Background

After years of failed attempts from mail order products, Wile E. Coyote decided to cash his 401K funds and rake out the big bucks to hire professionals. Until a recent assignment, ACME Surveying and Parameterization Services (SPS) was the only professional company that could help Wile E. Coyote catch the pesky fowl, they call "The Roadrunner." Unfortunately, Wile ran out of money before ACME SPS could complete the project. As a result, two cadets in Major Bowman's MA 153 class have been tasked to complete the mission. The mission includes launching an ACME Teflon-covered exploding net to land on the roadrunner while he makes his daily dash down the hill. The purpose of this report is to determine, based on given equations for the hill and the path of the roadrunner.

With the money Wile paid to ACME he was able to determine the speed of the roadrunner, the hiding place, the point where the net will explode to capture the roadrunner and all of the gravitational pulls on 16 October 2008, the day he plans to finally capture the roadrunner once and for all. The exploding net will travel in the air for a certain amount of time in a straight line once it is launched. The rest of this report is to determine the optimal amounts of eggs and crème sauce to use with a fixed budget, and figure out how many servings that the ingredients which include roadrunner, will make based upon the equation given. Based on the results, a recommendation must be given to Mr. Wile E. Coyote as to when and where to launch the net from, as well as the amount of eggs and crème sauce needed and the number of servings they can expect to make from the ingredients.

Facts

The facts influencing our assessment describe the limitations and necessary components to capturing the Road Runner. The Road Runner runs on a hill modeled by the expression

$$z = 400 - \frac{x^2}{400} - \frac{y^2}{400},$$

along a path modeled by the curve

$$p(u) = < u, uSin(\tfrac{u}{10}), 400 - \frac{u^2}{400} - \frac{u^2 \sin^2(\frac{u}{10})}{400} >$$

The Road Runner will run at a constant speed of 50 m/s. The best hiding place to launch the Exploding-Net Pod from is point (50, 30, 783/2) and the best place to capture the Road Runner is point (150, 97, 320)

Important information regarding the creation of the maximum servings of the Road Runner Soufflé is that eggs cost \$2 per dozen and that cream sauce costs \$3 per jar. Wile E. Coyote has a budget of \$35, and can make $6x^{2/3}y^{1/2}$ servings of Road Runner Soufflé when x represents dozens of eggs and y represents jars of cream sauce.

Assumptions

In order to capture the roadrunner with the exploding net in stride on October 16, 2008, it is necessary to make the following assumptions to simplify the mathematical model.

1. There is no wind on the given day, 16 October 2008.
2. Friction in cartoon land is negligible or non-existent.
3. The Exploding-Net Pod Launcher is accurate and does not malfunction
4. The Road Runner will not stop or alter his route in any way
5. The Road Runner will run down the road at a constant 50 m/s.

6. Wile E. Coyote is competent enough to perfectly follow our instructions.
7. None of the servings of Roadrunner Soufflé need to be thrown out because they happen to be a bad batch.

Analysis

Part I:

The parametric equation

$$a_{ENP}(x, y, z) =< g_x, g_y, g_z > \qquad (1)$$

describes the Exploding-Net Pod's acceleration due to gravity in the x, y, and z directions, with g_x as the acceleration due to gravity in the x direction, g_y as the acceleration due to gravity in y direction, and g_z as the acceleration due to gravity in the z direction. Equation 1 can be integrated with respect to t in order to obtain an equation for the Exploding-Net Pod's velocity:

$$v_{ENP}(x, y, z) =< v_{ix} + g_x t, v_{iy} + g_y t, v_{iz} + g_z t > \qquad (2)$$

The products $g_x t$, $g_y t$, $g_z t$ in Equation 2 represent the change in the x, y, and z components of the Exploding-Net Pod's velocity with respect to time. The variables v_{ix} , v_{iy} , v_{iz} represent the initial x, y, and z components of the Exploding-Net Pod's velocity, as they are the constants of integration when the acceleration equation is integrated with respect to time. Equation 2 can be integrated with respect to t in order to obtain an equation for the Exploding-Net Pod's position:

$$P_{ENP}(x,y,z) =< x_i + v_{ix}t + \frac{g_z t^2}{2}, y_i + v_{iy}t + \frac{g_z t^2}{2}, z_i + v_{iz}t + \frac{g_z t^2}{2} > \tag{3}$$

The segments of Equation 3 $\frac{g_z t^2}{2}$, $\frac{g_z t^2}{2}$, and $\frac{g_z t^2}{2}$ represent the change in x, y, and z components of the position of the Exploding-Net Pod over time due to the acceleration in the x, y, and z directions. The products $v_{ix}t$, $v_{iy}t$, and $v_{iz}t$ represent the change in the x, y, and z components of the Exploding-Net Pod over time due to the initial velocity of the Exploding-Net Pod in the x, y, and z directions. The variables x_i, y_i, z_i represent the initial x, y, and z components of the Exploding-Net Pod's position, as they are the constants of integration when Equation 2 is integrated with respect to time. In order to decrease the amount of variables in the equation for the Exploding-Net Pod's position v_{ix}, v_{iy}, and v_{iz} need to be rewritten in terms of the initial velocity:

$$P_{ENP}(x,y,z) =< x_i + v_i\cos(45°)\cos(\alpha)t + \frac{g_z t^2}{2}, y_i + v_i\cos(45°)\sin(\alpha)t + \frac{g_z t^2}{2}, z_i + v_i\sin(45°)t + \frac{g_z t^2}{2} > \tag{4}$$

This equation takes into account the fact that $v_{ix}t$, $v_{iy}t$, and $v_{iz}t$ can be expressed in terms of the initial velocity of the Exploding-Net Pod multiplied by the cosines and sines of certain angles. The product $v_i\sin(45°)$ represents the z component of the Exploding-Net Pod's velocity, because the Exploding-Net Pod is launched at an angle of 45° with respect to the horizontal, and thus equal to the initial velocity multiplied by sin(45°). The products $v_i\cos(\pi/4)\cos(\alpha)t$ and $v_i\cos(45°)\sin(\alpha)t$ represent the x and y components of the Exploding-Net Pod's position, respectively. Because the Exploding-Net Pod can only be launched at an angle of 45°, the acceleration of the Exploding-Net Pod in a horizontal direction will be the initial velocity multiplied by cos(45°). Because the horizontal component of velocity is split up between the x and y components of velocity, the angle α is used to denote the angle the Exploding-Net Pod is launched at with respect to the x-axis.

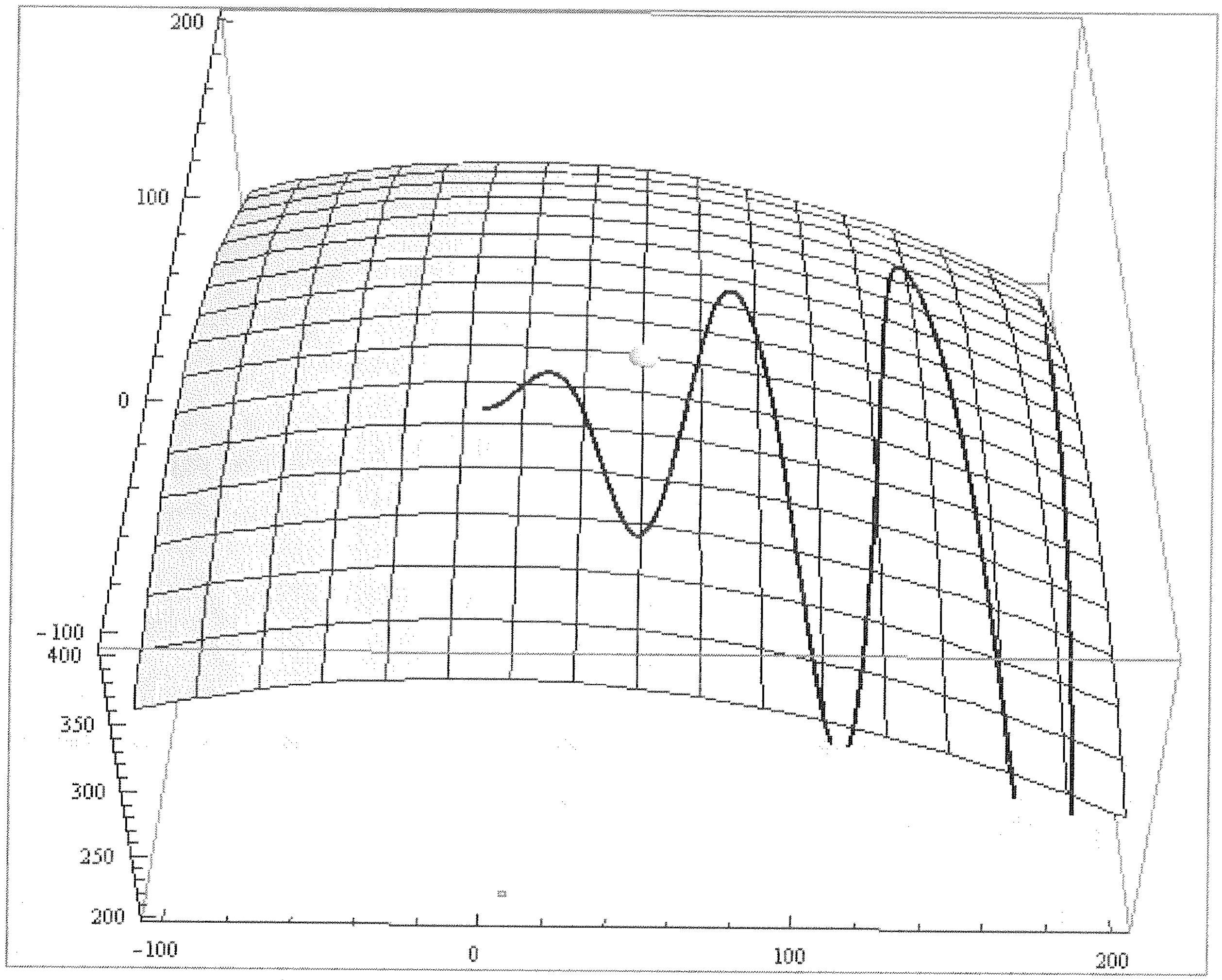

Figure (1) Graph of Road Runner's path down the hill.

The variables x_i, y_i, and z_i are equal to the x, y, and z components of Wile E. Coyote's launch point. The variables g_x, g_y, and g_z are equal to given values for 15 OCT 2008 and are known at other times. Also, Equation 4 is being solved so that its x, y, and z components are equal to the final position of the Exploding-Net Pod – the position where the Road Runner will be caught. Thus, Equation 4 has the 3 unknowns to be solved for – t, v_i, and α – and three different components – the x, the y, and the z. The x, y, and z components of Equation 4 can be solved simultaneously for the variables t, v_i, and α.

The equation

$$d_{RR}(t) = 50t \qquad (5)$$

represents the distance the Road Runner travels in any amount of time, and can be used to calculate the distance the Road Runner travels over the time the Exploding-Net Pod is in the air, because it is the speed the Road Runner travels at multiplied by the time it travels. Therefore, it also represents the distance on the road away from the capture point that the marker has to be placed on.

The equation

$$p(u) =< u, uSin(\tfrac{u}{10}), 400 - \frac{u^2}{400} - \frac{u^2 \sin^2(\frac{u}{10})}{400} > \qquad (6)$$

In which u is equal to the x component of the road corresponds to the road that the Road Runner will be traveling on. The distance along the road the Road Runner travels can be derived from the equation of the road by taking the square root of the summation of the squares of the road's partial derivatives:

$$D_{road}(u) = \int_{u_i}^{u_f} \sqrt{1^2 + \left(\frac{u\cos(\frac{u}{10})}{10} + \sin(\tfrac{u}{10})\right)^2 + \left(-\frac{u}{200} - \frac{u^2\cos(\frac{u}{10})\sin(\frac{u}{10})}{2000} - \frac{u\sin^2(\frac{u}{10})}{200}\right)^2}$$

(7)

Where u again is equal to the x component of the road, u_i is equal to a starting x value on the road, and u_f is equal to a final x value on the road.

The position to place the marker at can be found by solving for u_i when u_f is equal to the x components of the point the Road Runner will be captured at and the distance traveled on the road is equal to the distance found by using Equation 5.

Part II:

The given equation

$$f(x,y) = 6\,x^{2/3}y^{1/2} \tag{8}$$

describes the number of servings of Aunt Edna's famous Road Runner Soufflé that can be made when x dozen eggs and y jars of crème sauce are mixed with Road Runner meat. Because (9) is not picky, he does not care if the same proportion of eggs and crème sauce are used in every serving.

The given equation

$$2x + 3y = 35$$

helps determine how much of each ingredient is required. Since it costs \$2 per dozen of eggs and \$3 per jar of crème sauce and there is a budget of 35 dollars, we added up the products of the price of a dozen of eggs (\$2) by x dozen eggs and the price of a jar of crème sauce (\$3) by y jars of crème sauce and set that equal to our budget of \$35. Equation (8) when you in conjunction with the equation (2), the constraint equation, can be used to find the optimal amounts of each ingredient to buy for the soufflé by using the LaGrange method.

Below is the step by step LaGrange method:

$$4x^{-1/3}y^{1/2} = 2\lambda \tag{10}$$

$$3x^{2/3}y^{-1/2} = 3\lambda \tag{11}$$

$$2x + 3y = 35 \tag{2}$$

The first step of the LaGrange method is to make three separate equations. The first equation, equation (10), is the partial derivative with respect to x set equal to the partial derivative with respect to x of the constraint equation, equation (9), multiplied by lambda (λ). The second equation, equation (11), is the partial derivative with respect to y set equal to the partial derivative with respect to y of the constraint equation, equation (9), multiplied by lambda (λ). The third and final equation is simply the constraint equation, equation (9).

$$2x^{2/3}y^{-1/2} = 2\lambda \tag{12}$$

The next step of the LaGrange method is to get two of the equations to equal the same thing, so that you can set two equations equal to each other and solve for a variable, either x or y. To do this, multiply both sides of the second equation, equation (11) by $\frac{2}{3}$ to get equation (12). This will make both equations (10) and (11) equal to 2 λ.

$$4x^{-1/3}y^{1/2} = 2x^{2/3}y^{-1/2} \tag{13}$$

$$2y = x \tag{14}$$

Once equations (10) and (11) are set equal to each other, simply solve for a variable, preferably x. When you solve for x, you get equation (14).

$$x = 10$$

$$y = 5$$

Next, refer back to the constraint equation, equation (9), and substitute $2y$ for every instance of x, and then solve for y. When this is done correctly, you should get the value of $x = 10$. Next, substitute 10 for every instance of y in equation (9), and the result is $y = 5$.

$$6(10)^{2/3}(5)^{1/2} = 62.27$$

This means that we need to purchase 10 dozen eggs and 5 jars of crème sauce for the Soufflé. To determine the number of servings, we substitute the values of x and y that we just found and substitute them into equation (8). Once you calculate the answer, you should reach a final result of 62.27. This means that 10 dozen eggs, 5 jars of crème sauce, and one Road Runner will yield approximately 62 servings of Aunt Edna's famous Road Runner Soufflé.

Results

Part I:

Using Equation 4, we solved simultaneously for t, v_i, and α on 16 October 2008 by setting Equation 4 equal to (50, 30, 320), which is the point the Road Runner is captured at, setting g_x, g_y, and g_z equal to their values on Thursday, -2, -3, and -9.8 respectively, and by setting x_i, y_i, and z_i equal to the x, y, and z components of the launch point (150, 97, 391.5).

$$P_{ENP}(50,30,320) = < 150 + v_i\cos(45°)\cos(\alpha)t + \frac{2t^2}{2}, 97 + v_i\cos(45°)\sin(\alpha)t + \frac{3t^2}{2}, 391.5 + v_i\sin(45°)t + \frac{9.8t^2}{2} >$$

Solving this equation for v_i, t, and α, we found that the Exploding-Net Pod should be launched with an initial velocity of 40.92 meters/second at an angle of .77 radians with respect to the positive x-axis. The time the Exploding-Net Pod is airborne when launched at this speed and direction on 16 October 2008 is 7.78 seconds. Supporting work for these calculations can be found in Appendix A.

Once we found the time, it became possible to find the position to place the marker at. In order to find the distance on the road the Road Runner traveled in time t, we used Equation 5, multiplying the speed of the Road Runner by the time the Exploding-Net Pod travels in the air.

$$d_{RR}(7.78) = 50(7.78)$$

Using this equation, we found that on Wednesday, the Road Runner will travel 389 meters in the 7.78 seconds the Exploding-Net Pod is airborne.

Equation 6,

$$p(u) =< u, uSin(\tfrac{u}{10}), 400 - \frac{u^2}{400} - \frac{u^2 \sin^2(\tfrac{u}{10})}{400} >$$

tells us that the value u is equal to the x coordinate of the road. Thus, by using Equation 7

$$D_{road}(u) = \int_{u_i}^{u_f} \sqrt{1^2 + \left(\frac{u\cos(\tfrac{u}{10})}{10} + \sin(\tfrac{u}{10})\right)^2 + \left(-\frac{u}{200} - \frac{u^2\cos(\tfrac{u}{10})\sin(\tfrac{u}{10})}{2000} - \frac{u\sin^2(\tfrac{u}{10})}{200}\right)^2}$$

setting u_f to x component of the point the Road Runner will be captured at, and setting $D_{road}(u)$ equal to the distance the Road Runner travels while the Exploding-Net Pod is in the air, we can solve for u_i, the x component of the road that the marker should be placed at.

$$389 = \int_{u_i}^{150} \sqrt{1^2 + \left(\frac{u\cos(\tfrac{u}{10})}{10} + \sin(\tfrac{u}{10})\right)^2 + \left(-\frac{u}{200} - \frac{u^2\cos(\tfrac{u}{10})\sin(\tfrac{u}{10})}{2000} - \frac{u\sin^2(\tfrac{u}{10})}{200}\right)^2}$$

The guess-and-check method yields that u_i is approximately equal to 100 when the distance traveled on the road is 389 meters and the final x component of the point traveled to is 150. Thus, the u value and x component of the point on the Road Runner's path the marker should be placed on is 100.

$$p(100) =< 100, 100Sin(\tfrac{100}{10}), 400 - \frac{100^2}{400} - \frac{100^2 \sin^2(\tfrac{100}{10})}{400} >$$

Substituting 100 for u in Equation 6 then yields that the marker should be placed at point (100, -54.4, 367.6) on 16 October 2008. Supporting work for these calculations can be found in Appendix A.

However, despite the fact that our information is accurate and despite the fact that Wile E. Coyote has created the perfect plan, history tells us that Wile E. Coyote's plan will probably not work on the first try. As anyone who has ever seen a Wile E. Coyote cartoon knows, his schemes rarely ever work. To compensate for this, we solved for the initial velocity, direction of fire, time the Exploding-Net Pod is in the air, and position of the marker for two other days – A day with a g_x of 3, a g_y of 0, and a g_z of 9.8, as well as a day with a a g_x of 1, a g_y of 2, and a g_z of 9.8.

In order to solve for the new initial velocities, directions of fire, amounts of time the Exploding-Net Pos is airborne, and the positions of the new markers for the other two days, we simply inserted the modified g_x, g_y, and g_z values into Equation 4 and came up with new initial velocities, directions of fire, and amounts of time the Exploding-Net Pod is airborne.

For an acceleration vector of $<3, 0, 9.8>$ we found that $v_i = 37.15$, $t = 7.35$, and $\alpha = .35$. Using the new value of t, we used Equation 5 to solve for the distance the Road Runner travels while the ball is in the air, which is 357.5 meters. Then, using that distance and the x-component of the point the Road Runner will be captured at in Equation 7, we used the guess-and-check method to discover that the x component and u value of the point the marker should be placed at is approximately 103.5. We substituted 103.5 in for u in equation 6, which yielded the point the marker should be placed at: (103.5,-82.6711, 356.133). Supporting work for these calculations can be found in Appendix B.

For an acceleration vector of $<1, 2, 9.8>$ we found that $v_i = 34.32$, $t = 7.03$, and $\alpha = .75$. Using the new value of t, we used Equation 5 to solve for the distance the Road Runner travels while the ball is in the air, which is 351.5 meters. Then, using that distance and the x-component of the point the Road Runner will be captured at in Equation 7, we used the guess-and-check method to discover that the x component and u value of the point the marker should be placed at is approximately 104.5. We substituted 104.5 in for u in equation 6, which yielded the point the marker should be placed at: (104.5, -89.33, 352.75). Supporting work for these calculations can be found in Appendix B.

Part II:

Using equations (8) and (9), we used LaGrange multipliers to determine the number of dozens of eggs and jars of crème sauce, as well as the total number of servings that can be made with the purchased ingredients. The first step was to take the partial derivatives of the $f(x,y)$ equation and set that equal to the partial derivative of the constraint equation, which is equation (2). Next, we lined up all three of the equations, (9), (10), and (11) and looked to see how we could get two of the equations to equal each other. We noticed that equations (10) and (11) were very similar on the right side of the equations. With equation (10) being equal to 2λ and equation (11) being equal to 3λ, we determined that if we multiplied both sides of equation (11) by $\frac{2}{3}$, that the right side would equal 2λ, just as equation (10) does.

Because both equations are equal now, it is possible to set them equal to each other. The reason to set the equations equal to each other is so you can solve for an unknown variable, such as x or y. In our case, it was easier to solve in terms of x so we could prevent fractions when we perform the next step by substituting in the resulting equation, equation (14), $2y = x$. If we would have solved for y, then $y = \frac{x}{2}$, which is still a very workable equation, but equation (14) is easier.

Next, we substituted equation (14) back into the constraint equation in order to ob (1) numerical value for y. Once we did the simple algebra and determined that $y = 5$, we were able to substitute the 5 back into the constraint equation and again use simple algebra, this time solving to obtain a numerical value for x. This meant that with the \$35 budget, we will buy 10 dozen eggs and 5 jars or crème sauce to mix into the Road Runner Soufflé. Since we were given the original equation (below) which told us how much of each ingredient would go into a serving of soufflé,

$$f(x,y) = 6\,x^{2/3}y^{1/2}$$

we substituted our new values into the equation to get a final answer of 62.27. This number represents the number of servings of Aunt Edna's Road Runner Soufflé that can be made with the ingredients purchased. One probably would not make .27 of a serving, so for whole numbers, the ingredients would make enough for 62 complete servings.

Discussion

Notice figure (1) is a graph of the Road Runner's path when he runs down the hill at 50 m/s. This graph should give Wile a better idea visually of the path the Road Runner will take, and will provide him with a grid so he can listen to our recommendations and fire from the proper point so he successfully captures the Road Runner once and for all.

Conclusions and Recommendations

Through our research, we discovered that if on 16 October 2008 the Exploding-Net Pod is launched from the hiding place at point (50, 30, 391.5) with an initial velocity of 40.92 meters/second at an angle of .77 radians with respect to the x-axis, the Exploding-Net Pod will be airborne for 7.78 seconds before reaching the point (150, 97, 320). If a marker is placed at the point (100, 54.4, 367.6), and Wile E. Coyote launches the Exploding-Net Pod at the exact moment the Road Runner passes it, Wile E. Coyote will capture the Road Runner at point (150, 30, 391.5) and be able to make Road Runner Soufflé. In order for Wile E. Coyote to get the most servings of his Road Runner Soufflé for \$35, he will need to purchase 10 dozen eggs at \$2 per dozen and 5 jars of crème sauce at \$3 per jar.

Although we believe our plan will be successful, there is some chance of error due to variables that could not be accounted for. For instance, the slightest bit of wind would throw our calculations off and cause the Exploding-Net Pod to completely miss the Road Runner. Similarly, the Road Runner could see the Exploding-Net Pod coming at him, and simply stop to avoid it. Such variables will always exist, and thus the success of this plan can never be guaranteed. In case we miscalculated the effect of any of the variables that may affect the Exploding-Net Pod or the Road Runner's behavior on a given day, we calculated how to catch the Road Runner considering multiple gravity scenarios so Wile does not have to consult for advice again.