# MA153 Lesson 57

LESSON 57 - Block I - IV Review

12 December, 2008

# Outline

1. Admin
2. Block I - Vectors
3. Block II - Problem Solving with Partial Derivatives
   - Functions of Several Variables
   - Partial Derivatives
4. Multiple Integrals
   - Iterated Integrals
5. Vector Calculus

# Admin

1. Review

# Admin

1. Review
2. Next Week - TEE

# Admin

1. Review
2. Next Week - TEE
3. Homework 9 - Section 11.8 - 11.10 and Calc Labs Due Today 12 December

# Admin

4. Homework 9 a returned. You should feel comfortable with most of this material.

# Admin

5. MCM - February

# Admin

1. End of Course Survey on CIS due by 19 December.

# Admin

2. TEE INFO

# Admin

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   1. Time: 0735-1105, Tuesday, 16 December

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      1. B3 - BH304
      2. **C3 - BH322 - Change to previous**
      3. D3 - BH306

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3. Study Session Saturday Morning, 0830-1100, TH 342, LTC Outing.

# Three Dimensional Coordinate System

1. Distance Formula in Three Dimensions: The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2. Equation of a Sphere: An equation of a sphere with center $C(h, k, l)$ and radius $r$ is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

If the center is the origin O, then the equation is

$$x^2 + y^2 + z^2 = r^2$$

# Vectors

1. A **vector** has both magnitude and direction.
2. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude and the arrow points in the direction of the vector.
3. A vector is represented by a bold ($\mathbf{v}$) or an arrow above it ($\vec{v}$).
4. Suppose a particle or ball moves along a line segment from point $A$ to point $B$. The **displacement vector v**, has **initial point** $A$ (the tail) and **terminal point** $B$ (the tip) and we write this as $\mathbf{v} = AB$.
5. Another vector $\mathbf{u} = CD$ could have the same length and direction as $\mathbf{v}$ but be in a different position. In this case $\mathbf{u}$ and $\mathbf{v}$ are **equivalent** (or **equal**) and we can write $\mathbf{u} = \mathbf{v}$.
6. The **zero vector**, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

# Vector Addition

1. If a particle moves from A to B, displacement $\overrightarrow{AB}$. Then the particle moves from B to C, displacement $\overrightarrow{BC}$. The combined effect is that the particle has moved from A to C, displacement $\overrightarrow{AC}$, is called the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

2. The sum of $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.
3. An example mathematically. If the vector $\mathbf{u} = \langle 1, 2, 3 \rangle$ is added to $\mathbf{v} = \langle 2, 3, 4 \rangle$ the resulting vector is the addition of each of the components meaning add the $x$ components, the $y$ components, and the $z$ components to get $\mathbf{u} + \mathbf{v} = \langle 3, 5, 7 \rangle$.

# Scaler Multiplication

1. If $c$ is a scalar and $\mathbf{v}$ is a vector, then the scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same if $c > 0$ and opposite if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = 0$.

# Components - a review

1. Vectors have components

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

2. A vector from the origin $O$ to the point $P$ where $P(3, 2, 1)$ is called the position vector of the point $P$.
3. Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1 \rangle$$

# Magnitude or Length

1. The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by $|v|$ or $||v||$. We use the distance formula to compute the length.

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

2. The Magnitude is a number, not a vector!

# The Unit Vector!

1. A Unit Vector is a vector whose length is 1. For example $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are all unit vectors where $\mathbf{i} = \langle 1, 0, 0 \rangle$. If $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

# Dot Product

1. Can we multiply two vectors? Of course we can, we get the Dot Product.
2. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of **a** and **b** is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

3. So we multiply corresponding components and add. The result is a number not a vector. It is a real number known as a scaler.
4. The five properties of the Dot Product are given in your text on page 779

# Dot Product Angle

1. The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle $\theta$ between a and b**.
2. If $\theta$ is the angle between the vectors **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

3. If $\theta$ is the angle between the nonzero vectors **a** and **b**, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

4. Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

# Projections

1. Projections can be seen as the shadow of a vector onto another.
2. Scalar projection of **b** onto **a**: is a number $\text{comp}_a\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$
3. Vector projection of **b** onto **a**: is a vector $\text{proj}_a\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$
4. To find work $W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta$

# Cross Product

1. The **cross product** $\mathbf{a} \times \mathbf{b}$ is a vector and only works when the vector are three-dimensional.
2. The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular or orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
3. If $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

4. Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

5. The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

# Equation of a Line

1. $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$
2. $x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$
3. $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$
4. Line Segment from $\mathbf{r_0}$ to $\mathbf{r_1}$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r_0} + t\mathbf{r_1}$$

# Equation of a Plane

1. Equation of a plane is $\mathbf{n} \cdot \langle r - r_0 \rangle = 0$
   1. Where $r = (x, y, z)$ and $r_0 = (x_0, y_0, z_0)$ some given point on the plane.
   2. $\mathbf{n} = \langle a, b, c \rangle$ Which is a vector perpendicular to the plane.
      1. $\mathbf{n}$ can be given like find the plane perpendicular to the vector $\langle 1, 2, 3 \rangle$
      2. $\mathbf{n}$ can be derived by taking the cross product of two vectors in the plane like the vector $\mathbf{V}_1 = \langle 1, 2, 3 \rangle$ and $\mathbf{V}_2 = \langle 4, 5, 6 \rangle$. So $\mathbf{n} = \mathbf{V}_1 \times \mathbf{V}_2$ where we use mathematica to give us the Cross Product.
      3. $\mathbf{n}$ can be given by three points like $P_1 = (1, 2, 3), \quad P_2 = (4, 5, 6), \quad P_3 = (7, 8, 9)$ where we must make two vectors from the three points and then take their cross product. So $\overrightarrow{P_1 P_2} = P_2 - P_1 = (4, 5, 6) - (1, 2, 3) = \langle 4 - 1, 5 - 2, 6 - 3 \rangle = \langle 3, 3, 3 \rangle$ then we get $\mathbf{n} = \overrightarrow{P_1 P_2} \times \overrightarrow{P_1 P_3}$ which we do in mathematica again.

# Calculating the Equation of a Plane

1. Now we use the **n** we found and a point on the line $r_0 = (x_0, y_0, z_0)$ for example $r_0 = (1, 2, 3)$.
2. Again the equation is $\mathbf{n} \cdot \langle r - r_0 \rangle = 0$
3. So if we found $\mathbf{n} = \langle 8, 7, 6 \rangle$
4. We would write it like $\langle 8, 7, 6 \rangle \cdot \langle (x, y, z) - (1, 2, 3) \rangle = 0$
5. Simplified would be $\langle 8, 7, 6 \rangle \cdot \langle x - 1, y - 2, z - 3 \rangle = 0$
6. Taking the Dot Product would give $8(x - 1) + 7(y - 2) + 6(z - 3) = 0$
7. Simplifying again gives $8x + 7y + 6z - 40 = 0$ or $8x + 7y + 6z = 40$
8. Now if we want to know if a point is on the plane we put in the values for $x, y, z$ and if they equal 40 we are on the plane.

# Derivatives of Vector Functions

1. The derivative of a vector function is the derivative of its components.
2. The derivative describes the tangent of the given curve.
3. The derivative also gives the rate of change in each direction of the curve.
4. Review the Differential Rules on Page 826.

# Equation of a tangent line

The equation of the tangent line can be found using the derivative of the vector function.

1. Remembering that the equation of a line is $\overrightarrow{r}(t) = \overrightarrow{r}_0(t) + t\overrightarrow{v}$.
2. We can get $\overrightarrow{r}_0(t)$ from a curve. For example $\overrightarrow{r}(t) = \langle t, t^2, t^3 \rangle$, if we want to know what this looks like at $t = 1$ we would put this into our equation for $r$ and get $\overrightarrow{r}(1) = \langle 1, 1, 1 \rangle$.
3. We can get $\overrightarrow{v}$ by realizing that $\overrightarrow{r}'(t)$ is the slope of the tangent line of the curve or meaning $\overrightarrow{r}'(t) = \overrightarrow{v}$
4. In our example we would get $\overrightarrow{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ and at the point $t = 1$ we get $\overrightarrow{r}'(1) = \langle 1, 2, 3 \rangle$.
5. Now we substitute what we know into our equation of a line and get
   $\langle x, y, z \rangle = \langle 1, 1, 1 \rangle + t\langle 1, 2, 3 \rangle$
6. From here we can give either the vector or parametric equation.

# Integrating a Vector Function

1. Integrating a Vector Function is simply integrating its components.

# Arc-Length

1. If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a position vector in three dimensions
2. Then $\vec{r}'(t) = \{x'(t), y'(t), z'(t)\}$ is the velocity vector function or $\vec{v}(t)$
3. Then Speed $= |\vec{r}'(t)|$
   $= |\langle x'(t), y'(t), z'(t) \rangle| \ = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$
   $= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$
4. We know that Total Distance is : $D = R * T$ (constant rate or speed)
5. So $D =$speed$*T$
6. All we have left to do now is multiply by time
7. So $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$
8. Or $L = \int_a^b |\vec{r}'(t)| dt$

# Projectile Motion

1. $x(t) = x_0 + (v_0 \cos\alpha)t \qquad y(t) = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2$
2. Or $\mathbf{r}(t) = \langle x_0 + (v_0 \cos\alpha)t, \quad y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \rangle$
3. The velocity vector is the derivative of the position vector $\mathbf{v}(t) = \mathbf{r}'(t) = \langle v_0 \cos\alpha, \quad v_0 \sin\alpha - gt \rangle$
4. The acceleration vector is the derivative of the velocity vector $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle 0, \quad -g \rangle$
5. What is true at the maximum height of a projectile?
   1. The derivative of the y component $= 0$. So we take the derivative of $y(t)$ and set it equal to zero, $y'(t) = 0$. In our case $y(t) = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2$ so taking the derivative with respect to $t$ gives $y'(t) = 0 + \sin\alpha - gt$. Setting this equal to 0 and solving for $t$ gives $t = \dfrac{\sin\alpha}{g}$
   2. Now we have $t$ or time when the particle is at the highest point. If we substitute this $t$ back into our position vector we will get the $x$ and $y$ position at that time - giving us the highest $y$.

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   - Functions of Several Variables
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# Functions of Several Variables

1. Domain defines the values $x$ and $y$ can be that allow $f(x, y)$ to be defined.
2. Range defines the values that $f(x, y)$ can achieve.
3. For example find the domain and range of $f(x, y) = \sqrt{9 - x^2 - y^2}$:
    1. To be defined we cannot take the square root of a negative number so the domain looks like $D = \{(x, y)|9 - x^2 - y^2 \geq 0\}$ or simplified $D = \{(x, y)|x^2 + y^2 \leq 9\}$
    2. Now the range will describe what $f(x, y)$ can be. So the smallest $x^2 + y^2$ can be is zero and the greatest would be 9. This means that $f(x, y)$ can range from 0 to 3, or $R = \{z|0 \leq z \leq 3\}$ or $R = [0, 3]$
    3. ( means the range moves up to but not including the number next to it. [ means the range moves up to and includes the number next to it.

# Steps to find the Limit of a Function of Two Variables

1. Test the limit to see if the number is defined. For example $\lim_{(x,y)\to(1,1)} \sqrt{6-x^2-y^2}$. The limit would be $\sqrt{6-1-1}=2$.
2. If the limit gives you an undefined function then test the function on a few different paths - if they are not all equal there is no limit. For example if $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz^2}{x^2+y^2+z^2}$ We try the line along the $x$ axis which is $f(x,0,0)=\frac{0}{x^2}=0$. Now we try at a 45 degree angle in the x y plane $f(x,x,0)=\frac{x^2}{2x^2}=\frac{1}{2}$ So we get different limits Therefore no limit exists.

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# Partial Derivatives

1. Notations For Partial Derivatives. If $z = f(x, y)$ we write:
   1. $f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = \mathbf{D}_1 f = \mathbf{D}_x f$
   2. $f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = \mathbf{D}_2 f = \mathbf{D}_y f$
2. How to find a partial derivative of a function like $f(x, y) = x^2 + y^2 - 4xy + 5$
   1. To find $f_x$, regard $y$ as a constant and differentiate $f(x, y)$ with respect to $x$. So $f_x(x, y) = 2x - 4y$
   2. To find $f_y$, regard $x$ as a constant and differentiate $f(x, y)$ with respect to $y$. So $f_y(x, y) = 2y - 4x$
   3. To find the second or higher derivatives keep taking the partial derivative of the derivative. Or for a mixed partial take the partial of the function with respect to the inside variable then the outside. So $f_{xx}(x, y) = 2, \quad f_{yy} = 2, \quad f_{xy} = -4$
3. Clairaut's Theorem: Suppose $f$ is defined on a disk D that contains the point (a,b). If the functions $f_{xy}$ and $f_{yx}$, are both continuous on D, Then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

4. Laplace's Equation: Harmonic functions will satisfy the following equation, known as the Laplace Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

# Tangent Planes

1. Equation of a Tangent Plane:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

2. Make sure you take the partial derivatives and evaluate them at the point of interest $(x_0, y_0)$ then multiply those slopes times $x - x_0$. You should get an equation looking like $ax + by + cz = d$. Where a, b, and c are slopes and d is similar to your intercept in the equation of a line.

# Linear Approximations

1. Linear Approximation is using the equation of a plane to approximate a point some distance from your original point.
2. $$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
3. So you are approximating what $f$ is at the point $(x, y)$ based on a plane at the point $(a, b)$. Your approximation will me more accurate the closer you are to the point $(a, b)$.

# Total Differential - Can be found using one of the following.

1. $$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

2. $$dz = f_x(x, y)(x - a) + f_y(x, y)(y - b)$$

# The Chain Rule - Similar to what we remember from single variable derivatives.

1. Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of $t$. Then $z$ is a differentiable function of $t$ and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2. Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of $s$ and $t$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

# Gradient

1. The gradient $\nabla f(x, y, z)$ is a vector in the direction of the steepest slope. For example if $f(x, y, z) = x^4 + xy + z$ then $\nabla f(x, y, z) = \langle 4x^3 + y, x, 1 \rangle$
2. Rules for Finding the Gradient
   1. Get the gradient by taking the partial derivatives and put them in vector notation! To get the gradient at a point sub in the values of (x,y,z) into your gradient. For example if $f(x, y, z) = x^4 + xy + z$ and you want $\nabla f(2, 2, 5) = \langle 4(2)^3 + (2), (2), 1 \rangle = \langle 34, 2, 1 \rangle$.

# Directional Derivative

1. The Directional Derivative is a Number/Scaler giving the slope/rate of change in a specific direction.
2. Rules for Finding the Directional Derivative
   1. Get the gradient by taking the partial derivatives and put them in vector notation! Then get the gradient at the point you are at by subbing in the values of (x,y,z)
   2. Get the direction you want into vector notation
   3. Get the vector into a unit vector - by dividing the vector by its magnitude
   4. Take the dot product of the unit vector and the gradient - $D_u \cdot f_x f_y = D_u ff$

# Tangent Planes to Level Surfaces

1. $$F_x(x_0, y_0, z_0)(x-x_0)+F_y(x_0, y_0, z_0)(y-y_0)+F_z(x_0, y_0, z_0)(z-z_0) =$$

# Steps to find the Extrema of Functions of Two Variables

1. Find the critical points. If our function was $f(x,y) = x^2 + y^2 + x^2y + 4$
   1. find the partial derivatives of $f(x,y)$ $\quad f_x = 2x + 2xy, \quad f_y = 2y + x^2$
   2. Set the partial derivatives equal to zero and solve the system of two equations and two unknowns.
   $$2x + 2xy = 0, \quad 2y + x^2 = 0$$
   3. Solve one of the equations for $x$ or $y$ So lets solve the second equation for $y$ where we get $y = \frac{-x^2}{2}$.
   4. Sub your $x$ or $y$ into the other equation and solve for the variable. In our case we put our $y$ into the first equation which gives $2x + 2x(\frac{-x^2}{2}) = 0 \quad \Rightarrow \quad 2x - x^3 = 0 \quad \Rightarrow 2x = x^3$
   Now we need the roots of this equation. The obvious one is when $x = 0$. So we factor an $x$ out of the equation and are left with $2 = x^2 \quad \Rightarrow \quad x = \pm\sqrt{2}$. So our roots are $x = -\sqrt{2}, 0, \sqrt{2}$.
   5. Now we want to know what our critical points are so we sub our values of $x$ back into one of our equations. We get the following critical points $(-\sqrt{2}, -1), (0,0), (\sqrt{2}, -1)$.
2. Classify our critical points using the second derivative test.
   1. The discriminant $D = f_{xx}f_{yy} - (f_{xy})^2$
   2. So we find the second partial derivatives and the mixed partial. $f_{xx} = 2 + 2y, \quad f_{yy} = 2, \quad f_{xy} = 2x$. So $D(x,y) = (2y+2)(2) - (2x)^2$.
   3. We need to test $D(x,y)$ at each of our critical points.
      1. If $D(x,y) > 0$ and $f_{xx}(x,y) > 0$ we have a minimum.
      2. If $D(x,y) > 0$ and $f_{xx}(x,y) < 0$ we have a maximum.
      3. If $D(x,y) < 0$ we have a possible saddle point.
   4. Testing each critical point we get:
   $D(-\sqrt{2}, -1) = (2(-1)+2)(2) - 2*(-\sqrt{2})^2 \quad \Rightarrow \quad D(-\sqrt{2}, -1) = -8$ Therefore $(-\sqrt{2}, -1)$ is probably a saddle point.
   $D(0,0) = 2*(0)(2) - 2*(0)^2 \quad \Rightarrow \quad D(0,0) = 0$ Therefore $(0,0)$ is not definable by the second derivative test.
   $D(\sqrt{2}, -1) = (2(-1)+2)(2) - 2*(\sqrt{2})^2 \quad \Rightarrow \quad D(\sqrt{2}, -1) = -8$ Therefore $(\sqrt{2}, -1)$ is probably a saddle point.
3. Next we need to test the boundary conditions. If our boundary was $D = \{(x,y) | -1 \le x \le 1, -1 \le y \le 1\}$.
   1. We can make a box with equations of the lines as the four sides of the box. Along this box we have the bottom line where a point on $f(x,y)$ can be described as
   $f(x,-1) = x^2 + (-1)^2 + x^2*(-1) + 4 \Rightarrow f(x,-1) = x^2 - x^2 + 5$ or $f(x,-1) = 5$
   Similarly we get
   $f(x,1) = 2x^2 + 5, \quad f(-1,y) = y^2 + y + 5, \quad f(1,y) = y^2 + y + 5$
   2. Checking the end points of $x$ and $y$ we get
   $f(1,1) = 7, \quad f(-1,-1) = 5 \quad f(-1,1) = 7 \quad f(1,-1) = 5$
   3. We should also take the partial derivatives of the boundaries and set them equal to zero to get the max or min on the boundary, we treat this like max and min in two dimensions where we already tested the end points above.
      1. On the bottom line we get $f_x(x,-1) = 2x - 2x = 0 \Rightarrow$ No mater what $x$ is we get $f(x,-1) = 5$
      2. On the right line we get $f_y(1,y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ So $f(1, -\frac{1}{2}) = \frac{19}{4}$
      3. On the top line we get $f_x(x,1) = 4x = 0 \Rightarrow x = 0$ So $f(0,1) = 5$
      4. On the left line we get $f_y(-1,y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ So $f(-1, -\frac{1}{2}) = \frac{19}{4}$
4. We need to check our critical points. We realize the only critical point in our domain is $(0,0)$ which $f(0,0) = 4$
5. We compare all of our points and see that our Global Minimum is at $f(0,0) = 4$ and our Global Maximum is at $f(1,1) = 7$ and $f(-1,1) = 7$.

# Optimization

These steps are similar to many other models we have done. If we want a box without a top to have the maximum volume while using only 12 inches of cardboard we do the following:

1. Find the Objective Equation - The equation we want to maximize or minimize. $V = xyz$.
2. Find the Constraint Equation - Our limitations. Our Surface Area has to satisfy $2xz + 2yz + xy = 12$.
3. Use the Constraint Equation to reduce our Objective Equation to an equation of only two variables. $z = \frac{12-xy}{2(x+y)}$
4. Find the Critical Points of the new Objective Equation by taking the partial derivatives and setting them equal to zero then solving the system of equations. In this case using Mathematica can help. Our only positive critical point is $(2, 2)$
5. Use the Second Derivative Test to determine the classification of the critical points. Again Mathematica can make finding the Determinant easier. In our case we get $D > 0$ and $f_{xx} < 0$ proving we have a maximum.
6. Test the Critical Points in the function to determine the Max or Min you are seeking. In our case we get $f(2, 2) = 4$

# LaGrange Multipliers

1. Find the Objective Equation - The equation we want to maximize or minimize.
   $f(x, y) = 4x + 6y$.
2. Find the Constraint Equation - Our limitations. Our Surface Area has to satisfy
   $x^2 + y^2 = 13$.
3. Find the gradient of both the Objective and Constraint Equation $\nabla Ob, \nabla Co$ by taking the partial derivatives and putting them in vector notation.
   $\nabla Ob(x, y) = \langle 4, 6 \rangle$,
   $\nabla Co(x, y) = \langle 2x, 2y \rangle$
4. Set $\nabla Ob(x, y) = \lambda \nabla Co(x, y)$.
   $\langle 4, 6 \rangle = \lambda \langle 2x, 2y \rangle$
5. Solve the equations $\nabla Ob(x, y) = \lambda \nabla Co(x, y)$ and $Co(x, y) = k$ for all values of $x, y, \lambda$ by solving the system of three equations and three unknowns. In this case I would solve the first equation for $\lambda$ giving $\lambda = \frac{2}{x}$. I would then sub $\lambda$ into the second equation. $6 = 2y(\frac{2}{x})$. I would solve the second equation for $x$ giving $x = \frac{2y}{3}$. Finally I would sub the $x$ into the final equation $(\frac{2y}{3})^2 + y^2 = 13$ This would give $\frac{4y^2}{9} + y^2 = 13$. So $y = \pm 3$ then $x = \pm 2$.
6. Evaluate $Ob$ at all the points $(x, y)$. The largest of these values is the max; the smallest is the minimum. In our case $Ob(2, 3) = 26$ and $Ob(-2, -3) = -26$

# Outline

1. Admin
2. Block I - Vectors
3. Block II - Problem Solving with Partial Derivatives
   - Functions of Several Variables
   - Partial Derivatives
4. Multiple Integrals
   - Iterated Integrals
5. Vector Calculus

# Iterated Integrals over Rectangular Regions

1. Create a 3D plot of the surface over the region
2. Draw and label the region
3. Choose the order of integration
4. Compute/Evaluate the "inside" integral
5. Compute/Evaluate the "outside" integral
6. Does the answer make sense?

# Iterated Integrals over General Regions

1. Sketch the region
2. Label the boundaries
3. Solve for and label the intersections
4. Compute/Evaluate the "inside" integral
5. Compute/Evaluate the "outside" integral
6. If possible, check answer in Mathematica

# Polar Coordinates

1. $r^2 = x^2 + y^2$
2. $x = r\cos\theta$
3. $y = r\sin\theta$
4. $\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$

# Polar Coordinates Algorithm

1. If possible create a 3D plot of the surface over the region
   - Study diagram to determine if this is positive, negative, or mixed.
   - Establish a very rough idea of the volume of the space
2. Draw and Label the Region
3. Determine the limits of integration
   - Max and Min radial limits
   - Max and Min angular limits
4. Convert the integrand to an equivalent polar expression
5. Set up the iterated integral
6. Compute/Evaluate the inside integral
7. Compute/Evaluate the outside integral
8. Look back, does it make sense?

# Center of Mass

1. Draw and label the region and look at the density function $\rho(x, y)$
2. Compute Mass where $m = \iint\limits_D \rho(x, y)dA$
3. Compute the Moments for x and y where

$$M_x = \iint\limits_D y\rho(x, y)dA \quad M_y = \iint\limits_D x\rho(x, y)dA$$

4. Finally compute $\overline{x}$ and $\overline{y}$ which are the exact coordinates of the center of mass.

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_D x\rho(x, y)dA, \quad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint\limits_D y\rho(x, y)dA$$

# Triple Integrals

1. Cubic Regions -
$$\int\int\int_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dx\ dy\ dz$$
2. General Regions - $\int\int\int_E f(x,y,z)dV =$
$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\ dy\ dx$$

# Triple Integrals Cylindrical Coordinates

1. $x = r\cos\theta \quad y = r\sin\theta \quad z = z$
2. $r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad z = z$
3. $\int\int\int_E f(x,y,z)dV =$
$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r \; dz \; dr \; d\theta$$

# Cylindrical Algorithm

1. If possible create a 3D plot of the region then label it.
2. Determine the limits of integration
   - Max and Min $z$ values. Where
     $z_1 = u_1(x, y) = u_1(r\cos\theta, r\sin\theta)$
     $z_2 = u_2(x, y) = u_2(r\cos\theta, r\sin\theta)$
   - Max and Min radial limits.
     Where $r_1 = h_1(\theta), \quad r_2 = h_2(\theta)$
   - Max and Min angular limits
3. Convert the integrand to an equivalent cylindrical expression
4. Set up the triple integral - where $dV = r\ dz\ dr\ d\theta$
5. Compute/Evaluate the integrals from inside out.
6. Look back, does it make sense?

# Triple Integrals Spherical Coordinates

1. $x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$
2. The distance formula gives us:

$$\rho^2 = x^2 + y^2 + z^2$$

3. $\int\int\int_E f(x,y,z)dV =$
$\int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \; d\rho \; d\theta \; d\phi$
4. Where E is a spherical wedge given by:

$$E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

# Spherical Coordinates Algorithm

1. If possible create a 3D plot of the region then label it.
2. Determine the limits of integration
   - Max and Min $\rho$ values. Similar to the radius.
   - Max and Min $\theta$ angle values. The angle of rotation in the $xy$ plane.
   - Max and Min $\phi$ angle values. The angle of rotation from the positive $z$ axis.
3. Convert the integrand to an equivalent spherical expression
4. Set up the triple integral - where $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$
5. Compute/Evaluate the integrals from inside out.
6. Look back, does it make sense?

# Vector Fields

1. $F(x,y) = P(x,y)\,\widehat{i} + Q(x,y)\,\widehat{j} = \langle P(x,y), Q(x,y)\rangle$
2. $F(x,y,z) = P(x,y,z)\,\widehat{i} + Q(x,y,z)\,\widehat{j} + R(x,y,z)\,\widehat{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$
3. Gradient Fields
   $\nabla f(x,y,z) = f_x(x,y,z)\,\widehat{i} + f_y(x,y,z)\,\widehat{j} + f_z(x,y,z)\,\widehat{k}$

# Line Integrals 2 dimensions

1. $\int_C f(x,y)ds$
2. $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$
3. Similar arguments help us define the line integral as the following:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

4. What if we want to parameterize with something other than $t$?
5. Do a similar process, but instead:

$$\int_C f(x,y)ds = \int_a^b f(x,y(x))\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}dx$$

6. We may also have to remember the equation of a line segment:

$$\overrightarrow{r}(t) = (1-t)\overrightarrow{r_0} + t\overrightarrow{r_1}$$

# Line Integrals 3 dimensions

1. Start with a plane curve $C$ given by the parametric equations $x = x(t) \quad y = y(t) \quad z = z(t) \quad a \leq t \leq b$
2. $\int_C f(x,y,z)ds =$

$$\int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

3. $$\int_a^b f(\overrightarrow{r}(t))|\overrightarrow{r}'(t)|dt$$
4. If we talk about work we remember $W = \overrightarrow{F} \cdot \overrightarrow{D}$
5. We suppose that $\overrightarrow{D} = \overrightarrow{PQ}$ the displacement vector, and that $\overrightarrow{F} = P\widehat{i} + Q\widehat{j} + R\widehat{k}$.
6. Now work can be described by:

$$W = \int_C \overrightarrow{F}(x,y,z) \cdot \overrightarrow{T}(x,y,z)ds = \int_C \overrightarrow{F} \cdot \overrightarrow{T}ds$$

7. Also

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_a^b \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)dt = \int_C \overrightarrow{F} \cdot \overrightarrow{T}ds$$

8. So

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C Pdx + Qdy + Rdz \quad \text{where} \quad \overrightarrow{F} = P\widehat{i} + Q\widehat{h} + R\widehat{k}$$

# FTC and the line integral

1. $\int_C \nabla f \cdot d\vec{r} = f(x_2, y_2) - f(x_1, y_1)$
2. $\int_C \vec{F} \cdot d\vec{r}$ Path Independent if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$
3. $\vec{F}$ is a conservative vector field on $D$ if $\int_C \vec{F} \cdot d\vec{r}$ is path independent.
4. If $\vec{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j}$ is a conservative field then:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

5. Page 1050 - Theorem 6

# Green's Theorem

1. $$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
2. $$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx$$
3. Holes and Green's Theorem

# Curl

1. If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field in three dimensions and the partial derivatives of $P$, $Q$, and $R$ all exist, then the curl of $\vec{F}$ is the vector field in three dimensions defined by:
2. $$\text{curl}\,\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

# More about Curl

1. The Curl of a gradient field is **0**. This is because the curl is the cross product of the gradient with the gradient.

$$\text{curl}(\nabla f) = \nabla \times (\nabla f)$$

2. So we have an extension of what we learned in Green's Theorem. If $\vec{F}$ is a vector field defined on all reals in three dimensions whose component functions have continuous partial derivatives and curl $\vec{F} = \mathbf{0}$, then $\vec{F}$ is a conservative vector field.

# Divergence

1. If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field in three dimensions and the partial derivatives of $P$, $Q$, and $R$ all exist, then the **divergence of F** is the function of three variables defined by:
2. $$\text{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$
3. Similar to the dot product where we get a number, divergence is:
$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$$

I

# More about Divergence

1. If $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is a vector field in the real three dimensions and $P$, $Q$, and $R$ have continuous second-order partial derivatives, then
2. $$\text{div curl}\vec{F} = 0$$

# Physical description of Divergence

1. Fluid flow again is one of the best examples. Divergence can be seen as the net rate of change of the mass of the fluid flowing from the point P per unit volume. Or the divergence measures the tendency of the fluid to diverge from point P.
2. If div $\vec{F} = 0$ at a point P, the the fluid is said to be incompressible.

# Laplace Operator

1. If we take the divergence of the gradient vector of a function of three variables, we get:
2. $$\text{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
3. This is often called the Laplace operator and looks like:
4. $$\nabla^2 = \nabla \cdot \nabla$$

# Vector Form of Green's Theorem

1. $$\oint_C \vec{F} \cdot d\vec{r} = \oint_C Pdx + Qdy$$
2. If we regard $\vec{F}$ as a vector field in three dimensions with the third component 0, we get:
3. $$\text{curl}\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

# Vector Form of Green's Theorem

1. So:

$$(\text{curl}\vec{F}) \cdot \hat{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \hat{k} \cdot \hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

2. We can rewrite the Green's Theorem in vector form:
3. $$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl}\vec{F}) \cdot \hat{k} dA$$

# Second Vector Form of Green's Theorem

1. $$\oint_C \vec{F} \cdot \vec{n} ds = \iint_D \text{div}\vec{F}(x, y) dA$$

# Parametric Surface

1. $$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

# Tangent Planes

1. $$\vec{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\hat{i} + \frac{\partial y}{\partial v}(u_0, v_0)\hat{j} + \frac{\partial z}{\partial v}(u_0, v_0)\hat{k}$$
$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\hat{i} + \frac{\partial y}{\partial u}(u_0, v_0)\hat{j} + \frac{\partial z}{\partial u}(u_0, v_0)\hat{k}$$
2. $$\vec{n} = \vec{r}_u \times \vec{r}_v$$
3. $$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

# Surface Area

1. $$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| dA$$

2. $$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

# Surface Integrals

1. $$\iint_S f(x,y,z)dS = \iint_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|dA$$

2. $$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dA$$

# Surface Integrals of Vector Fields

1. $$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS$$
2. This is called the **flux** of $\vec{F}$ across $S$.
3. $$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$
4. $$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \left( -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$