MA104

Board Sheet - 12 February 2008, Optimization, Lesson 17

These Steps Should Help You Optimize

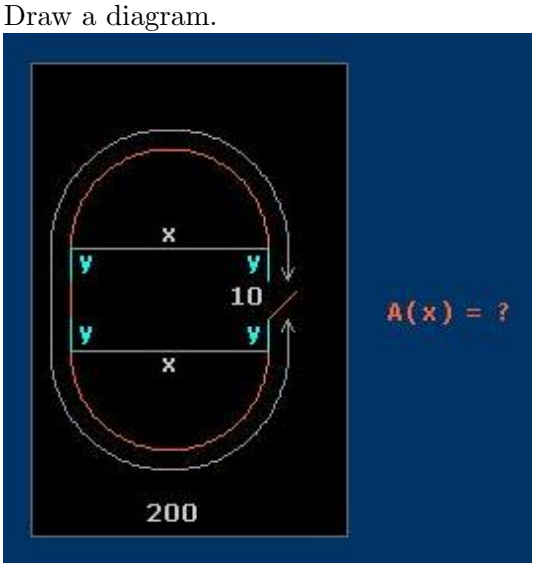
1. Understand the Problem: What is unknown? What are the given quantities? What are the given conditions?
2. Draw a diagram identifying the given and required quantities.
3. Introduce Notation. Assign a symbol to the quantity to be minimized or maximized such as Q. Select symbols for the unknown quantities like $(a, b, c, ..., x, y)$ use symbols that signify the variables to you.
4. Express Q in terms of the other symbols.
5. Find relationships that reduce Q to a function of only one variable.
6. Find the absolute maximum or minimum.

An Example

Area Enclosed by a Fence

You want to enclose an area consisting of a rectangle with a 10 foot gate and semicircular ends. What is the largest area that can be enclosed with 200 feet of fencing and what are its dimensions?

1. Understand the problem: What is unknown (x and y). What are the given quantities (200 feet of fence, 10 foot gate). What are the given conditions (area consists of semicircular ends and a rectangle).
2. Draw a diagram.



3. Introduce Notation. We want to maximize A the area of the fenced in area. Let x equal the diameter of the semicircle and y equal the sides of the rectangle.

4. Express A in terms of x and y. $A = \pi\left(\frac{x}{2}\right)^2 + x(2y+10)$

5. Find relationships that reduce A to a function of only one variable. $x\pi + 4y + 10 = 200$ so $y = \frac{200-10-x\pi}{4}$. Of course based on our picture $y \geq 0$. Now we substitute y into our equation for A which gives us

$$A = \pi\left(\frac{x}{2}\right)^2 + x\left(2\left(\frac{200-10-x\pi}{4}\right)+10\right)$$

6. Find the absolute maximum of A

   (a) take the derivative of A with respect to x.

$$A'(x) = \pi 2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right) + \left(2\left(\frac{200-10-x\pi}{4}\right)+10\right) + x\left(2\left(\frac{-\pi}{4}\right)\right)$$

   (b) Use Mathematica to find when $A'(x) = 0$ which gives us a negative value for y.

   (c) So knowing y must be greater or equal than 0. We look back at our relationship equation for x and y where $x\pi + 4y + 10 = 200$. So now $x\pi + 10 = 200$. Which gives us

$$x = 190/\pi$$

   .

   (d) Putting this into our equation for our area A gives us an area of $A = 3477.54$. To check this answer we could substitute in a value of x that is slightly smaller than our value and see if the area is smaller.

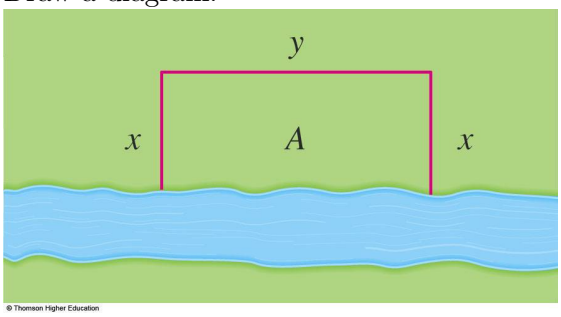
   (e) If we use $x = 185/\pi$ we get about 3460 which is smaller than our previous answer.

An Easier Example

Now we have 200 more feet of fencing and want to fence of a rectangular field that borders a straight river. What are the dimensions that give us the most area.

1. Understand the problem: What is unknown (x and y). What are the given quantities (200 feet of fence, one side is a river). What are the given conditions (area consists of a rectangle with three sides).

2. Draw a diagram.



3. Introduce Notation. We want to maximize A the area of the fenced in area. Let y equal the width of the rectangle and x equal the height of the rectangle.

4. Express A in terms of x and y. $A = xy$

5. Find relationships that reduce A to a function of only one variable.
   $2x + y = 200$ so $y = 200 - 2x$. Of course based on our picture $y \geq 0$. Now we substitute y into our equation for A which gives us

$$A = (200 - 2x)x$$

6. Find the absolute maximum of A

   (a) take the derivative of A with respect to x.

$$A'(x) = (200 - 2x) - 2x$$

   (b) Setting $A' = 0$ We find $200 - 4x = 0$ and solving for x we get $x = 50$.

   (c) Putting this into our equation for our area A gives us an area of $A = (50)(100) = 5000$. To check this answer we could substitute in a value of x that is slightly smaller or larger than our value and see if the area is smaller.

   (d) If we use $x = 49$ or $x = 51$ we get about 4998 which is smaller than our previous answer.

Now Try it on a Problem

No summer visit to London is complete without having lunch on sunny Goodge Street. There, for a mere pound coin, you can purchase the best fish and chips youve ever tasted from any one of a number of friendly street vendors. One of the reasons that the prices are so reasonable is that they give you no silverware, nor even a plate: they just roll up a piece of paper into a cone, and toss your food in. (They do give you a little packet of vinegar, though.) Neither a long, skinny cone nor a wide fat cone would hold enough fish and chips to make anybody happy. The vendors must be trained to roll a cone of the perfect size. Many students will never get a chance to see sunny Goodge Street, but by trying to solve the vendors problem of optimizing the volume of a cone, we can feel as if we are there now. For modeling purposes, assume that the piece of paper is a circle of radius 5 inches, and that we are cutting a wedge out of it whose central angle is θ.