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Ants, Tunnels, and Calculus:

An Exercise in Mathematical Modeling.

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ANTS, TUNNELS, AND CALCULUS: AN EXERCISE IN MATHEMATICAL MODELING

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ABSTRACT: A simple mathematical model describing the time it takes for an ant to construct a linear tunnel of length x is produced from five intuitively acceptable assumptions. Using the definition of the derivative and the notion of an antiderivative, the model permits the investigator to discover the effects of change in tunnel length on the overall time it takes to complete the tunnel.

KEYWORDS: Mathematical modeling, calculus, derivative, antiderivative, tunnel, ants.

INTRODUCTION

The process of mathematical modeling consists of a number of steps. These include accepting a situation, making some assumptions, converting the assumptions into mathematical statements, solving the mathematics, and interpreting the results in light of the situation.

We discuss a modeling activity which has proven to be of interest to regular college level calculus classes as well as high school groups. We have used it at various times and for a number of different purposes: to motivate the definition of the derivative in practice, to show the modeling process and to get students involved in that process, to confirm the notion of the antiderivative, and to permit students to see that mathematics can be applied to other areas of interest, in this case, entomology.

THE SETTING

Consider a class of students who have some exposure to calculus, preferably the definition of the derivative and the uniqueness of the antiderivative up to a constant, i.e., the indefinite integral. We present the problem the following problem.

How long does it take an ant to build a tunnel?

We open the discussion by asking the students to respond with questions about this question. They do just that - every time!

- How long is the tunnel?
- Who cares?
- What kind of stuff are they digging in?
- Just one ant? Or a bunch of ants?
- What shape is the tunnel - curved, up, down, straight?
- How big is the tunnel? You know, is it 1 inch around, 1/2 inch around, or real tiny for the ant to just get through?
- Why do ants build tunnels?
- Could we watch one in our ant village?

Some of the questions we can address and others we have to put off. But we always acknowledge the question and respond positively by asking for more detail and writing it on the board. Engaging the student is what this activity is about. From the questions there evolves a sense that some precision in responding to our original question and the class' list of questions may be appropriate.

When it comes to precision the students usually seek authority. We do not represent authority on ant tunnel building in our role as mathematics teacher. Accordingly, we encourage the students to "wing it" just a bit, to make some assumptions in response to some of their own questions. And thereby begins the modeling process.

MODELING THE BUILDING OF AN ANT TUNNEL

From our questions, our own experience and observations about ants, and our desire to emphasize the simplicity of the situation rather than the complexity involved, we usually come down to five assumptions:

1. The ant will build a level straight tunnel.

2. The ant will build in uniform, moist, fine sand.

3. The ant will walk as fast when she is carrying something as when she is unburdened. It would take more energy to do so, but pace is the same.

4. The tunnel's cross-sectional area is constant.

5. The ant is digging into the side of a sand wall.

This last assumption we throw in so that the issue of what to do with the sand when we get it to the mouth of the tunnel does not complicate the modeling activity too much.

When considering our original problem, "How long does it take an ant to build a tunnel?" we are now prepared to offer a more precise statement of the problem:

> "How long does it take an ant to dig a straight tunnel of length x in uniform, moist, fine sand into the side of a vertical wall?"

We begin to identify more expressions, having identified the length as x in our revision of the original question. We let $T(x)$ be the time necessary to build a straight tunnel of length x in uniform, moist, fine sand into the side of a vertical wall. And we now know that we are seeking a precise description for $T(x)$. We offer up a crude drawing on an overhead. (See Figure 1.)

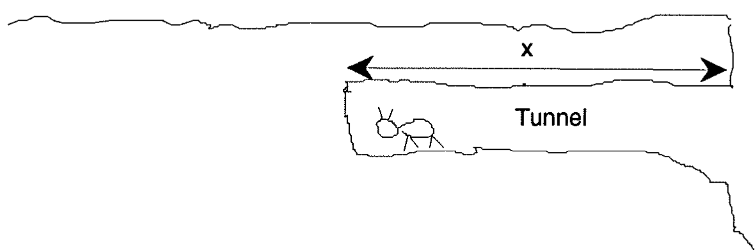

Figure 1. Crude drawing for ant tunnel building model. x is the length of the tunnel and $T(x)$ is the time it takes this ant to build this tunnel.

We encourage the students to speak up about any questions or suggestions they have at ANY TIME. At this point several make conjectures as to the nature of $T(x)$. "I know. It is some constant times x." "Why?" is our response. It sure sounds good - the longer the tunnel the more time it takes. We all make a note of that as a characteristic which any solution for $T(x)$ must possess. We do not offer any reasons why it is not appropriate. But we point out that perhaps we had better take a closer look at the situation.

In Figure 2 we see the sketch we offer the students to help us build a mathematical model to determine just how much time it would take to build a small section of the tunnel. Perhaps from knowing the time it takes to build a small section of the tunnel we can build up to a larger section of the tunnel from the assumptions more directly rather than just jumping on a function offered "by the Class," e.g. $T_C(x) = kx$.

With the diagram of Figure 2 we ask the question, "How long does it take to dig out the section from distance x to distance $x + h$?" Quickly the students respond, "The time it takes to dig to $x + h$ less the time to dig to x. That's $T(x + h) - T(x)$."

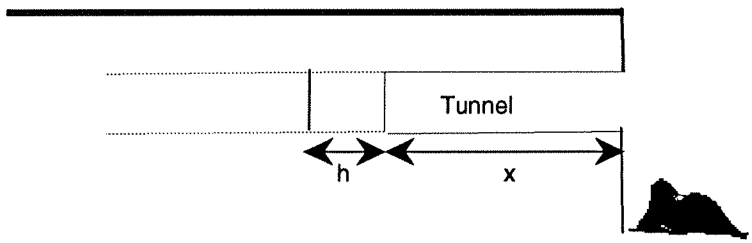

Figure 2. Diagram used to discover the time it takes to build a small section of the tunnel from distance x to $x + h$.

Thus we have the time to extend the tunnel from distance x to $x + h$ as

$$T(x + h) - T(x).$$

And what might we be able to say about this time?

1. Since the ant must bring all the sand in this little section from x to $x + h$ out a distance x and since we assume that the ant travels at a constant rate - with or without baggage - we can assume that the time to cart out all the stuff from x to $x + h$ is proportional to x, i.e.

$$T(x + h) - T(x) = \beta x. \tag{1}$$

2. Based on our previous discussion and the proposed linear model for $T(x)$ itself which was offered earlier we postulate that the time it takes to extend the tunnel h units (for small h) is also proportional to h, for the longer the section, the longer the time needed to dig out the section. In fact, we make it reasonable by saying, "If h is 10 little granules high by 1 granule deep, then the ant has to make 10 trips back out the tunnel. If h is 10 little granules high by 2 granules deep, then the ant has to make 20 trips back out the tunnel - all over the same trip distance, x." Thus it seems reasonable that

$$T(x + h) - T(x) = \gamma h. \tag{2}$$

3. Since this is in the side of the hill, the ant merely drops the sand over the edge at the tunnel opening and this does not effect the time for digging.

Now we point out that we have $T(x+h)-T(x)=\gamma h$ and we also have $T(x+h)-T(x)=\beta x$. This means $\gamma h=\beta x$ for all x and all h. The only way this can happen is if $\gamma=\alpha x$ and $\beta=\alpha h$.

Thus, we produce the equation

$$T(x+h)-T(x)=\alpha x h. \tag{3}$$

At this point, if the class is studying the derivative, we ask if they see the makings of a derivative in this equation. In all cases we point out that our objective is to get a full description of $T(x)$, perhaps to solve for $T(x)$.

We again check our assumptions. Have we used them? Used them well?

We jokingly mention the developing salivary glands for differentiation which demand the form

$$\frac{f(x+h)-f(x)}{h}.$$

In this case we get

$$\frac{T(x+h)-T(x)}{h}=\alpha x. \tag{4}$$

We recall that we wish information about $T(x)$, not $T(x)$ *and* $T(x+h)$, and recalling the definition of derivative if the students do not mention it (and someone usually does) we proceed to obtain

$$T'(x)=\lim_{h\to 0}\frac{T(x+h)-T(x)}{h}=\alpha x. \tag{5}$$

Thus to find $T(x)$ we need to find a function $T(x)$ which has a derivative equal to αx for some constant α.

In the classes in which the formal antiderivative or indefinite integral concept has been covered we know that the answer is

$$T(x)=\frac{\alpha}{2}x^2+c. \tag{6}$$

And in those classes in which we have not covered the indefinite integral we ask them to think of a function which has x as a derivative, then one which has αx as a derivative. They usually get the quadratic part, and upon being prompted, "Could we add, say 4, or 5, or c to your answer?" they see that the form could be that of Equation (6).

Now we are getting somewhere. And the group usually feels that way, for they have a candidate for $T(x)$ which was built upon the simple assumptions and not just thrown out to us. But there is concern about the two constants α and c. We ask the students how they might determine them. We sometimes need to prompt them with such questions as, "How long does it take the ant to build a tunnel of length $x = 0$?" And will this knowledge give us information about these constants? Sure enough,

$$0 = T(0) = \frac{\alpha}{2}(0)^2 + c = c. \tag{7}$$

Thus we have a simpler model and one less constant

$$T(x) = \frac{\alpha}{2}x^2. \tag{8}$$

But what about the constant α? If someone knew some other data point, e.g., how long it takes to build a tunnel of length, say $x = 1$, then we could determine $T(1) = \frac{\alpha}{2}(1)^2$ and thus the fact that $\alpha = 2 \cdot T(1)$. But we do not have any more realistic, known data. Thus we are comfortable with the constant α and our model (a one parameter model), for $T(x)$, the time necessary for an ant to build a straight tunnel of length x into the side of a vertical wall, as expressed in Equation (8).

REASONABLENESS AND INTERPRETATION

We look at Equation (8) and ask, "Is this a reasonable model?" and "Why is it more reasonable than the 'Class offered' model $T_C(x) = kx$?"

Perhaps we can answer both questions with the same issue, namely, what if we double the length of the tunnel from x to $2x$? How does that effect the time to build the tunnel?

With the model $T_C(x) = kx$, we see that doubling the length from x to $2x$ has the following effect:

$$T_C(2x) = k(2x) = 2(kx) = 2T_C(x). \tag{9}$$

Thus it takes twice as long to build a tunnel of length $2x$ as it does to build a tunnel of length x. This sounds good – but wait. Consider the diagram in Figure 3. This says it takes $T(x)$ time to build the initial tunnel of length x and it takes *the same amount of time* to build the next x units of the tunnel. This is unreasonable for the ant has to carry all the material from the inner section of tunnel (AB) through the outer section (OA) of tunnel in addition to the distance in the inner section. So, surely the time to build from O to A cannot be the same as the time to build from A to B. With this, we finally dismiss the linear model. (We could have done this earlier, even before we arrived at our *answer*, and, indeed, we have done this in some instances.)

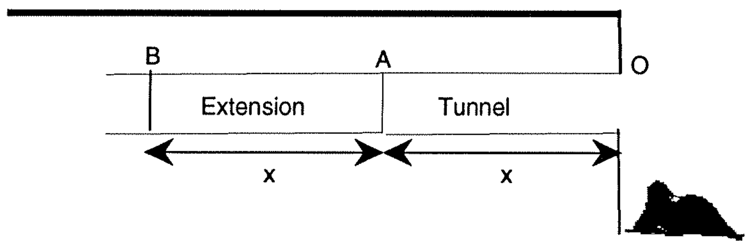

Figure 3. Diagram to help analyze tunnel building model in cases of tunnels of length x and $2x$.

Now let us consider our derived model, $T(x) = \frac{\alpha}{2}x^2$, and ask the questions, "What if we double the length of the tunnel from x to $2x$? How does that effect the time to build the tunnel?"

$$T(2x) = \frac{\alpha}{2}(2x)^2 = \frac{\alpha}{2}4x^2 = 4\frac{\alpha}{2}x^2 = 4T(x). \tag{10}$$

Thus it takes *four times as long* to build a tunnel of length $2x$ as it does to build a tunnel of length x. We suggest that this *IS* interesting and it is reasonable now that we believe doubling the length of the tunnel from x to $2x$ should *more than* double the time it takes ($T(x)$) to build a tunnel of length x.

These discussions make the model appear reasonable as it seems to fit well with our perceptions of reality. We interpret the model as only a rough estimation, for unless we have more data, we cannot confirm or deny the model. But when we teach calculus for engineering students this analysis suggests issues for the civil engineering students who will eventually be making bids on projects - namely doubling the length of a tunnel will not merely double the cost!

CONCLUSION

Students in course evaluations point to this exercise as useful, for it shows them the modeling process and it shows how their mathematics can be used. Since we only use this as an interjection, we do not build upon this model, but rather return to the course material. Perhaps readers would like to suggest extensions, more realistic applications, sources of data, other aspects of the modeling process (e.g. energy), and similar experiences. We would welcome such contributions.