MA104

Board Sheet - 01 April 2008, Block III

Vectors So Far

1. Parametric Equations, Intersection, and Collision

   (a) Parametric Equations are equations with the x, y and z components represented as a function of t.

   (b) Collision and/or intersection.

      i. Set the $x(t)$ components equal to each other and solve for t. You may have more than one t.

      ii. Substitute these t's into the y and z components.

      iii. If both vectors are equal you have a collision and intersection at these times.

      iv. If they are not equal check for intersection.

      v. Replace your t with an r in one of the vectors.

      vi. Set the $x(t) = x(r)$ and $y(t) = y(r)$ and solve the system of equations for t and r.

      vii. These are your possible times for intersection where t is the time the first particle will go through the intersection point and r is the time for the second particle to go through the intersection point.

      viii. Substitute your t and r into the respective vectors and check if you have intersection and where it will be.

2. Three Dimensional Coordinate System

   (a) Distance Formula in Three Dimensions: The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

   $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

   (b) Equation of a Sphere: An equation of a sphere with center $C(h, k, l)$ and radius r is

   $$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

   If the center is the origin O, then the equation is

   $$x^2 + y^2 + z^2 = r^2$$

3. Vectors

   (a) A **vector** has both magnitude and direction.

   (b) A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude and the arrow points in the direction of the vector.

   (c) A vector is represented by a bold ($\mathbf{v}$) or an arrow above it ($\vec{v}$).

(d) Suppose a particle or ball moves along a line segment from point A to point B. The **displacement vector v**, has **initial point** A (the tail) and **terminal point** B (the tip) and we write this as $\mathbf{v} = AB$.

(e) Another vector $\mathbf{u} = CD$ could have the same length and direction as $\mathbf{v}$ but be in a different position. In this case $\mathbf{u}$ and $\mathbf{v}$ are **equivalent** (or **equal**) and we can write $\mathbf{u} = \mathbf{v}$.

(f) The **zero vector**, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

4. Vector Addition:

(a) If a particle moves from A to B, displacement $\overrightarrow{AB}$. Then the particle moves from B to C, displacement $\overrightarrow{BC}$. The combined effect is that the particle has moved from A to C, displacement $\overrightarrow{AC}$, is called the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

(b) The sum of $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

(c) An example mathematically. If the vector $\mathbf{u} = \langle 1, 2, 3\rangle$ is added to $\mathbf{v} = \langle 2, 3, 4\rangle$ the resulting vector is the addition of each of the components meaning add the x components, the y components, and the z components to get $\mathbf{u} + \mathbf{v} = \langle 3, 5, 7\rangle$.

5. Scaler Multiplication:

(a) If c is a scalar and $\mathbf{v}$ is a vector, then the scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same if $c > 0$ and opposite if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = 0$.

6. Components - a review:

(a) Vectors have components

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle$$

(b) A vector from the origin O to the point P where $P(3, 2, 1)$ is called the position vector of the point P.

(c) Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1\rangle$$

7. Magnitude or Length:

(a) The **magnitude** or **length** of the vector $\mathbf{v}$ is the length of any of its representations and is denoted by $|v|$ or $||v||$. We use the distance formula to compute the length.

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

(b) The Magnitude is a number, not a vector!

8. The Unit Vector!

   (a) A Unit Vector is a vector whose length is 1. For example $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are all unit vectors where $\mathbf{i} = \langle 1, 0, 0 \rangle$. If $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

9. Dot Product

   (a) Can we multiply two vectors? Of course we can, we get the Dot Product.

   (b) If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

   (c) So we multiply corresponding components and add. The result is a number not a vector. It is a real number known as a scaler.

   (d) The five properties of the Dot Product are given in your text on page 779

10. Dot Product Angle

   (a) The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle** θ **between a and b**.

   (b) If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

   (c) If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

   (d) Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

11. Projections

   (a) Projections can be seen as the shadow of a vector onto another.

   (b) Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: is a number $\text{comp}_a \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

   (c) Vector projection of $\mathbf{b}$ onto $\mathbf{a}$: is a vector $\text{proj}_a \mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^{\mathbf{2}}}\,\mathbf{a}$

   (d) To find work $W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos\theta$

12. Cross Product

   (a) The **cross product** $\mathbf{a} \times \mathbf{b}$ is a vector and only works when the vector are three-dimensional.

   (b) The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular or orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

(c) If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

(d) Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

(e) The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

13. Equation of a Line

(a) $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$

(b) $x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$

(c) $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

(d) Line Segment from $\mathbf{r_0}$ to $\mathbf{r_1}$ is given by the vector equation

$$\mathbf{r}(t) = (1-t)\mathbf{r_0} + t\mathbf{r_1}$$

14. Equation of a Plane

(a) Equation of a plane is $\mathbf{n} \cdot \langle r - r_0 \rangle = 0$

i. Where $r = (x, y, z)$ and $r_0 = (x_0, y_0, z_0)$ some given point on the plane.

ii. $\mathbf{n} = \langle a, b, c \rangle$ Which is a vector perpendicular to the plane.

A. $\mathbf{n}$ can be given like find the plane perpendicular to the vector $\langle 1, 2, 3 \rangle$

B. $\mathbf{n}$ can be derived by taking the cross product of two vectors in the plane like the vector $\mathbf{V}_1 = \langle 1, 2, 3 \rangle$ and $\mathbf{V}_2 = \langle 4, 5, 6 \rangle$. So $\mathbf{n} = \mathbf{V}_1 \times \mathbf{V}_2$ where we use mathematica to give us the Cross Product.

C. $\mathbf{n}$ can be given by three points like $P_1 = (1, 2, 3), \quad P_2 = (4, 5, 6), \quad P_3 = (7, 8, 9)$ where we must make two vectors from the three points and then take their cross product. So $\overrightarrow{P_1P_2} = P_2 - P_1 = (4, 5, 6) - (1, 2, 3) = \langle 4-1, 5-2, 6-3 \rangle = \langle 3, 3, 3 \rangle$ then we get $\mathbf{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ which we do in mathematica again.

15. Calculating the Equation of a Plane

(a) Now we use the $\mathbf{n}$ we found and a point on the line $r_0 = (x_0, y_0, z_0)$ for example $r_0 = (1, 2, 3)$.

(b) Again the equation is $\mathbf{n} \cdot \langle r - r_0 \rangle = 0$

(c) So if we found $\mathbf{n} = \langle 8, 7, 6 \rangle$

(d) We would write it like $\langle 8, 7, 6 \rangle \cdot \langle (x, y, z) - (1, 2, 3) \rangle = 0$

(e) Simplified would be $\langle 8, 7, 6 \rangle \cdot \langle x - 1, y - 2, z - 3 \rangle = 0$

(f) Taking the Dot Product would give $8(x - 1) + 7(y - 2) + 6(z - 3) = 0$

(g) Simplifying again gives $8x + 7y + 6z - 40 = 0$ or $8x + 7y + 6z = 40$

   (h) Now if we want to know if a point is on the plane we put in the values for x, y, z and if they equal 40 we are on the plane.

16. Derivatives of Vector Functions
   (a) The derivative of a vector function is the derivative of its components.
   (b) The derivative describes the tangent of the given curve.
   (c) The derivative also gives the rate of change in each direction of the curve.
   (d) Review the Differential Rules on Page 826.

17. Projectile Motion
   (a) $x(t) = x_0 + (v_0 \cos \alpha)t \qquad y(t) = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
   (b) Or $\mathbf{r}(t) = \langle x_0 + (v_0 \cos \alpha)t, \quad y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \rangle$
   (c) The velocity vector is the derivative of the position vector $\mathbf{v}(t) = \mathbf{r}'(t) = \langle v_0 \cos \alpha, \quad v_0 \sin \alpha - gt \rangle$
   (d) The acceleration vector is the derivative of the velocity vector $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle 0, \quad -g \rangle$
   (e) What is true at the maximum height of a projectile?
      i. The derivative of the y component $= 0$. So we take the derivative of $y(t)$ and set it equal to zero, $y'(t) = 0$. In our case $y(t) = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2$ so taking the derivative with respect to t gives $y'(t) = 0 + \sin \alpha - gt$. Setting this equal to 0 and solving for t gives $t = \dfrac{\sin \alpha}{g}$
      ii. Now we have t or time when the particle is at the highest point. If we substitute this t back into our position vector we will get the x and y position at that time - giving us the highest y.