MA153

Block IV Review Sheet

Vector Calculus - Stewart Chapter 16

1. Vector Fields

   (a) $F(x,y) = P(x,y)\,\widehat{i} + Q(x,y)\,\widehat{j} = \langle P(x,y), Q(x,y)\rangle$
   (b) $F(x,y,z) = P(x,y,z)\,\widehat{i} + Q(x,y,z)\,\widehat{j} + R(x,y,z)\,\widehat{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$
   (c) Gradient Fields $\nabla f(x,y,z) = f_x(x,y,z)\,\widehat{i} + f_y(x,y,z)\,\widehat{j} + f_z(x,y,z)\,\widehat{k}$

2. Line Integrals 2 dimensions

   (a) $\displaystyle\int_C f(x,y)ds$
   (b) $\displaystyle L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
   (c) Similar arguments help us define the line integral as the following:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

   (d) What if we want to parameterize with something other than t?
   (e) Do a similar process, but instead:

$$\int_C f(x,y)ds = \int_a^b f(x,y(x))\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx$$

   (f) We may also have to remember the equation of a line segment:

$$\overrightarrow{r}(t) = (1-t)\overrightarrow{r_0} + t\overrightarrow{r_1}$$

3. Line Integrals 3 dimensions

   (a) Start with a plane curve C given by the parametric equations $x = x(t) \quad y = y(t) \quad z = z(t) \quad a \le t \le b$
   (b) $\displaystyle\int_C f(x,y,z)ds =$

$$\int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

(c)

$$\int_a^b f(\overrightarrow{r}(t))|\overrightarrow{r}'(t)|dt$$

(d) If we talk about work we remember $W = \overrightarrow{F} \cdot \overrightarrow{D}$

(e) We suppose that $\overrightarrow{D} = \overrightarrow{PQ}$ the displacement vector, and that $\overrightarrow{F} = P\widehat{i} + Q\widehat{j} + R\widehat{k}$.

(f) Now work can be described by:

$$W = \int_C \overrightarrow{F}(x, y, z) \cdot \overrightarrow{T}(x, y, z)ds = \int_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds$$

(g) Also

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_a^b \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)dt = \int_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds$$

(h) So

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C Pdx + \ Qdy + \ Rdz \quad \text{where} \quad \overrightarrow{F} = P\widehat{i} + Q\widehat{h} + R\widehat{k}$$

4. FTC and the line integral

(a) $\displaystyle\int_C \nabla f \cdot d\vec{r} = f(x_2, y_2) - f(x_1, y_1)$

(b) $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ Path Independent if and only if $\displaystyle\int_C \vec{F} \cdot d\vec{r} = 0$

(c) $\vec{F}$ is a conservative vector field on D if $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ is path independent.

(d) If $\vec{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j}$ is a conservative field then:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

(e) Page 1050 - Theorem 6

5. Green's Theorem

(a)

$$\int_C P\, dx \ + \ Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

(b)

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2} \oint_C xdy - ydx$$

(c) Holes and Green's Theorem

6. Curl

(a) If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field in three dimensions and the partial derivatives of P, Q, and R all exist, then the curl of $\vec{F}$ is the vector field in three dimensions defined by:

(b)
$$\operatorname{curl} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\widehat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\widehat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\widehat{k}$$

7. More about Curl

   (a) The Curl of a gradient field is **0**. This is because the curl is the cross product of the gradient with the gradient.
   $$\operatorname{curl}(\nabla f) = \nabla \times (\nabla f)$$

   (b) So we have an extension of what we learned in Green's Theorem. If $\vec{F}$ is a vector field defined on all reals in three dimensions whose component functions have continuous partial derivatives and curl $\vec{F} = \mathbf{0}$, then $\vec{F}$ is a conservative vector field.

8. Divergence

   (a) If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field in three dimensions and the partial derivatives of P, Q, and R all exist, then the **divergence of F** is the function of three variables defined by:

   (b)
   $$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

   (c) Similar to the dot product where we get a number, divergence is:
   $$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

9. More about Divergence

   (a) If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field in the real three dimensions and P, Q, and R have continuous second-order partial derivatives, then

   (b)
   $$\operatorname{div}\ \operatorname{curl}\vec{F} = 0$$

10. Physical description of Divergence

    (a) Fluid flow again is one of the best examples. Divergence can be seen as the net rate of change of the mass of the fluid flowing from the point P per unit volume. Or the divergence measures the tendency of the fluid to diverge from point P.

    (b) If div $\vec{F} = 0$ at a point P, the the fluid is said to be incompressible.

11. Laplace Operator

    (a) If we take the divergence of the gradient vector of a function of three variables, we get:

(b)

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

(c) This is often called the Laplace operator and looks like:

(d)

$$\nabla^2 = \nabla \cdot \nabla$$

12. Vector Form of Green's Theorem

(a)

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P dx + Q dy$$

(b) If we regard $\vec{F}$ as a vector field in three dimensions with the third component 0, we get:

(c)

$$\text{curl}\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

13. Vector Form of Green's Theorem

(a) So:

$$(\text{curl}\vec{F}) \cdot \hat{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} \cdot \hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

(b) We can rewrite the Green's Theorem in vector form:

(c)

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl}\vec{F}) \cdot \hat{k} dA$$

14. Second Vector Form of Green's Theorem

(a)

$$\oint_C \vec{F} \cdot \vec{n} ds = \iint_D \text{div}\vec{F}(x, y) dA$$

15. Parametric Surface

(a)

$$\vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$$

16. Tangent Planes

(a)

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\hat{i} + \frac{\partial y}{\partial v}(u_0, v_0)\hat{j} + \frac{\partial z}{\partial v}(u_0, v_0)\hat{k}$$
$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\hat{i} + \frac{\partial y}{\partial u}(u_0, v_0)\hat{j} + \frac{\partial z}{\partial u}(u_0, v_0)\hat{k}$$

(b)

$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

(c)

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

17. Surface Area

(a)

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| dA$$

(b)

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

18. Surface Integrals

(a)

$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA$$

(b)

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

19. Surface Integrals of Vector Fields

(a)

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS$$

(b) This is called the **flux** of $\vec{F}$ across S.

(c)

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$

(d)

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

20. Stoke's Theorem

(a) Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in the real three dimensions that contains S. Then

(b)

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}$$

(a) Another way of saying this

$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

(b) If the surface is flat and lies in the xy-plane we get:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot d\vec{S} = \iint_S (\text{curl}\vec{F}) \cdot \hat{k} dA$$

The vector form of Green's Theorem.

(a)

$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} =$$

$$\iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

(a) One more way

$$\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \text{curl}\vec{F} \cdot d\vec{S}$$

21. Divergence Theorem

(a) Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F} dV$$