to Euler [3].

On January 1, 1972, the question of revising the leap year rule became moot, for Coordinated Universal Time (UTC) then became effective internationally.

Leap seconds are introduced (or deleted) to account for variations in the Earth's rotation. In 1972—a leap year—two leap seconds were added, making it the longest year since 46 B.C., the "last year of confusion", which had 445 days.

Why 97/400?

Having shown that the Gregorian leap year isn't the best, it remains to explain where that rule came from. The only good explanation is due to Swerdlow [11]. The Gregorian year is 365 97/400 days = 365.2425 = 365 days 5 hours 49 minutes 12 seconds. This value is about four seconds shorter than the best values then available:

Alphonsine tables (1518)	365d 5; 49, 15, 58 . . . h
De revolutionibus (1543)	365d 5; 49, 16, 28 . . . h
Prutenic tables (1551)	365d 5; 49, 15, 45 . . . h

Note that these differ by less than a second. If converted to continued fractions, none will give rise to 97/400. Rounding to seconds doesn't help, but truncating the first and last to 5; 49, 15 will do the trick—but this seems an unlikely procedure.

The explanation of 97/400 comes from restating these values in sexagesimal fractions of a day, not of an hour:

Alphonsine tables	365; 14, 33, 9, 57 . . . days
De revolutionibus	365; 14, 33, 11, 12 . . . days
Prutenic tables	365; 14, 33, 9, 24 . . . days

Now if these are rounded to 365; 14, 33—a much more reasonable procedure than above—you get

$$0;\,14,\,33 = \frac{14}{60}\ \frac{33}{60^2} = \frac{97}{400}\,.$$

Perverse Months

There are 14 different calendars that are needed to have a complete set since the year can begin on any day of the week and the year can be common or leap. These calendars are used in a 28 year cycle unless it is interrupted by a common century year.

What is the greatest number of years that it can take to collect a complete set of calendars? (The minimum is 25 years.)

Kirby Baker calls a month perverse if it requires six lines to print on a calendar without double-dating (squeezing two dates into one box).

Each year contains at least two and at most four perverse months [4].

The next really perverse year, with four perverse months, is the leap year 2000. There is an inverse relationship between Friday-the-thirteenths and perverse months; so what is good for the calendar makers is bad for the superstitious. The exact relationship between the two is a bit complicated to state: There are a total of four perverse months and Fridays-the-thirteenths each year, except for common years beginning on Sunday or Thursday, and leap years beginning on Sunday or Saturday, when the sum is five. We end on a happier note.

A year and a month are selected at random. What is the probability that the calendar for the month requires only four lines, with no "double-dating"? (Kravitz [6])

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Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, Ohio 43403

Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half.

Leibniz