MA205

Board Sheet - Block III - Summary - ODEs

1 Lesson 41 - Modeling with ODEs

1. A Differential Equation looks like:

$$\frac{dy}{dx} + 2xy = e^x \quad \text{or} \quad y' + 2xy = e^x$$

   In this case y is the dependent variable and x is the independent variable.

2. The order of the differential equation is determined by the highest derivative in the equation. For example: The first equation is first order and the second is third order:

$$\frac{dy}{dx} - 2y = 0$$

$$y''' - 2y'' + 2y' + 2y = \cos(t)$$

3. Linearity of a Differential Equation: An ODE is linear if it does not contain a product, power, or function of the dependent variable or any of its derivatives: Linearity does not apply to the independent variable: For example the first equation is linear the second is not:

$$2xy' - y = xe^{-x}$$

$$y' - e^{-2y} = 0$$

4. Homogeneity of a Differential Equation: A differential equation is homogeneous if all the terms involve the dependent variable or one of its derivatives. For example the first equation is homogeneous and the second is non-homogeneous.

$$y'' - 4y' + 2xy = 0$$

$$y'' - 3y' - 10y = e^x$$

5. Solutions of ODEs - Three Methods

   (a) Analytical

      i. Given $y' = y - x$ and the initial condition $y(0) = 2$, verify that $y = e^x + x + 1$ is a solution.

      ii. Take the appropriate amount of derivatives of the proposed solution:

$$y = e^x + x + 1 \qquad \frac{dy}{dx} = e^x + 1$$

iii. Substitute the derivatives into the differential equation:

$$\frac{dy}{dx} = y - x$$

$$e^x + 1 = e^x + x + 1 - x$$

$$e^x + 1 = e^x + 1$$

So we have proved it is a solution.

iv. Now we need to see if it satisfies the initial condition:

$$y(x) = e^x + x + 1 \quad y(0) = e^0 + 0 + 1 = 2 \quad \text{so} \quad y(0) = 2$$

So we have satisfied both the differential equation and the initial condition.

(b) Graphical Solutions

(c) Numerical Solutions

2 Lesson 42 - Slope Fields

Be able to use Slope Fields in Mathematica!

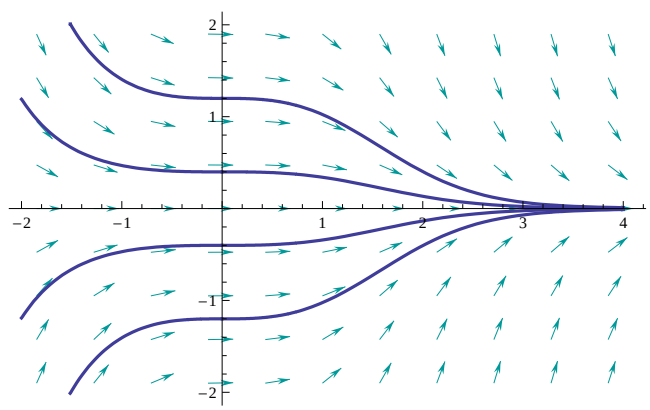
Make sure you have installed the Slope Field or DE package as listed in the UPDATED course guide!

For Example:

```
In[2]:= << DiffEqs`DEGraphics`

In[3]:= DEPlot[y * Cos[t] - y, {t, -2, 4}, {y, -2, 2}]
```

Out[3]=

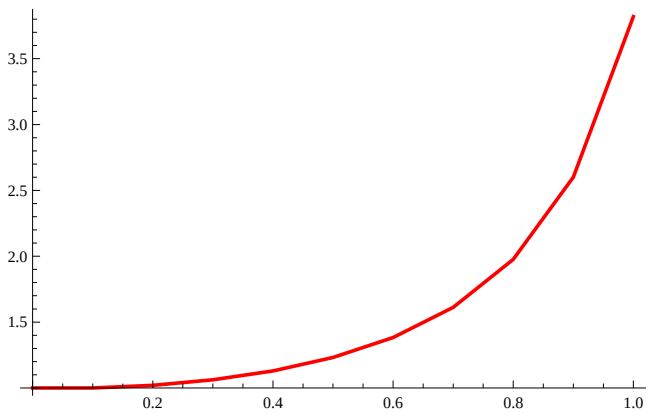


3 Lesson 43 - Numerical or Eulers's Method

Make sure you have a functioning Euler's Method working on Mathematica!

In[1]:=
```
EulerStep[{t_, y_}] := {t, y} + {h, h * f[t, y]}
f[t_, y_] := 2 t * y^2; (*function is f'[t]=2ty^2*)
starttime = 0;
stoptime = 1;
h = .1;
IC = {0, 1};
NumberofSteps = IntegerPart[(stoptime - starttime)/h];
points = NestList[EulerStep, IC, NumberofSteps];
approxsoln = ListPlot[points, Joined → True, PlotStyle → {Thick, Red}]
TableForm[Table[points], TableHeadings → {None, {"t-values", "y-values"}}]
(*exact=DEPlot[f[t,y],{t,0,stoptime},{y,1,4},InitialPoints→{IC}];
Show[exact,approxsoln]*)
```

Out[8]=



Out[9]//TableForm=

t-values	y-values
0	1
0.1	1
0.2	1.02
0.3	1.06162
0.4	1.12924
0.5	1.23125
0.6	1.38285
0.7	1.61232
0.8	1.97626
0.9	2.60116
1.	3.81906

4 Lesson 44 - Analytical Solution - Separation of Variables

Example $\frac{dy}{dx} = -ky$

1. We start by separating the equation meaning get everything that is a function of x on one side of the equation and everything that is a function of y on the other side of the equation.

   (a) So that the equation looks like $h(y)dy = g(x)dx$

   (b) In our case we multiply both sides of the equation by dx

   (c) This gives us $dy = -kydx$

   (d) Next since we need all terms with y on the other side we divide both sides by y.

(e) Giving us $\frac{1}{y}dy = -kdx$

2. Next we integrate "take antiderivative" both sides.

   (a) This should look like $\int \frac{1}{y}dy = -k\int dx$
   (b) Integrating gives us $\ln|y| + C_1 = -kx + C_2$
   (c) Combining the constants on the right side of the equation to give us one constant C gives us:
   (d) $\ln|y| = -kx + C$

3. Now we need only to evaluate!

   (a) To get ride of the ln we raise both sides of the equation to powers of e
   (b) $e^{\ln|y|} = e^{-kx+C}$
   (c) Simplifying further gives:
   (d) $|y| = e^{-kx+C}$
   (e) $|y| = e^{-kx}e^{C}$
   (f) If we are given initial conditions like when $x = 0$ then $y = A$ then our equation looks like $y = e^0e^c = A$
   (g) So Finally we get $y = \pm Ae^{-kx}$

Separation of Variables - Summary

1. Separate
2. Integrate
3. Evaluate

DSolve for Difficult ODEs

- Given $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 5$ use the DSolve

$$\text{DSolve}[\{y'[x] == x^2 + (y[x]), y[0] == 5\}, y[x], x]$$

5 Lesson 45 - Exponential Growth and Decay

1. Exponential Growth:

   The ODE looks like: $\dfrac{dP}{dt} = kP \quad k > 0$ So the solution is $P(t) = P_0e^{kt} \quad P(0) = P_0$

2. Exponential Decay:

   The ODE looks like: $\dfrac{dP}{dt} = kP \quad k < 0$ So the solution is $P(t) = P_0e^{kt} \quad P(0) = P_0$

3. Newton's Law of Cooling or Exponential Growth/Decay with Carrying Capacity:

$$\text{The ODE looks like:} \quad \frac{dT}{dt} = k(T - T_s)$$

$$\text{So the solution is} \quad T(t) = (T_0 - T_s)e^{-kt} + T_s, \quad T(0) = T_0 = A + T_s, \quad A = T_0 - T_s$$

T is the temperature of the object, T_s is the ambient temperature. This can be solved by separation of variables or DSolve.

4. Mixing Problems:

$$\frac{dC}{dt} = Ri - Ro$$

Where $C = C(t)$ is the concentration at time t, Ri = rate of product in, and Ro = rate of product out. This will usually look something like:

$$\frac{dC}{dt} = 4\frac{g}{min} - \frac{C(t)}{200}\frac{g}{min}$$

Where 200 is the amount of fluid in the tank.

MAKE SURE YOU CAN SOLVE MODELS WITH ALL THREE METHODS!

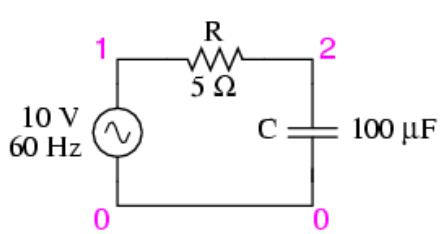
6 Logistical Equation - Lesson 46 - Write in the equation and know it

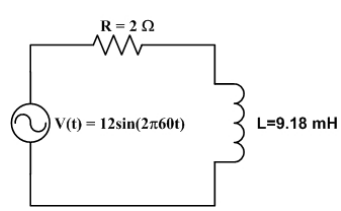
$$\frac{dP}{dt} = kP(K - P) \quad \text{or} \quad \frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

$$\text{The Solution is} \quad P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-kt}}$$

Where $P = P(t)$ is the population at time t, k is the growth rate, $P(0) = P_0$ is the initial population, and K is the carrying capacity.

7 Electrical Circuits - Lesson 47 - Know how to model electrical circuits and the three kinds.





Object	Variable	Units	Symbol
Resistor	R	Ω	write in
Inductor	L	H - Henry	write in
Capacitor	I= $\frac{dQ}{dt}$	Amps	write in
Charge	Q	Columbs	$+ \quad -$
EMF	E	Volts	write in

1. Kirchhoff's Laws
   (a) $\sum_{\text{loop}} E = 0$
   (b) $\sum_{\text{node}} I = 0$
2. Ohm's Laws
   (a) Resistor Circuit - $V = IR$ or $E = IR$
   (b) Inductor Circuit - $E = L\frac{dI}{dt}$
   (c) Capacitor Circuit - $E(t) = \frac{1}{C}\int I dt, \quad \text{so } \frac{1}{C}\int \frac{dQ}{dt}dt, \quad \text{finally } \frac{Q}{C}$
3. An RC Circuit like the first picture would look like:

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

$$\text{With a solution:} \quad Q(t) = Ae^{-\frac{t}{CR}} + CE$$

4. An RL Circuit like the second picture would look like:

$$L\frac{dI}{dt} + RI = E(t)$$

$$\text{With a solution:} \quad I(t) = Ae^{-\frac{R}{L}t} + \frac{E}{R}$$

5. So to solve the RC circuit above we substitute into a first order equation the following:

$$100\mu F = 100 * 10^{-6} \quad \text{or} \quad 0.001F$$

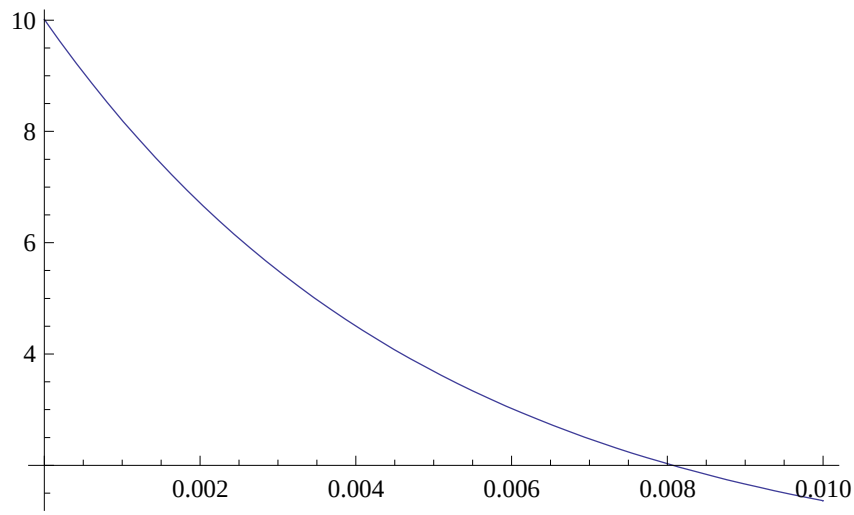
$$5\frac{dQ}{dt} + \frac{1}{.001}Q = 10 \quad \text{or} \quad 5\frac{dQ}{dt} + 1000Q = 10$$

At this point we can separate variables, do a DEPlot, use Euler's, or DSolve and plot.

In[21]:=
```
DSolve[5 Q'[t] + 1000 Q[t] == 10, Q[t], t]
Plot[1/100 + e^(-200 t) * 10, {t, 0, .01}]
```

Out[21]= $\left\{\left\{Q[t] \to \frac{1}{100} + e^{-200\,t}\,C[1]\right\}\right\}$

Out[22]=



8 Second Order Differential Equations -Lesson 49 - Characteristic Equation

$$ay'' + by' + cy = 0 \quad \text{Characteristic Equation} \quad ar^2 + br + c = 0$$

Solve for the roots using the quadratic equation.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three cases plus spring mass:

1. real different roots: Over-damped - $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
2. repeated roots: Critically-damped - $y(t) = C_1 e^{rt} + C_2 t e^{rt}$
3. imaginary roots: Under-damped - where $r = \alpha \pm \beta i$ solution is $y(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

9 Spring Mass - Lesson 50 and 51

1. $my'' + \delta y' + ky = F(t)$

2. m =mass:
   - English Units : $m = w/g \Rightarrow m = \frac{lb}{32\frac{ft}{sec}} = slug$
   - Metric Units : $m = kg$
3. k = spring constant
   - English Units : $k = \frac{mg}{ft} \Rightarrow k = \frac{lb \cdot 32\frac{ft}{sec}}{ft} \Rightarrow k = \frac{lb}{ft}$
   - Metric Units : $k = \frac{N}{m}$
4. $F(t)$ = any driving force
5. $y(0)$ = location of mass at time zero.
6. $y'(0)$ = velocity of mass at time zero.
7. Example Problem: Coarse Wide Board Problem:

   A mass weighing 2 pounds stretches a spring 6 inches beyond its natural length. At t = 0 the mass is released from a point 8 inches below the equilibrium position with an upward velocity of 43 ft/s through a medium which offers a resistance to motion that is equivalent to one half of its instantaneous velocity. Determine a differential equation and initial conditions which model this motion at anytime t. What is the long-term behavior of this system?

   (a) Find all the variables

   $F = ma$ We were given F as weight in pounds so $m = W/a$ or $m = 2/32 slugs$

   $$\delta = \frac{1}{2}\frac{ft}{s}$$

   $$k = \frac{2lb}{6in} \quad \text{so} \quad k = \frac{2lb}{6in}\frac{12in}{1ft} = 4\frac{lb}{ft}$$

   (b) Substitute into our equation:

   $$\frac{1}{16}y'' + \frac{1}{2}y' + 4y = 0$$

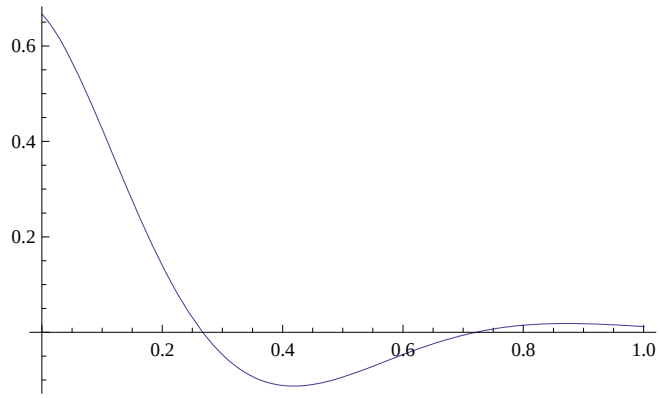
   (c) Use the Characteristic Equation or DSolve and plot to help understand.

In[1]:=
```
DSolve[{1/16 y''[t] + 1/2 y'[t] + 4 y[t] == 0, y[0] == 2/3, y'[0] == -4/3}, y[t], t]
DSolve[1/16 y''[t] + 1/2 y'[t] + 4 y[t] == 0, y[t], t]
Plot[1/9 e^(-4 t) (6 Cos[4 Sqrt[3] t] + Sqrt[3] Sin[4 Sqrt[3] t]), {t, 0, 1}]
```

Out[1]= $\left\{\left\{y[t] \to \frac{1}{9} e^{-4t} \left(6 \operatorname{Cos}\left[4 \sqrt{3}\, t\right] + \sqrt{3} \operatorname{Sin}\left[4 \sqrt{3}\, t\right]\right)\right\}\right\}$

Out[2]= $\left\{\left\{y[t] \to e^{-4t} C[2] \operatorname{Cos}\left[4 \sqrt{3}\, t\right] + e^{-4t} C[1] \operatorname{Sin}\left[4 \sqrt{3}\, t\right]\right\}\right\}$

Out[3]=



This system is under-damped, but only barely!