MA205 Graded Homework Due 14 November

Instructions:

For questions (1-3), do the following:

- State the Differential Equation.
- Solve the Differential Equation for $x(t)$ IOT find the Particular Solution.

**Note: Calculations must be done by hand-show all work. Confirm calculations by using Mathematica DSolve command. Turn-in both calculations done by hand and Mathematica work.

- Plot the particular solution in Mathematica. Comment on the long-term behavior of the function.

Question 1-Spring Mass System No-Damping

A spring with a 7-kg mass is held stretched 1.4 m beyond its natural length by a force of 80 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2 m/s. What is the position when t=5 seconds.

Question 2-Spring Mass System Damping

A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is stretched 1m beyond its natural length and then released with zero velocity. What is the position when t=5 seconds.

Question3-Spring Mass System Damping

A mass weighing 32lbs stretches a spring 8 feet beyond its natural length. The mass is released with an upward velocity of -0.5 ft/sec, from a position 2 feet below its equilibrium position. The subsequent motion of the mass takes place in a medium that offers a damping force of .5 times the instantaneous velocity.

Question 4

Two tanks each containing 50 gallons of salt solution. Initially tank A contains 10 lbs of Salt, and tank B contains 20 lbs of Salt. A solution containing 3 lbs of salt per gallon enters tank A at the rate of 2 gallon per minute and the well-stirred mixture drains from tank A to tank B at the same rate. The mixture in tank B is kept well-stirred and is drained off at the rate of 2 gallons per minute. **Setup the differential equations and put in matrix form.**

61

$m = 7\ kg$

$x = 1.4$

$F = kx$

$80N = 1.4k$

$57.14 = k$

$x(0) = 0$

$x'(0) = 1.2\ \frac{m}{s}$

No dampening

$mx'' + kx = 0$

$7x'' + 57.14x = 0$

$7r^2 + 57.14 = 0$

$r^2 + 8.1 = 0$

$r = \pm\sqrt{8.1}i$ so $\alpha = 0$, $\beta = 8$

Case III imaginary results

general solution

Case III

No damping

$\alpha = 0$

$y = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$

$y = e^{\alpha t}(C_1 \cos(\sqrt{8.1}t) + C_2 \sin(\sqrt{8.1}t))$

$y(0) = C_1 = 0$

$y'(t) = -\sqrt{8.1}\, C_1 \sin(\sqrt{8.1}t) + \sqrt{8.1}\, C_2 \cos(\sqrt{8.1}t)$

$y'(0) = \sqrt{8.1}\, C_2 = 1.2$

$C_2 = .42$

So $y = .42 \sin(\sqrt{8.1}t)$ ANS

at $t = 5$ $y(5) = .42 \sin(\sqrt{8.1}(5)) = .41$ ft

Q2

Spring Mass w/ dampening

$M = 2kg$

$C = 14$

$F = Kx$ $\begin{cases} F = 6N \\ x = \text{distance spring stretched} = .5 \end{cases}$

$6 = .5K$

$12 = K$

$mx'' + cx' + kx = 0$

$2x'' + 14x' + 12x = 0$

$2r^2 + 14r + 12 = 0$

$2(r^2 + 7r + 6) = 0$

$2(r+1)(r+6) = 0$

2 Real Roots $\Rightarrow$ $r = -1, -6$

Case I $\rightarrow$ Over Damping

$x = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$x = C_1 e^{-t} + C_2 e^{-6t}$

$x(0) = 1$

$x'(0) = 0$

$x(0) = C_1 + C_2 = 1$

$x'(t) = -C_1 e^{-t} - 6C_2 e^{-6t}$

$x'(0) = -C_1 - 6C_2 = 0$

$C_1 + C_2 = 1$

$-C_1 - 6C_2 = 0$

$-5C_2 = 1$

$C_2 = -\frac{1}{5}$

$C_1 = \frac{6}{5}$

$x = \frac{6}{5}e^{-t} - \frac{1}{5}e^{-6t}$ ANS

$x(5) = \approx .008$

Q3 $x = 8\,ft$

$F = 32\,lbs$

Spring w/ dampening

$F = kx$

$32 = k\,8$

$k = 4$

$mg = F, \; m = \frac{F}{g} \quad g = 32\frac{ft}{s^2}$

$\frac{32}{32} = m$

$m = 1$

$mx'' + dx' + kx = 0$

$x'' + .5x' + 4x = 0$

$r^2 + \frac{1}{2}r + 4 = 0$

$r = -.25 \pm 1.98i$

$\alpha \pm \beta i$

imaginary $\Rightarrow$ case III

under damping.

$$x = e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

$$x = e^{-.25t}\left(C_1 \cos(1.98t) + C_2 \sin(1.98t)\right)$$

$x(0) = 2$

$x'(0) = -.5$

$x(0) = C_1 = 2$

$$x'(t) = \left(-.25e^{-.25t}\right)\left(C_1 \cos(1.98t)\right) + e^{-.25t}(1.98)C_2 \cos(1.98t)$$

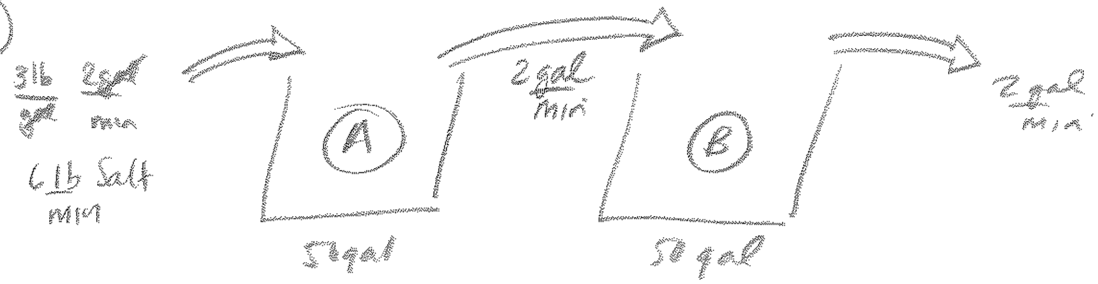
* Note: did not write down sin terms, $\sin(0) = 0$

$x'(0) = -.5 = -.25(C_1) + 1.98\,C_2$

$C_2 = 0$

S. $y = e^{-.25t}\left(2\cos(1.98t)\right)$

(4)



$A(0) = 10 \qquad B(0) = 20$

$A(t)$ - amount of Salt at t in tank A (lbs)

$B(t)$ - amount of Salt at t in tank B (lbs)

$$\frac{dR}{dt} = \text{Rate in} - \text{Rate Out}$$

$$\frac{dA}{dt} = 6 - \frac{2}{50}A$$

$$\frac{dB}{dt} = \frac{2}{50}A - \frac{2}{50}B$$

In Matrix Form

$$\begin{bmatrix} \frac{dA}{dt} \\ \frac{dB}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{2}{50} & 0 \\ \frac{2}{50} & -\frac{2}{50} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Let $x = \begin{bmatrix} A \\ B \end{bmatrix}$

$$x' = \frac{1}{50}\begin{bmatrix} -2 & 0 \\ 2 & -2 \end{bmatrix} x + \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

ANS

Question 1

In[135]:=
```
DSolve[{7 x''[t] + 57.14 x[t] == 0, x[0] == 0, x'[0] == 1.2}, x[t], t]
```

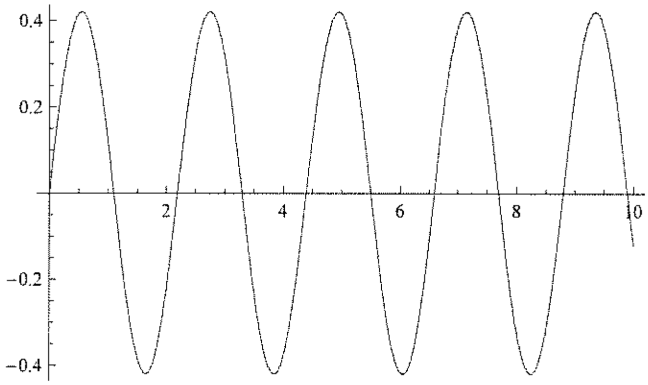
In[141]:=
```
{{x[t] → 0.42001050039376636` Sin[2.8570714276785494` t]}}
q1[t_] = 0.42001050039376636` Sin[2.8570714276785494` t];
q1[5] // N
```

Out[141]= {{x[t] → 0.420011 Sin[2.85707 t]}}

Out[143]= 0.415407

In[144]:=
```
Plot[q1[t], {t, 0, 10}]
```

Out[144]=



Question 2

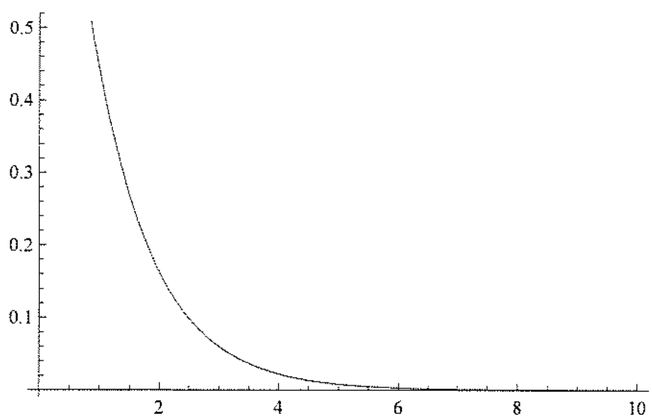
In[140]:=
```
DSolve[{2 x''[t] + 14 x'[t] + 12 x[t] == 0, x[0] == 1, x'[0] == 0}, x[t], t]
```

Out[140]= $\left\{\left\{x[t] \to \frac{1}{5} e^{-6t} \left(-1 + 6 e^{5t}\right)\right\}\right\}$

$q2[t\_] = \frac{1}{5} e^{-6t} \left(-1 + 6 e^{5t}\right);$

```
Plot[q2[t], {t, 0, 10}]
```

Out[146]=



In[147]:=
```
q2[5] // N
```

Out[147]= 0.00808554

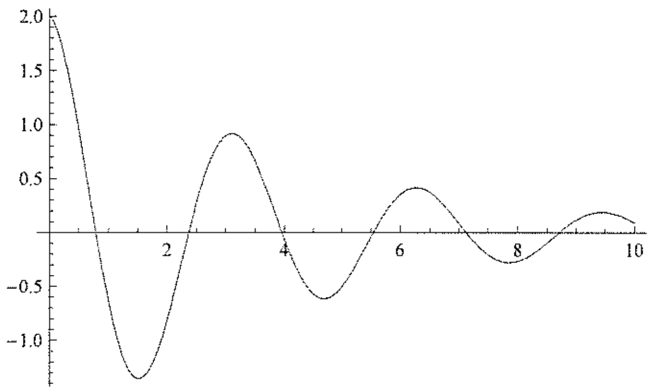
Question 3

In[148]:=
```
DSolve[{1 x''[t] + .5 x'[t] + 4 x[t] == 0, x[0] == 2, x'[0] == -.5}, x[t], t]
```

In[150]:=
```
{{x[t] → 2.` e^(-0.25` t) Cos[1.984313483298443` t]}}
q3[t_] = 2.` e^(-0.25` t) Cos[1.984313483298443` t];
Plot[q3[t], {t, 0, 10}]
```

Out[150]= {{x[t] → 2. e^(-0.25 t) Cos[1.98431 t]}}

Out[152]=



In[153]:=
```
Solve[r^2 + .5 r + 4 == 0, r]
```

Out[153]= {{r → -0.25 - 1.98431 i}, {r → -0.25 + 1.98431 i}}