Background

As everyone knows Wiley Coyote and the Road Runner have had it out for one another from the beginning of time. Wiley Coyote has finally had enough and wants to finish the Road Runner once and for all. He wants to have a full proof plan backed up by math to ensure that his time he will nab the Road Runner and be able to make his Aunt Edna's famous Road Runner Soufflé consisting of Road Runner, eggs, and cream sauce. Naturally, since the Road Runner is such a delicacy Wiley wants to maximize Soufflé with a budget of 35 dollars and would like to know how many servings he can make. Unfortunately Wiley Coyote is not the greatest when it comes to modeling surfaces and using projectile motion to solve his problem. Initially Wiley turned to the ACME Surveying and Parametrization Services, Ltd. but their services proved to be a little too expensive. However, ACME was able to devise the initial plan and find the surface modeling the hill along with the graph corresponding to the road that Road Runner will be traveling. The purpose of this report is to take the information given by ACME and Wiley Coyote to calculate: the direction in which he should aim the launcher, the initial speed at which he should launch the pod, how many seconds he should set the timer for so that the pod will open just as it reaches the Road Runner, and the maximum servings of Aunt Edna's Road Runner Soufflé. In the report we calculate the direction, speed, time according to the gravity on October 15^{th}, October 16^{th}, and October 17^{th} in case something goes wrong with Wiley's plan and the plan is not executed on the 16^{th}.

Facts

Wiley Coyote provides the facts that influence the solution of the problem. Wiley states that the Road Runner will come speeding down the road at 50 m/s. The hill where the plan will take place can be modeled by the surface

$$z = 400 - \frac{x^2}{400} - \frac{y^2}{400}$$

The road that the Road Runner will speed down corresponds to the graph of

$$p(u) = \left(u, u\sin\left(\frac{u}{10}\right), 400 - \frac{u^2}{400} - \frac{u^2 \sin(\frac{u}{10})^2}{400}\right)$$

Wiley will be hiding in a level area at the point (50, 30, 783/2) and he wants to capture the Road Runner on the road at the point (150, 97, 320). Wiley could only afford the ACME Exploding-Net Pod Launcher that will only launch the net at an angle of 45 degrees to the horizontal. On Thursday, October 16^{th} gravity will be pulling south at 3 m/s^2, west at 2 m/s^2, and down at 9.8 m/s^2. In case something goes wrong we need the values for the 15^{th} and 17^{th}. On the 15^{th} gravity will be pulling south at 2 m/s^2, west at 1 m/s^2, and down at 9.8 m/s^2. On the 17^{th} gravity will be pulling west at 3 m/s^2, and down at 9.8 m/s^2. In terms of maximizing the Soufflé Wiley states that eggs cost \$2 per dozen and cream sauce costs \$3 per jar and he has a budget of \$35. Also every x dozens eggs and y jars of cream sauce make $6x^{2/3}y^{1/2}$ servings.

Assumptions

In order to solve for the direction, initial speed, time setting on the launcher and maximize the amount of Aunt Edna's Road Runner Soufflé, it is necessary to make following assumptions and focus in only on the variables we are concerned with.

- There is no wind which will affect the flight and trajectory of the Exploding-Net Pod.
- The Road Runner will run at exactly 50 m/s even when making turns along the road.
- The hill and the road can be perfectly modeled by the given equations.
- The Road Runner will not speed up or slow down when he sees the net on target to hit him.
- The ratio of ingredients in the maximization of the Soufflé is not important to Wiley.
- We will account for only whole servings when maximizing Aunt Edna's famous Road Runner Soufflé.

Analysis

The given equation

$$z = 400 - \frac{x^2}{400} - \frac{y^2}{400} \tag{1}$$

models the surface which represents the hill where Wiley Coyote plans to capture the Road Runner. The road that the Road Runner will be running is located on the hill and can be represented by the graph of

$$p(u) = \left(u, u sin\left(\frac{u}{10}\right), 400 - \frac{u^2}{400} - \frac{u^2 \sin(\frac{u}{10})^2}{400} \right) \tag{2}$$

Wiley Coyote is hiding in a level area at the point (50, 30, 783/2) and plans to capture the Road Runner at the point (150, 97, 320). The first step to solving Wiley's problem was to create a graph of the surface, and the road, marked with the hiding point and the capture point.

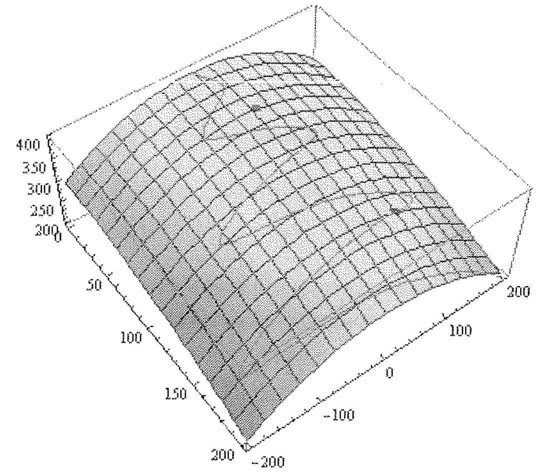

Figure 1: Plot of the surface representing the hill, graph representing the road, Wiley Coyote's hiding point, and the point where the Road Runner will be captured (Enlarged figure can be found in Appendix A)

In this graphic the hill is represented by the orange surface, the road is the single black line, Wiley's hiding point at (50, 30, 783/2) is the blue point, and the capture point on the road at (150, 97, 320) is the red point. Since Wiley Coyote could only afford the ACME Exploding-Net Pod Launcher that launches the pod at an angle of 45 degrees to the horizontal, we can take into account for gravity and derive the projectile motion equations for this situation. The projectile motion equation states

$$v = v_0 + gt \tag{3}$$

Wiley states that there are three components to gravity so we need to derive a velocity vector with a component in the x, y, and z directions or it can be thought of as having a component of gravity acting west to east, south to north, and downwards. Since the Exploding-Net Pod Launcher can only launch at an angle of 45 degrees to the horizontal the initial velocity can be described by

$$v_0 = \frac{v_0}{\sqrt{2}} \tag{4}$$

Since sine and cosine of 45 degrees are both equal to ($1/\sqrt{2}$). Wiley desires to find not only the initial launch velocity, and the time to set the net for, but also the direction θ where he should point the launcher. In order to bring in this component of direction the x, and y components of velocity must be multiplied cosine θ and sine θ respectively. This will derive the equations

$$v_{0x} = \frac{v_0}{\sqrt{2}} cos\theta \tag{5}$$

$$v_{0y} = \frac{v_0}{\sqrt{2}} sin\theta \tag{6}$$

By combining equations 3, 4, 5, and 6 the velocity vector with x, y, and z components can be derived.

$$v(t) = \{\frac{v_0}{\sqrt{2}} coso + gt, \frac{v_0}{\sqrt{2}} sino + gt, \frac{v_0}{\sqrt{2}} - gt\} \tag{7}$$

Where g is the x, y and z component of gravity respectively as specified by Wiley Coyote for each particular day. Once the velocity vector is known calculus can be used to find the distance from Wiley's hiding point to the point at which the Road Runner will be captured. In order to find position take the definite integral of the velocity with respect to time

$$d(t) = \int v(t)dt \tag{8}$$

This gives us the position from the origin with respect to time but in order to solve the problem it is crucial to account for the fact that the desired distance is from Wiley's hiding point to the point where Road Runner will be captured. In order to set this up we need to add the position to Wiley's hiding point (this takes into account the fact that Wiley is not at the origin) and set it equal to the point where the Road Runner will be captured.

$$d(t) + \left(50, 30, \frac{783}{2}\right) = (150, 97, 320) \tag{9}$$

This process using equations 7, 8, and 9 can be repeated for the values of gravity the day before and the day after. Using a program such as Mathamatica the initial velocity, time setting on the net, and the direction of launch can be found.

Wiley also needs to know where to set a marker ahead of time in order to have a visual reference prompting the launch of the net. This means that the distance the Road Runner travels in t seconds is equal the length of the road from the marker to the capture point at (150, 97, 320). Wiley says that the Road Runner travels at 50 m/s so setting $50t$ equal to the arc length from u to 150 and solving for u (the time t that Wiley should set the net for was found using equation 9) and then substituting back into equation 2 will result in the point at which Wiley should set the marker. The arc length of the road from u to 150 can be expressed by the integral of the square root of the derivative of the road function , $p'(u)$ or the derivative of equation 2, squared. It is probably easier to understand with the expression

$$arc\ length = \int_u^{150} \sqrt{p'(u) * p'(u)}\, du \tag{10}$$

Then setting this arc length equal to the distance traveled by the Road Runner in t seconds

$$50t = \int_u^{150} \sqrt{p'(u) * p'(u)}\, du \tag{11}$$

Using guess and check it is possible to find u and substitute it into equation 2 to find the point where Wiley should place the marker.

The final task Wiley asks for is the maximization of Aunt Edna's Famous Road Runner Soufflé. The given equation

$$6x^{2/3}y^{1/2} \tag{12}$$

describes the number of servings of Road Runner Soufflé that can be produced where x is the number of dozen eggs and y is the number of jars of cream sauce. The constraint of this equation is that the total cost of the ingredients must not exceed thirty-five dollars, with a dozen eggs costing two dollars and a jar of crème sauce costing three dollars. This equation can be shown as

$$2x + 3y \leq 35 \tag{13}$$

Essentially by taking the partial derivative of the equation with respect to both x and y we can set the gradient equal to zero and use the constraints identified in equation 13 to find the critical values of equation 12 . Then we can check using the second derivative test to ensure that this value is a maximum. The partial derivatives of equation 12 are

$$f_x(x, y) = 4x^{-1/3}y^{1/2} \tag{14}$$

and

$$f_y(x, y) = 3x^{2/3}y^{-1/2} \tag{15}$$

These can then be used along with the constraints to maximize the number of servings of Aunt Edna's Famous Road Runner Soufflé. It is possible to use the Mathamatica maximize function and bypass all of the partial derivatives and solving but essentially this is what Mathamatica is doing to maximize equation 12 under the constraints of equation 13.

Results

On Thursday October 16th Wiley Coyote states that gravity will be pulling south at 3 m/s^2, west at 2 m/s^2, and down at 9.8 m/s^2. This can be interpreted as -2 m/s^2 in the direction of the x-axis, -3 m/s^2 in the direction of the y-axis, and -9.8 m/s^2 in the direction of the z-axis. Taking the values for gravity and substituting them into equation 7 results in

$$v(t) = \{\frac{v_0}{\sqrt{2}} coso - 2t, \frac{v_0}{\sqrt{2}} sino - 3t, \frac{v_0}{\sqrt{2}} - 9.8t\}$$

In order to find the position we want to integrate $v(t)$ with respect to t

$$d(t) = \int v(t)\, dt = \{t \frac{v_0}{\sqrt{2}} coso - t^2, t\frac{v_0}{\sqrt{2}} sino - \frac{3t^2}{2}, t\frac{v_0}{\sqrt{2}} - 4.9t^2\}$$

Using equation 9 the position $d(t)$ plus the Wiley's hiding point can be set equal to the point the Road Runner will be captured.

$$\left\{t\frac{v_0}{\sqrt{2}}coso - t^2, t\frac{v_0}{\sqrt{2}}sino - \frac{3t^2}{2}, t\frac{v_0}{\sqrt{2}} - 4.9t^2\right\} + \left(50, 30, \frac{783}{2}\right) = (150, 97, 320)$$

Using Mathamatica to solve this equation for o (direction to aim launcher), v_0 (initial launch velocity in m/s^2), and t (number of seconds the timer should be set for) we found that o = .7768707 radians, v_0 = 40.9168 m/s^2, and t = 7.78012 seconds when gravity is equal to -2 m/s^2,-3 m/s^2, and -9.8 m/s^2 in the direction x, y, and z axis respectively. In order to make sense of these results we need to convert ө from radians to degrees: multiplying ө by ($180/\pi$) yields ө = 44.51 degrees in the x-y plane. Supporting work for these calculations can be found in Appendix B.

On Wednesday October 15th Wiley Coyote states that gravity will be pulling south at 2 m/s^2, west at 1 m/s^2, and down at 9.8 m/s^2. This can be interpreted as -1 m/s^2 in the direction of the x-axis, -2 m/s^2 in the direction of the y-axis, and -9.8 m/s^2 in the direction of the z-axis. Taking the values for gravity and substituting them into equation 7 results in

$$v(t) = \{\frac{v_0}{\sqrt{2}}coso - t, \frac{v_0}{\sqrt{2}}sino - 2t, \frac{v_0}{\sqrt{2}} - 9.8t\}$$

In order to find the distance we want to integrate $v(t)$ with respect to t

$$d(t) = \int v(t)\,dt = \left\{t\frac{v_0}{\sqrt{2}}coso - \frac{t^2}{2}, t\frac{v_0}{\sqrt{2}}sino - t^2, t\frac{v_0}{\sqrt{2}} - 4.9t^2\right\}$$

Using equation 9 the distance $d(t)$ plus the Wiley's hiding point can be set equal to the point the Road Runner will be captured.

$$\left\{t\frac{v_0}{\sqrt{2}}coso - \frac{t^2}{2}, t\frac{v_0}{\sqrt{2}}sino - t^2, t\frac{v_0}{\sqrt{2}} - 4.9t^2\right\} + \left(50, 30, \frac{783}{2}\right) = (150, 97, 320)$$

Using Mathamatica to solve this equation for o (direction to aim launcher), v_0 (initial launch velocity in m/s^2), and t (number of seconds the timer should be set for) we found that o = .751001 radians, v_0 = 34.3226 m/s^2, and t = 7.02896 seconds when gravity is equal to -1 m/s^2,-2 m/s^2, and -9.8 m/s^2 in the direction x, y, and z axis respectively. In order to make sense of these results we need to convert ө from radians to degrees: multiplying ө by ($180/\pi$) yields ө = 43.03 degrees in the x-y plane. Supporting work for these calculations can be found in Appendix C.

On Friday October 17th Wiley Coyote states that gravity will be pulling west at 3 m/s^2, and down at 9.8 m/s^2. This can be interpreted as -3 m/s^2 in the direction of the x-axis, and -9.8 m/s^2 in the direction of the z-axis. Taking the values for gravity and substituting them into equation 7 results in

$$v(t) = \{\frac{v_0}{\sqrt{2}}coso - 3t, \frac{v_0}{\sqrt{2}}sino, \frac{v_0}{\sqrt{2}} - 9.8t\}$$

[redacted]

In order to find the position we want to integrate $v(t)$ with respect to t

$$d(t) = \int v(t)\,dt = \{t\frac{v_0}{\sqrt{2}}coso - \frac{3t^2}{2}, t\frac{v_0}{\sqrt{2}}sin, t\frac{v_0}{\sqrt{2}} - 4.9t^2\}$$

Using equation 9 the position $d(t)$ plus the Wiley's hiding point can be set equal to the point the Road Runner will be captured.

$$\left\{t\frac{v_0}{\sqrt{2}}coso - \frac{3t^2}{2}, t\frac{v_0}{\sqrt{2}}sino, t\frac{v_0}{\sqrt{2}} - 4.9t^2\right\} + \left(50, 30, \frac{783}{2}\right) = (150, 97, 320)$$

Using Mathamatica to solve this equation for o (direction to aim launcher), v_0 (initial launch velocity in m/s^2), and t (number of seconds the timer should be set for) we found that o = .354601 radians, v_0 = 37.1454 m/s^2, and t = 7.34657 seconds when gravity is equal to -3 m/s^2, and -9.8 m/s^2 in the direction x, and z axis respectively. In order to make sense of these results we need to convert ɵ from radians to degrees: multiplying ɵ by ($180/\pi$) yields ɵ = 20.32 degrees in the x-y plane. Supporting work for these calculations can be found in Appendix D.

In order to calculate where to set the marker we can use equation 11 to set the distance the Road Runner travels equal to the arc length from u to 150

$$50t = \int_u^{150} \sqrt{p'(u) * p'(u)}\,du$$

On Thursday October 16th the net will be set for 7.78012 seconds

$$50(7.78012) = \int_u^{150} \sqrt{p'(u) * p'(u)}\,du$$

$$389.006 = \int_u^{150} \sqrt{p'(u) * p'(u)}\,du$$

Taking the derivative of $p(u)$ and dotting it with itself allows us to take this integral from u to 150. Using simple guess and check to solve for you we are able to find that u = 100.017. Substituting u back into equation 2 calculates the point where Wiley should place his marker

$$p(u) = \left(u, usin\left(\frac{u}{10}\right), 400 - \frac{u^2}{400} - \frac{u^2 \sin(\frac{u}{10})^2}{400}\right)$$

$$p(100.017) = (100.017, -54.5566, 367.55)$$

Supporting work for these calculations can be found in Appendix E.

To solve Wiley Coyote's final problem of maximizing Aunt Edna's Famous Road Runner Soufflé we found the gradient of the objective function

$$f(x, y) = 6x^{2/3}y^{1/2}$$

by taking the partial derivative with respect to x and y

$$f_x(x,y) = 4x^{-1/3}y^{1/2}$$

and

$$f_y(x,y) = 3x^{2/3}y^{-1/2}$$

Then by setting the partial derivatives equal to zero and using the equation 13, $2x + 3y \leq 35$, the constraint equation, the critical values can be found. These critical values can be checked using the second derivative test in order to find which point is the maximum of the function. We found a critical point at (10,5) with a value of 62.2734. This means the recipe is maximized with a budget of $35 when Wiley Coyote uses 10 dozen eggs and 5 jars of cream sauce to yield a maximum of 62 Soufflé servings. Supporting work for these calculations can be found in Appendix F.

Discussion

After analyzing Wiley Coyote's problem it was determined that in order to capture the Road Runner, Wiley needs to know the direction in the x-y plane to point the Exploding-Net Pod Launcher, the initial launch velocity, how many seconds to set the pod for, and the point to set the visual reference marker which will allow Wiley to know when to launch. Using the given information including that the fact that Exploding-Net Pod Launcher can only launch at a 45 degree angle, it was possible to derive the projectile motion equations and use calculus to provide Wiley with the necessary values. In a real situation it would be hard to replicate these results since winds and other variables that we assumed negligible in this situation, would have an effect on the projectile.

Conclusion and Recommendations

In order for Wiley Coyote to successfully capture the Road Runner on this particular hill and road on Thursday October 16th , we found that he should: aim the launcher 44.51 degrees in the x-y plane, have an initial velocity of 40.1968 m/s^2, and set the timer on the Exploding-Net Pod for 7.78012 seconds. If something were to go wrong in Wiley's plan and he had to attempt a capture on Wednesday October 15th or Friday October 17th Wiley would have to alter his launch configuration. On Wednesday October 15th Wiley should: aim the launcher 43.03 degrees in the x-y plane, have an initial velocity of 34.3226 m/s^2, and set the timer on the Exploding-Net Pod for 7.02896 seconds. On Friday October 17th Wiley should: aim the launcher 20.32 degrees in the x-y plane, have an initial velocity of 40.1968 m/s^2, and set the timer on the Exploding-Net Pod for 7.78012 seconds. Also it is recommended that Wiley place a marker at the point (100.017, -54.5566, 367.55) in order to cue the launch of the Exploding-Net Pod. Finally, in order for Wiley to maximize his Aunt Edna's famous Road Runner Soufflé he needs to purchase 10 dozen eggs and 5 jars of cream sauce to achieve a maximum of 62 servings of Soufflé.