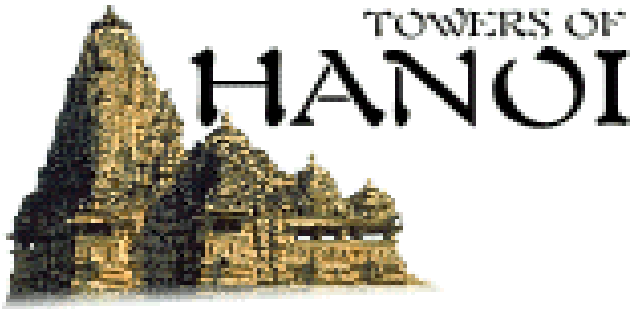
The Big Thought of the Day…

"Sequences & Series."



Admin Notes

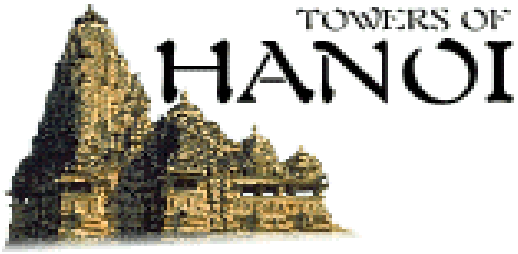
- Project 2.
- Block V Schedule.
- WPR IV Stats…

WPR IV Stats

Question	Avg %
1a	95%
1b	85%
2	75%
3a	89%
3b	90%
3c	86%
3d	65%
4a	91%
4b	84%
5a	95%
5b	95%
6	66%

	Points	Percent
Highest	300	100%
Average	255	85%
Lowest	0	0%
Stand Dev		11.7

Sequences



Legend has it that in Benares, an ancient city in India, there was a temple with a dome which marked the center of the world. At the beginning of time, God placed 64 gold disks on one needle. Within the dome, priests moved golden disks between **two** diamond needle-points (original and destination locations). It was said that when they completed their task, the universe would come to an end!

The disks are fragile; only one can be carried at a time. A disk may **not** be placed on top of a smaller, less valuable disk. And, there is only one other location in the temple (besides the original and destination locations) sacred enough that a pile of disks can be placed.

1.) Suppose there was only one disk, how many moves would it take?

2.) How about two? Three, Four, Five?

3.) Suppose it took one second to move a single disk. How long would it take to move all 64?

http://www.mazeworks.com/hanoi/index.htm

Sequences

$$M_d = \text{number of moves required for } d \text{ disks}$$

$$M_d = 2M_{d-1} + 1$$

$$M_0 = 0$$

"Initial Value" or *"Initial Condition"*

"Iteration Equation" or *"Recursion Equation"*

- Notice the above *"Iteration Equation"* moves in *"discrete"* steps… number of moves are *"integer"* values!
- We must *"iterate"* to find a solution related to a specific discrete step… thus *"Iteration Equation"*… Present=Past+Change
- We can define the *"sequence"* formed by the above *"Iteration Equation"* as…

$$\{M_1, M_2, M_3, \ldots, M_{64}\} = \{1, 3, 7, \ldots, 18446744073709551615\}$$

Sequences

Sequence vs. Series…

A ***"sequence"*** is a **list of numbers**. The order in which the numbers are listed is important, so for instance $\{1, 2, 3, 4, 5, ...\}$ is one sequence, and $\{2, 1, 4, 3, 6, 5, ...\}$ is an entirely different sequence!

A ***"series"*** is a **sum of numbers**. For example, $\{1 + 1/2 + 1/4 + 1/8 + 1/16 + ...\}$ is an example of a series. A series is composed of a sequence of terms that are added up. The order in which the terms appear is sometimes important.

Famous Sequences

A arithmetic sequence…

… is a sequence such that the differences between successive terms is a constant.

general form…

$$p_k = p_{k-1} + m$$

when…

$$p_0 = a$$

$$p_k = a + km$$

$$k = 1,2,3,\ldots$$

Famous Sequences

A geometric sequence…

… is a sequence such that each term is given by a multiple of the previous one. If the multiplier is ***r***, then the $\boldsymbol{k^{th}}$ term is given by the recursion equation,

general form…

$$p_k = rp_{k-1}$$

when…

$$p_0 = a$$

$$p_k = ar^k$$

$$k = 1,2,3,\ldots$$

Famous Sequences

Recall "Towers of Hanoi"…

$$M_d = \text{number of moves required for } d \text{ disks}$$

$$M_d = 2M_{d-1} + 1 \quad \Rightarrow \quad M_d = 2^d - 1$$

$$M_0 = 0$$

"difference equation" → $M_d - M_{d-1} = M_{d-1} + 1$

"sequence" → $\{M_0, M_1, M_2, M_3, \ldots, M_{64}\} = \{0, 1, 3, 7, \ldots, 18446744073709551615\}$

How are they related? →

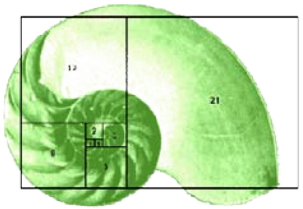
$$\begin{cases} M_1 - M_0 = M_0 + 1 = 1 = 2^0 \\ M_2 - M_1 = M_1 + 1 = 2 = 2^1 \\ M_3 - M_2 = M_2 + 1 = 4 = 2^2 \\ M_4 - M_3 = M_3 + 1 = 8 = 2^3 \\ \vdots \\ M_d - M_{d-1} = M_{d-1} + 1 = 2^{d-1} \end{cases}$$

Famous Sequences

A fibonacci sequence…

… is a sequence of numbers defined by the recursion equation…

$$F_n = F_{n-1} + F_{n-2}$$



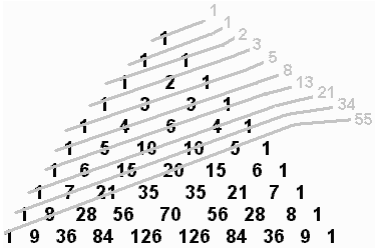
when…

$$F_0 = 0$$

$$F_1 = F_2 = 1$$

$$n = 1,2,3,\ldots$$

$$sequence = \{1,1,2,3,5,8,13,21,\ldots\}$$



A scrambled version 13, 3, 2, 21, 1, 1, 8, 5 of the first eight Fibonacci numbers appear as one of the clues left by murdered museum curator Jacque Saunière in D. Brown's novel *The Da Vinci Code*.

Sequences

A sequence $\{a_n\}$ has the limit L, and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like, by taking n sufficiently large.

– If $\lim_{n\to\infty} a_n$ exists, the sequence ***converges*** (or is convergent).

– Otherwise, it ***diverges*** (or is divergent).

Example 1…

Determine whether the sequence a_n is convergent or divergent… if convergent find the limit.

$$a_n = \frac{n^3}{n+1}$$

Example 2…

Determine whether the sequence a_n is convergent or divergent… if convergent find the limit.

$$a_n = \tan\left(\frac{2n\pi}{1+8n}\right)$$

Sequences

A sequence $\{a_n\}$ is ***bounded***…

- Below, if there is a number $\boldsymbol{m}$ such that $\boldsymbol{m} \le \boldsymbol{a_n}$ for all $\boldsymbol{n} \ge 1$… ***i.e.*** $\boldsymbol{a_n} = \boldsymbol{n}$.

- Above, if there is a number $\boldsymbol{M}$ such that $\boldsymbol{a_n} \le \boldsymbol{M}$ for all $\boldsymbol{n} \ge 1$… ***i.e.*** $\boldsymbol{a_n} = \boldsymbol{n}/(\boldsymbol{n}+1)$.

- But, not every ***bounded*** sequence is ***convergent***… ***i.e.*** $\boldsymbol{a_n} = (-1)^{\boldsymbol{n}}$.

Sequences

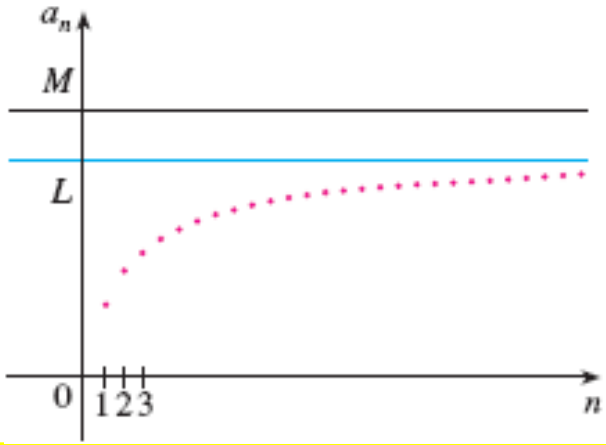
A sequence $\{a_n\}$ is called…

– ***Increasing***, if $a_n < a_{n+1}$ for all $n \geq 1$,
that is, $a_1 < a_2 < a_3 < \cdots$

– ***Decreasing***, if $a_n > a_{n+1}$ for all $n \geq 1$

– ***Monotonic***, if it is either increasing or decreasing.

– If a sequence is both ***bounded*** & ***monotonic***, then it must be ***convergent***.



Example 3…

Determine whether the sequence a_n is increasing, decreasing, or not monotonic… is the sequence bounded?

$$a_n = n(-1)^n$$

Example 4…

Determine whether the sequence a_n is increasing, decreasing, or not monotonic… is the sequence bounded?

$$a_n = \frac{n}{n^2+1}$$

Series & Sequences

Sequence vs. Series…

A ***"sequence"*** is a **list of numbers**. The order in which the numbers are listed is important, so for instance $\{1, 2, 3, 4, 5, ...\}$ is one sequence, and $\{2, 1, 4, 3, 6, 5, ...\}$ is an entirely different sequence!

A ***"series"*** is a **sum of numbers**. For example, $\{1 + 1/2 + 1/4 + 1/8 + 1/16 + ...\}$ is an example of a series. A series is composed of a sequence of terms that are added up. The order in which the terms appear is sometimes important.

Series

Consider the partial sums…

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

– In general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

– We write the infinite series as…

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Series

If the sequence $\{s_n\}$ is ***convergent*** and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series Σa_n is called ***convergent*** and we write…

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

– The number s is called the ***sum*** of the series.

– Otherwise, the series is called ***divergent***.

Geometric Series

An important example of an infinite series is the ***geometric series***…

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

$$= \frac{a}{1-r}, \quad a \neq 0$$

- The geometric series is ***convergent*** if $|r| < 1$.

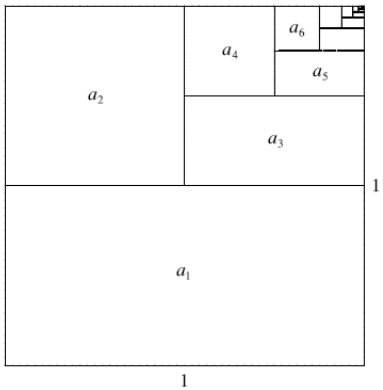
- The geometric series is ***divergent*** if $|r| \geq 1$.

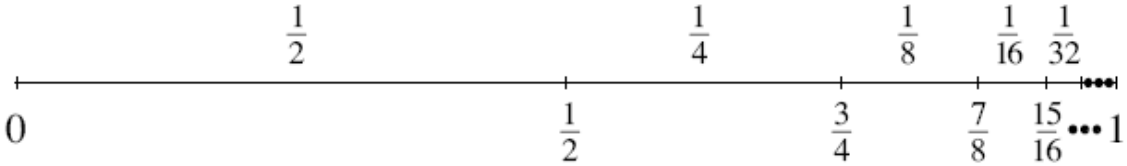
Geometric Series

Example 5…

Consider the geometric series below… is it convergent? If so, for what value s does the series converge?

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$





Geometric Series

Example 6…

How about the new geometric series below… is it convergent? If so, for what value s does the series converge?

$$\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$$

Harmonic Series

Another important infinite series is the ***harmonic series***…

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

$$s_1=1$$

$$s_2=1+\tfrac{1}{2}$$

$$s_4=1+\tfrac{1}{2}+\left(\tfrac{1}{3}+\tfrac{1}{4}\right)>1+\tfrac{1}{2}+\left(\tfrac{1}{4}+\tfrac{1}{4}\right)$$

$$=1+\tfrac{2}{2}$$

$$s_8=1+\tfrac{1}{2}+\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\left(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\right)$$

$$>1+\tfrac{1}{2}+\left(\tfrac{1}{4}+\tfrac{1}{4}\right)+\left(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}\right)$$

$$=1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}$$

$$=1+\tfrac{3}{2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\quad\Rightarrow\quad s_{2^n}>1+\frac{n}{2}\quad\Rightarrow\quad \text{divergent!}$$

Series Convergence/Divergence

If Σa_n is ***convergent***, then…

– The limit of the sequence $\{s_n\}$ is s, $\sum_{n=1}^{\infty} a_n = s$.

– The limit of the sequence $\{a_n\}$ is 0, $\lim_{n\to\infty} a_n = 0$.

Divergence Test…

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$,

then the series $\sum_{n=1}^{\infty} a_n$ is ***divergent***.

Example 7…

Check the convergence/divergence of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{k^n},\ k > 1 \qquad \sum_{n=1}^{\infty} \frac{4 \cdot 5^n - 5 \cdot 4^n}{6^n} \qquad \sum_{n=1}^{\infty} (-1)^n$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{n}{n+1}\right) \qquad \sum_{n=1}^{\infty} (-1)^{2n} \qquad \sum_{n=1}^{\infty} \left[\frac{5}{n(n+1)} - \left(-\frac{1}{2}\right)^n\right]$$

Answers:

$\sum_{n=1}^{\infty} \frac{1}{k^n}$, $k > 1$ is geometric and converges to $\frac{1}{k-1}$.

$\sum_{n=1}^{\infty} \frac{4 \cdot 5^n - 5 \cdot 4^n}{6^n}$ is a sum of two geometric series and converges to 10.

$\sum_{n=1}^{\infty} (-1)^n$ diverges by the Test for Divergence.

$\sum_{n=1}^{\infty} \sin\left(\frac{n}{n+1}\right)$ diverges by the Test for Divergence.

$\sum_{n=1}^{\infty} (-1)^{2n}$ diverges by the Test for Divergence.

$\sum_{n=1}^{\infty} \left[\frac{5}{n(n+1)} - \left(-\frac{1}{2}\right)^n\right]$ is the sum of a geometric series and 5 times the series from Example 6, and converges to $\frac{16}{3}$.