MA153

Review Sheet - Block II

Problem Solving with Partial Derivatives

1. Functions of Several Variables

   (a) Domain defines the values x and y can be that allow $f(x, y)$ to be defined.

   (b) Range defines the values that $f(x, y)$ can achieve.

   (c) For example find the domain and range of $f(x, y) = \sqrt{9 - x^2 - y^2}$:

      i. To be defined we cannot take the square root of a negative number so the domain looks like $D = \{(x, y)|9 - x^2 - y^2 \geq 0\}$ or simplified $D = \{(x, y)|x^2 + y^2 \leq 9\}$

      ii. Now the range will describe what $f(x, y)$ can be. So the smallest $x^2 + y^2$ can be is zero and the greatest would be 9. This means that $f(x, y)$ can range from 0 to 3, or $R = \{z|0 \leq z \leq 3\}$ or $R = [0, 3]$

      iii. (means the range moves up to but not including the number next to it. [means the range moves up to and includes the number next to it.

2. Steps to find the Limit of a Function of Two Variables

   (a) Test the limit to see if the number is defined. For example $\lim_{(x,y)\to(1,1)} \sqrt{6 - x^2 - y^2}$. The limit would be $\sqrt{6 - 1 - 1} = 2$.

   (b) If the limit gives you an undefined function then test the function on a few different paths - if they are not all equal there is no limit. For example if $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz + xz^2}{x^2 + y^2 + z^2}$ We try the line along the x axis which is $f(x, 0, 0) = \frac{0}{x^2} = 0$. Now we try at a 45 degree angle in the x y plane $f(x, x, 0) = \frac{x^2}{2x^2} = \frac{1}{2}$ So we get different limits Therefore no limit exists.

3. Partial Derivatives

   (a) Notations For Partial Derivatives. If $z = f(x, y)$ we write:

      i. $f_x(x, y) = f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x} f(x, y) = \dfrac{\partial z}{\partial x} = f_1 = \mathbf{D}_1 f = \mathbf{D}_x f$

      ii. $f_y(x, y) = f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y} f(x, y) = \dfrac{\partial z}{\partial y} = f_2 = \mathbf{D}_2 f = \mathbf{D}_y f$

   (b) How to find a partial derivative of a function like $f(x, y) = x^2 + y^2 - 4xy + 5$

      i. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x. So $f_x(x, y) = 2x - 4y$

      ii. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y. So $f_y(x, y) = 2y - 4x$

iii. To find the second or higher derivatives keep taking the partial derivative of the derivative. Or for a mixed partial take the partial of the function with respect to the inside variable then the outside.
So $f_{xx}(x,y) = 2, \quad f_{yy} = 2, \quad f_{xy} = -4$

(c) Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx}, are both continuous on D, Then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

(d) Laplace's Equation: Harmonic functions will satisfy the following equation, known as the Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

4. Tangent Planes

(a) Equation of a Tangent Plane:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(b) Make sure you take the partial derivatives and evaluate them at the point of interest (x_0, y_0) then multiply those slopes times $x - x_0$. You should get an equation looking like $ax + by + cz = d$. Where a, b, and c are slopes and d is similar to your intercept in the equation of a line.

5. Linear Approximations

(a) Linear Approximation is using the equation of a plane to approximate a point some distance from your original point.

(b)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$$

(c) So you are approximating what f is at the point (x,y) based on a plane at the point (a,b). Your approximation will me more accurate the closer you are to the point (a,b).

6. Total Differential - Can be found using one of the following.

(a)

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

(b)

$$dz = f_x(x,y)(x - a) + f_y(x,y)(y - b)$$

7. The Chain Rule - Similar to what we remember from single variable derivatives.

(a) Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

(b) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

8. Gradient

   (a) The gradient $\nabla f(x, y, z)$ is a vector in the direction of the steepest slope. For example if $f(x, y, z) = x^4 + xy + z$ then $\nabla f(x, y, z) = \langle 4x^3 + y, x, 1\rangle$

   (b) Rules for Finding the Gradient

      i. Get the gradient by taking the partial derivatives and put them in vector notation! To get the gradient at a point sub in the values of (x,y,z) into your gradient. For example if $f(x, y, z) = x^4 + xy + z$ and you want $\nabla f(2, 2, 5) = \langle 4(2)^3 + (2), (2), 1\rangle = \langle 34, 2, 1\rangle$.

9. Directional Derivative

   (a) The Directional Derivative is a Number/Scaler giving the slope/rate of change in a specific direction.

   (b) Rules for Finding the Directional Derivative

      i. Get the gradient by taking the partial derivatives and put them in vector notation! Then get the gradient at the point you are at by subbing in the values of (x,y,z)

      ii. Get the direction you want into vector notation

      iii. Get the vector into a unit vector - by dividing the vector by its magnitude

      iv. Take the dot product of the unit vector and the gradient - $D_u \cdot f_x f_y = D_u f f$

10. Tangent Planes to Level Surfaces

   (a)

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

11. Steps to find the Extrema of Functions of Two Variables

   (a) Find the critical points. If our function was $f(x, y) = x^2 + y^2 + x^2y + 4$

      i. find the partial derivatives of $f(x, y)$ $\quad f_x = 2x + 2xy, \quad f_y = 2y + x^2$

      ii. Set the partial derivatives equal to zero and solve the system of two equations and two unknowns.

$$2x + 2xy = 0, \quad 2y + x^2 = 0$$

      iii. Solve one of the equations for x or y So lets solve the second equation for y where we get $y = \frac{-x^2}{2}$

      iv. Sub your x or y into the other equation and solve for the variable. In our case we put our y into the first equation which gives
      $2x + 2x(\frac{-x^2}{2}) = 0 \quad \Rightarrow \quad 2x - x^3 = 0 \quad \Rightarrow 2x = x^3$
      Now we need the roots of this equation. The obvious one is when $x = 0$. So we factor an x out of the equation and are left with $2 = x^2 \quad \Rightarrow \quad x = \pm\sqrt{2}$. So our roots are $x = -\sqrt{2}, 0, \sqrt{2}$.

v. Now we want to know what our critical points are so we sub our values of x back into one of our equations. We get the following critical points $(-\sqrt{2}, -1), (0,0), (\sqrt{2}, -1)$.

(b) Classify our critical points using the second derivative test.

i. The discriminant $D = f_{xx}f_{yy} - (f_{xy})^2$

ii. So we find the second partial derivatives and the mixed partial.
$f_{xx} = 2 + 2y, \quad f_{yy} = 2, \quad f_{xy} = 2x$. So $D(x, y) = (2y + 2)(2) - (2x)^2$.

iii. We need to test $D(x, y)$ at each of our critical points.

A. If $D(x, y) > 0$ and $f_{xx}(x, y) > 0$ we have a minimum.

B. If $D(x, y) > 0$ and $f_{xx}(x, y) < 0$ we have a maximum.

C. If $D(x, y) < 0$ we have a possible saddle point.

iv. Testing each critical point we get:
$D(-\sqrt{2}, -1) = (2(-1) + 2)(2) - 2 * (-\sqrt{2})^2 \quad \Rightarrow \quad D(-\sqrt{2}, -1) = -8$ Therefore $(-\sqrt{2}, -1)$ is probably a saddle point.
$D(0, 0) = 2 * (0)(2) - 2 * (0)^2 \quad \Rightarrow \quad D(0, 0) = 0$ Therefore $(0, 0)$ is not definable by the second derivative test.
$D(\sqrt{2}, -1) = (2(-1)+2)(2) - 2*(\sqrt{2})^2 \quad \Rightarrow \quad D(\sqrt{2}, -1) = -8$ Therefore $(\sqrt{2}, -1)$ is probably a saddle point.

(c) Next we need to test the boundary conditions. If our boundary was
$D = \{(x, y)| - 1 \le x \le 1, -1 \le y \le 1\}$.

i. We can make a box with equations of the lines as the four sides of the box. Along this box we have the bottom line where a point on $f(x, y)$ can be described as
$f(x, -1) = x^2 + (-1)^2 + x^2 * (-1) + 4 \Rightarrow f(x, -1) = x^2 - x^2 + 5$ or $f(x, -1) = 5$
Similarly we get $f(x, 1) = 2x^2 + 5, \quad f(-1, y) = y^2 + y + 5, \quad f(1, y) = y^2 + y + 5$

ii. Checking the end points of x and y we get
$f(1, 1) = 7, \quad f(-1, -1) = 5 \quad f(-1, 1) = 7 \quad f(1, -1) = 5$

iii. We should also take the partial derivatives of the boundaries and set them equal to zero to get the max or min on the boundary, we treat this like max and min in two dimensions where we already tested the end points above.

A. On the bottom line we get $f_x(x, -1) = 2x - 2x = 0 \Rightarrow$ No mater what x is we get $f(x, -1) = 5$

B. On the right line we get $f_y(1, y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ So $f(1, -\frac{1}{2}) = \frac{19}{4}$

C. On the top line we get $f_x(x, 1) = 4x = 0 \Rightarrow x = 0$ So $f(0, 1) = 5$

D. On the left line we get $f_y(-1, y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ So $f(-1, -\frac{1}{2}) = \frac{19}{4}$

(d) We need to check our critical points. We realize the only critical point in our domain is $(0, 0)$ which $f(0, 0) = 4$

(e) We compare all of our points and see that our Global Minimum is at $f(0, 0) = 4$ and our Global Maximum is at $f(1, 1) = 7$ and $f(-1, 1) = 7$.

12. Optimization
These steps are similar to many other models we have done. If we want a box without a top to have the maximum volume while using only 12 inches of cardboard we do the following:

(a) Find the Objective Equation - The equation we want to maximize or minimize. $V = xyz$.

(b) Find the Constraint Equation - Our limitations. Our Surface Area has to satisfy $2xz + 2yz + xy = 12$.

(c) Use the Constraint Equation to reduce our Objective Equation to an equation of only two variables. $z = \frac{12-xy}{2(x+y)}$

(d) Find the Critical Points of the new Objective Equation by taking the partial derivatives and setting them equal to zero then solving the system of equations. In this case using Mathematica can help. Our only positive critical point is $(2, 2)$

(e) Use the Second Derivative Test to determine the classification of the critical points. Again Mathematica can make finding the Determinant easier. In our case we get $D > 0$ and $f_{xx} < 0$ proving we have a maximum.

(f) Test the Critical Points in the function to determine the Max or Min you are seeking. In our case we get $f(2, 2) = 4$

13. LaGrange Multipliers

(a) Find the Objective Equation - The equation we want to maximize or minimize.
$f(x, y) = 4x + 6y$.

(b) Find the Constraint Equation - Our limitations. Our Surface Area has to satisfy
$x^2 + y^2 = 13$.

(c) Find the gradient of both the Objective and Constraint Equation $\nabla Ob, \nabla Co$ by taking the partial derivatives and putting them in vector notation.
$\nabla Ob(x, y) = \langle 4, 6\rangle$,
$\nabla Co(x, y) = \langle 2x, 2y\rangle$

(d) Set $\nabla Ob(x, y) = \lambda \nabla Co(x, y)$.
$\langle 4, 6\rangle = \lambda\langle 2x, 2y\rangle$

(e) Solve the equations $\nabla Ob(x, y) = \lambda \nabla Co(x, y)$ and $Co(x, y) = k$ for all values of x, y, λ by solving the system of three equations and three unknowns. In this case I would solve the first equation for λ giving $\lambda = \frac{2}{x}$. I would then sub λ into the second equation. $6 = 2y(\frac{2}{x})$. I would solve the second equation for x giving $x = \frac{2y}{3}$. Finally I would sub the x into the final equation $(\frac{2y}{3})^2 + y^2 = 13$ This would give $\frac{4y^2}{9} + y^2 = 13$. So $y = \pm 3$ then $x = \pm 2$.

(f) Evaluate Ob at all the points (x, y). The largest of these values is the max; the smallest is the minimum. In our case $Ob(2, 3) = 26$ and $Ob(-2, -3) = -26$