Peak-Frequency Responses and Tuned Mass Dampers— Exciting Applications of Systems of Differential Equations

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Abstract

We present two applications of systems of ordinary differential equations concerning excitation of a structure through resonance and we investigate peak-frequency responses and stabilization by means of a tuned mass damper (TMD). The physics, engineering, and mathematics associated with these phenomena are presented and discussed along with illustrations of how to use this material with students.

Introduction

The study of differential equations, while often taught in a mathematics department as a course in solution techniques sprinkled with applications, could serve as a tool for modeling physical phenomena, thus motivating students to want to learn more about solution strategies and their interpretation.

Most such courses get to *systems* of ordinary differential equations (ODEs). Sometimes the study focuses on two-by-two systems of first-order linear

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ODEs, often in the abstract, to introduce notions of eigenvalues, equilibria, and stability. However, this material can also occur as a result of inquiry into a simple mixing problem or a modest compartment flow application, say, to study concurrent or countercurrent diffusion in a kidney dialysis system. The study of systems gets a bit unwieldy without a computer algebra system, such as Mathematica or Maple, which permits elegant graphical outputs and analytic and numerical inquiries that are otherwise tedious if confined to paper-and-pencil work. We present several such inquiries and the role that technology plays in discovery.

In the study of motion, Newton's Second Law plays a major role in building *second-order* linear ODEs. Courses usually use an analogy such as a spring-mass system (with or without damping) to interpret parameters and describe what solutions mean. Eventually, through the cases offered by the quadratic characteristic equation of such an equation, along with nonhomogeneous or driver terms, one can study effects such as beats and resonance. Indeed, in our courses we have "played" the solution of such differential equations using Mathematica's `Play` command to produce the "sound" of the solution.

We introduce a second spring-mass system for the purpose of reducing resonance, or the peak-frequency response, in a structure, so as to reduce or eliminate risk of damage to the structure and discomfort for those in the structure. This addition leads to a system of second-order linear constant-coefficient ODEs. Such systems can be studied, their solutions rendered, and their behaviors plotted, to analyze displacements for the structure. These tasks can be done readily, using technology to offer closed-form solutions (or numerical solutions), for a system of two second-order equations or by converting this system into a system of four first-order equations.

Modeling Structures as Spring-Mass Damper Systems

In most differential equations curricula, the spring-mass damper system for a suspended mass with spring and damper (dashpot) or for a horizontal slider on a frictionless surface, is modeled using Newton's Second Law of Motion.

From the wisdom of Wikipedia [2007], we see a working presentation of Newton's Second Law:

> Observed from an inertial reference frame, the net force on a particle is proportional to the time rate of change of its linear momentum. Momentum is the product of mass and velocity. This law is often stated as $F = ma$ (the force on an object is equal to its mass multiplied by its acceleration).

Since in our applications the mass remains constant, we can use the $F = ma$ form of Newton's Second Law.

Figure 1 depicts a spring-mass dashpot in which a mass is hung on a spring attached to a fixed surface. As the mass moves up and down, a plunger in the liquid of the dashpot dampens the motion by offering resistance to the motion.

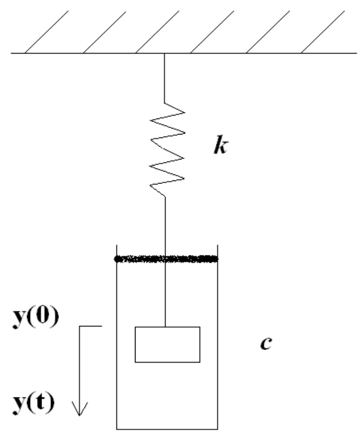


Figure 1. Spring-mass dashpot with mass m, spring constant k, and damping coefficient c, and initial displacement of mass $y(0)$ and general displacement $y(t)$ at time t.

It is worth taking the time to develop differential equations from first principles in physics, that is, to make immediate use of Newton's Second Law using a free-body diagram (FBD); see **Figure 2**. In an engineering mathematics course, one can presume that the students have seen the FBD technique; but even if students have had physics, they may not have practiced FBDs sufficiently well to launch out on their own. Thus, we patiently present the problem-formulation strategy involved in drawing and using FBDs: One needs to find all the forces acting on the mass, sum them, and set the sum equal to mass times acceleration.

In our situation, we assume that the mass is at static equilibrium, that is, the downward force due to gravity acting on the mass has balanced out the upward force due to the restoring force of the spring and the spring is at rest. This means that we take the displacement of the spring from this equilibrium point and say that the spring has been vertically displaced a distance of $y(t)$ m (usually we use MKS units). Our convention is to denote downward displacement positively (spring extended) and upward displacement negatively (spring compressed).

We recall Hooke's Law from high school physics: The restoring force $F = k \cdot y$ of a spring is proportional to the displacement y with constant of proportionality k, which in SI units has units Newtons per meter (N/m) for small displacements. This means that the farther (within reason) that we stretch (or compress) the spring, the greater the restoring force will be. By convention, k is positive. (One can actually validate this law with a ring stand, a set of masses, and a spring—all borrowed from your local science department stock room, for they always have stuff like this, even if in the back room!)

All phenomena of this type lose energy, else they would go forever! One model of such loss is *damping*, which presumes that there is a force directly proportional to the velocity y' acting opposite to the direction of motion, which causes the system to lose energy. When we model the spring-mass system, we envision a dashpot or container of some liquid into which the mass (or a plunger attached to it) is submerged. As the mass moves, it meets resistance from the medium; and the faster it travels, the greater the resistance. The term $c \cdot y'$ (where c has units N·s/m) is used to reflect this damping and is an aggregate term that may involve friction in the spring as well as in the medium. By convention, c is positive.

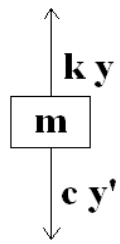


Figure 2. Free Body Diagram for developing the differential equation for motion of the mass in a spring mass dashpot configuration.

In a free-body diagram (**Figure 2**), we use the conventions in which the positive displacement and positive velocity are downward. In terms of the mass's motion, we see that if the spring were extended and traveling upward—that is, displacement is positive, $y > 0$ and velocity is negative, $y' < 0$—there would be a spring restoration force $-k \cdot y$ in the upward direction due to the spring trying to contract, and there would also be a resistance or damping force $-c \cdot y'$ in the downward direction (recall that c is positive by convention and $y' < 0$ in this case). Thus, the forces acting on the mass add to $(-k \cdot y - c \cdot y')$. Similar analysis in other cases (spring extended with mass traveling downward, spring compressed with mass traveling upward, and spring compressed with mass traveling upward) yield the same result. Analysis of the situation leads to an application of Newton's Second Law and to a second-order differential equation. We use $y(t)$ instead of just y, because position now changes over time:

$$m \cdot y''(t) = -k \cdot y(t) - c \cdot y'(t)\,.$$

Quite often, this second-order linear, constant-coefficient ODE is written with all the terms involving the variable y on the left-hand side:

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = 0\,.$$

Of course, there need to be initial conditions of position, $y(0) = y_0$, and velocity, $y'(0) = v_0$, for the problem to be well-posed.

To solve such differential equations, we can use the usual approach of conjecturing a solution of the form $y(t) = e^{\lambda t}$, determining the corresponding eigenvalues of the resulting characteristic equation, and building solutions from eigenvalues, while doing the appropriate actions for the various cases where $c^2 - 4km$ is < 0, $= 0$, or > 0 [Edwards and Penney 1992, 126–135].

The case $c^2 - 4km \neq 0$ leads to a general solution; and depending on whether $c^2 - 4km < 0$ or $c^2 - 4km > 0$, one obtains corresponding *underdamped* or *overdamped* solutions. The special case $c^2 - 4km = 0$ is called *critically damped* and the solution is slightly different in form.

When the situation is underdamped, $c^2 - 4km < 0$, the solutions look like

$$y(t) = c_1 e^{-\left(\frac{c}{2m}t\right)} \cos\left(\frac{\sqrt{4km - c^2}\, t}{2m}\right) + c_2 e^{-\left(\frac{c}{2m}t\right)} \sin\left(\frac{\sqrt{4km - c^2}\, t}{2m}\right),$$

with c_1 and c_2 depending upon the initial conditions.

If $c = 0$, that is, there is no damping and the system continually oscillates without losing any amplitude, the solutions look like

$$y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right) = \sqrt{c_1^2 + c_2^2} \cos\left(\sqrt{\frac{k}{m}}\, t - \phi\right).$$

Here $\omega_0 = \sqrt{k/m}$ is called the *natural frequency* of the system and, if the second form is used, ϕ is called the *phase*.

A good approach to such differential equations is to study the effect of a driving force (or nonhomogeneous input term), say $f(t)$, on the solution. Accordingly, the driven differential equation with initial conditions is

$$m \cdot y''(t) + c \cdot y'(t) + k \cdot y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0\,. \qquad \textbf{(1)}$$

Quite often in engineering applications, the function $f(t)$ is sinusoidal, say $f(t) = F_0 \cos(\omega t)$, and is used to model earthquake or wind forces acting on a structure. An alternating voltage imposed on a circuit can be modeled using these same differential equations [Kreyszig 1999, 118–124].

In the case where $c = 0$, assuming $\omega_0 \neq \sqrt{k/m}$, the solution to **(1)** is

$$y(t) = A \cos\left(\sqrt{\frac{k}{m}}\, t - \phi_1\right) + B \cos(\omega t - \phi_2)\,,$$

with A and B determined by initial conditions. If the driver frequency ω is the same as the natural frequency $\omega_0 = \sqrt{k/m}$, then we have *resonance* and in this case (again with $c = 0$) the solution to **(1)** with initial conditions $y(0) = 0$ and $y'(0) = 0$ is

$$y(t) = \frac{F_0}{2m\omega}\, t \sin(\omega t)$$

and the solution grows without bound. This means that the mass will continue to oscillate with ever-increasing displacements.

Of course, c can never actually be 0—otherwise we might have perpetual motion—so resonance is not really possible; but the resulting displacements can lead to system failure. However, there is something comparable to resonance, namely, the frequency that gives the maximum amplitude for the response. We call this the *maximum frequency response*.

Here is the steady-state (nonhomogeneous) solution for **(1)** with $c \neq 0$:

$$\mathrm{ss}(t) = -\frac{F_0 m \cos(t\omega)\omega^2 + F_0 k \cos(t\omega)}{m^2\omega^4 + c^2\omega^2 - 2km\omega^2 + k^2} + \frac{cF_0 \sin(t\omega)\omega}{m^2\omega^4 + c^2\omega^2 - 2km\omega^2 + k^2}.$$

The amplitude of the steady-state solution as a function of input frequency, ω, is

$$\mathrm{amp}(\omega) = F_0 \sqrt{\frac{1}{m^2\omega^4 + c^2\omega^2 - 2km\omega^2 + k^2}}. \qquad \textbf{(2)}$$

Upon factoring, we see that the denominator in **(2)** can never be zero and hence there can never be true resonance or an infinite response in the damped case.

We obtain $\mathrm{amp}(\omega)$ in **(2)** from combining the terms of $\mathrm{ss}(t)$ into one phase-shifted cosine term and denoting its amplitude by $\mathrm{amp}(\omega)$. The frequency ω that maximizes the amplitude of the steady-state solution is

$$\omega_{\mathrm{max}} = \frac{\sqrt{2km - c^2}}{\sqrt{2}m}.$$

This is a calculus optimization problem in the one variable, ω. **Figure 3** shows ω_{max} the corresponding peak amplitude for fixed values of m, k, F_0, and several values of c, with explicit values given in **Table 1**.

Table 1.
Peak-frequency response for **(1)** with $m = 1$, $k = 1$, $F_0 = 1$, for $c = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

c value	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
Peak response frequency, ω_{max}	0.71	0.94	0.98	1.00
Maximum response amplitude	1.15	2.07	4.03	8.02

From **Table 1**, we see that when we have damping, i.e., $c \neq 0$, we cannot have resonance, but we can reach a maximum response amplitude for an input frequency equal to the peak frequency of

$$\omega_{\mathrm{max}} = \frac{\sqrt{4km - c^2}}{\sqrt{2}m}$$

in the case of **(2)**.

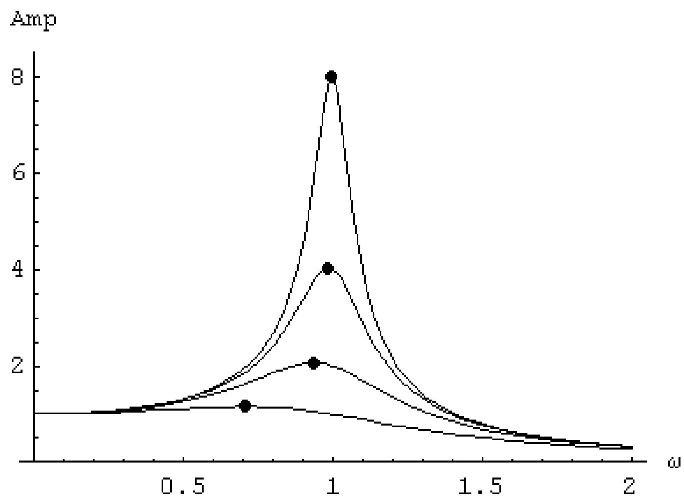


Figure 3. Peak-frequency response: Amplitude, of steady-state solution vs. input frequency ω, for $c = 1, \frac{1}{2}, \frac{1}{4}, and \frac{1}{8}$. The other parameters are fixed in all cases at $m = 1$, $k = 1$, $\omega_0 = 1$, and $F_0 = 1$. The peak is higher for lower values of c, that is, $\omega_{\max} \to \omega_0$ as c decreases.

Tuned Mass Dampers—Notions

We now turn our attention to mitigating both resonance and peak-frequency responses by reducing the amplitude of the system response to drivers at or near the respective resonance and peak-frequency response frequencies using a tuned mass damper.

A *tuned mass damper (TMD)* is a passive mechanical counterweight for a structure, consisting of a moving mass (roughly 1%–2% of the structure's mass) that is usually placed in the upper portion of the structure. The purpose of the TMD is to reduce the effects of motion caused by wind or earthquake. The first uses of TMDs in the United States for large structures were in the John Hancock Building in Boston in 1977 and in the Citicorp Center [Morgenstern 1995] in New York in 1978. Since then, many different styles, including active TMDs and pendulum TMDs, have been employed, while diverse applications have been found through retrofitting on large-span bridges and highways. Indeed, the current TMD exemplar is the 800-ton wind-compensating damper built into the center of the 508-meter-tall Taipei 101 in Taiwan [Haskett et al. n.d.], consisting of a huge spherical mass hung as a pendulum visible from the restaurant on the 88th and 89th floors. A very recent use of a TMD is in the building of the Grand Canyon Skywalk [Motioneering 2006]. As an example of the diversity of uses, TMDs are also used in the design of surgery tables to mitigate the vibrations of surroundings [Ming-Lai 1996].

In our engineering mathematics course, we introduce the notion of a TMD and model simple situations while addressing a few additional issues; but the topic can be used in any course that teaches systems of ODEs. Tying the mathematics to real applications—BIG buildings ARE real—interests the students and motivates their learning. Moreover, such approaches reinforce the mathematics of the moment and the power of mathematics to model worthwhile situations.

While engineering mathematics texts do not yet treat TMDs, the practicing engineering community has studied them for years and the engineering education community is bringing the study into a number of its courses, e.g., vibrations, structures, and dynamics courses [Koo et al. 2005].

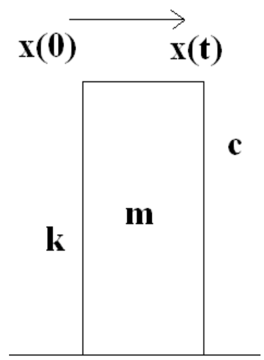


Figure 4. Structure modeled as a spring mass dashpot system with mass m, spring constant or stiffness k representing a restoration force, and damping coefficient c representing loss of energy. At the top of the structure, the initial displacement is $x(0)$ and general displacement is $x(t)$ at time t seconds.

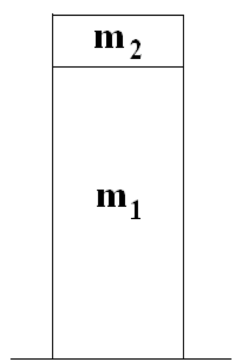


Figure 5. Structure of mass m_1 with additional mass m_2 mounted atop in preparation for design of tuned mass damper.

We examine a structure depicted in **Figure 4** and consider the spring constant k as the stiffness of the structure, i.e., the ability of the structure to restore itself to vertical, and again units will be $\mathrm{N/m}$. We can discuss both situations, no damping or damping with a term c measured in $\mathrm{N \cdot s/m}$. For simplification in studying the impact of an external force $F(t)$ applied horizontally to the building, we assume no damping ($c = 0$) at first, so resonance can occur. *We want to show that we can stop this resonance* with the addition of an additional mass atop the structure as seen in **Figure 5**.

Figures 6 and **7** show configurations that model the two-mass system in which there is no damping in either mass and there is only damping in the added mass, respectively. We depict the configuration horizontally because it is easier to see and create free-body diagrams. The horizontal displacements in **Figures 6** and **7** correspond to horizontal displacements of mass m_1 and m_2 in the building presentation of **Figures 4** and **5**.

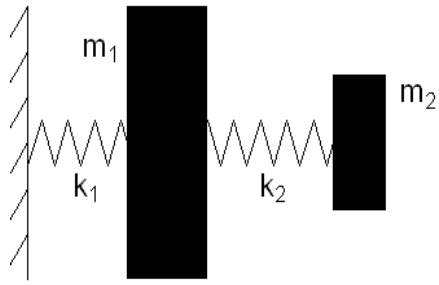


Figure 6. Horizontal depiction of two-mass spring system (no damping on either mass) with smaller mass m_2 serving as tuned mass damper.

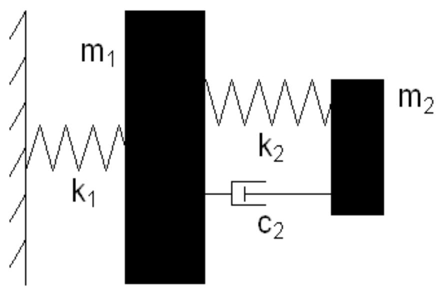


Figure 7. Horizontal depiction of two-mass spring system (damping on mass m_2) with smaller mass m_2 serving as tuned mass damper.

A TMD is modeled as a secondary spring-mass system appended to the primary one. From **Figure 6**, we construct a free-body diagram—this is a challenge for students. Using **Figure 6** and Newton's Second Law, we build the system of differential equations for our two-mass system, our TMD:

$$\begin{aligned} m_1 x_1''(t) + (k_1 + k_2)x_1 - k_2 x_2 &= F(t) \\ m_2 x_2''(t) + -k_2 x_1 + k_2 x_2 &= 0\,. \end{aligned}$$

In a similar manner, we construct a free-body diagram and the system of differential equations for **Figure 7**:

$$m_1 x_1''(t) + c_2 x_1'(t) - c_2 x_2'(t) + k_1 x_1(t) + k_2 x_1(t) - k_2 x_2(t) = F(t) \quad \textbf{(3)}$$

$$m_2 x_2''(t) - c_2 x_1'(t) + c_2 x_2'(t) - k_2 x_1(t) + k_2 x_2(t) = 0\,. \quad \textbf{(4)}$$

Usually, we study $F(t) = F_0 \cos(\omega t)$ to simulate earthquakes or wind.

Tuned Mass Dampers—Analysis

The real interest in studying the effects of TMDs on a structure is in minimizing the steady-state response—the long-term effect on the structure. We seek to minimize "swaying" or oscillations of mass m_1, especially excessive swaying, because of danger to the structure and discomfort to the occupants.

There are several cases:

- neither the structure nor the TMD has damping,
- the structure does not have damping but the TMD does, and
- both the structure and the TMD have damping.

Neither Structure Nor TMD Has Damping

We consider the case in which the governing equations for a driven structure of mass m_1 has a TMD of mass m_2 and neither has damping:

$$m_1 x_1''(t) + k_1 x_1(t) + k_2 x_1(t) - k_2 x_2(t) = F_0 \cos(\omega t), \tag{5}$$

$$m_2 x_2''(t) - k_2 x_1(t) + k_2 x_2(t) = 0\,. \tag{6}$$

Since our interest is in the steady state, we can take one of two routes. We can use

- real-valued conjectures for our steady state, determine the coefficients, and compute the amplitude of the steady state as a function of the input frequency ω, to see the effect that changing the input frequency has on our structure and TMD; or
- complex-valued conjectures for our steady state and perform a similar analysis.

Usually, engineers use the complex-valued approach; however, we pursue the real-valued approach. Thus, we conjecture the steady-state solutions as $x_1(t) = a\cos(\omega t) + b\sin(\omega t)$ and $x_2(t) = c\cos(\omega t) + d\sin(\omega t)$ and place these into **(5)** and **(6)**.

Let us assume the following parameters for the purpose of illustration: $m_1 = 10$ with $m_2 = 0.01m_1$, $k_1 = 90$ with $k_2 = 0.01k_1$, that is, the TMD has a mass of 1% of that of the structure.

The natural frequency of the structure is

$$\sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{90}{10}} = 3,$$

and the natural frequency of the TMD is the same,

$$\sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{0.90}{0.10}} = 3.$$

In this way, we obtain the following equations from **(5)** and **(6)**:

$$\begin{aligned}\text{mass } m_1: \quad & -\cos(t\omega) + \tfrac{909}{10}(a\cos(t\omega) + b\sin(t\omega)) \\ & \qquad -\tfrac{9}{10}(c\cos(t\omega) + d\sin(t\omega)) \\ & \qquad +10\left(-a\cos(t\omega)\omega^2 - b\sin(t\omega)\omega^2\right) = 0 \\ \text{mass } m_2: \quad & -\tfrac{9}{10}(a\cos(t\omega) + b\sin(t\omega)) + \tfrac{9}{10}(c\cos(t\omega) + d\sin(t\omega)) \\ & \qquad +\tfrac{1}{10}\left(-c\cos(t\omega)\omega^2 - d\sin(t\omega)\omega^2\right) = 0\,.\end{aligned}$$

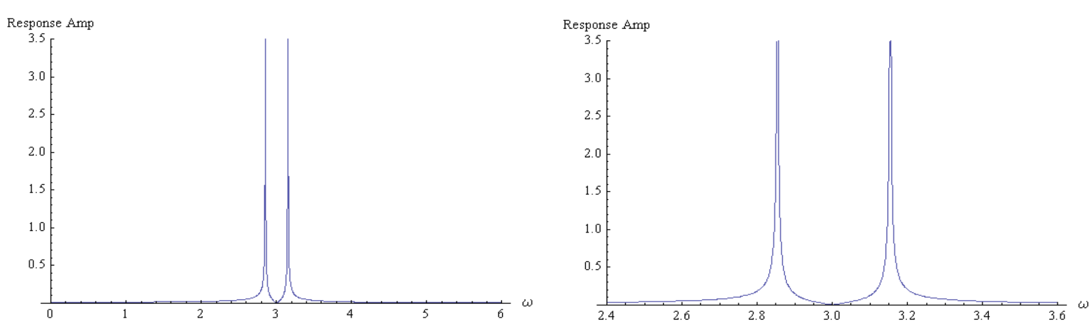
By comparing coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ in each of the equations for mass m_1 and mass m_2, we have four equations in the four unknowns a, b, c, d. We compute

$$a = -\frac{10\left(\omega^2 - 9\right)}{100\omega^4 - 1809\omega^2 + 8100}$$

and $b = 0$. This means that the amplitude of the steady-state solution for $x_1(t)$ is

$$\operatorname{amp}(\omega) = 10\left|\frac{\omega^2 - 9}{100\omega^4 - 1809\omega^2 + 8100}\right|\,.$$

In **Figure 8**, notice that there is *no* amplitude for the structure m_1 if the driver frequency is the same as that of the structure and the TMD's natural frequency of $\omega_0 = 3$. Also notice that there is a band of frequencies about $\omega = 3$ for which the response amplitude of the structure is relatively small.



(a) **(b)**

Figure 8. The case when neither the structure nor the TMD has a damper: Response amplitudes (in meters) of the structural mass m_1 vs. driver frequencies $F(t) = F_0\cos(\omega t)$. The two plots differ in horizontal scale.

Figure 8a shows a wide range for ω and **Figure 8b** shows a narrow range for ω. However, these give way to "large" amplitudes when the denominator of $\operatorname{amp}(\omega)$ is close to zero, that is, when ω is near 2.85 or 3.15.

Thus, the effect of employing a TMD whose natural frequency is the same as the structure's is to remove all oscillation in the structure when the structure is excited by a force with that frequency. Moreover, there is a frequency region (operating range) surrounding this natural frequency in which there is very little oscillation of the structural mass. The broader this operating range is, the better our system functions, for it means that the system is "safe" from large-response amplitudes over a wider range of frequencies. The spikes in **Figure 8** could cause trouble in our system; we could be in a "power up, power down" situation of system response as the input frequency passes through the two critical frequencies.

A natural question to ask is, "What if the size of the TMD's mass, m_2, is changed but the frequency of the TMD is kept the same as that of the structure, that is, we keep

$$\sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{90}{10}} = 3,$$

but increase (or decrease) m_2, changing k_2 appropriately?" We explore that situation in **Figure 9**. In all three cases, where added secondary mass m_2 (as a percentage of primary mass m_1) goes from 1%, to 2%, to 5%, (thin to thicker), we get total damping at the resonant frequency $\omega = 3$ by adding the TMD. However, the distance between peaks (the safe region about the resonant frequency of $\omega = 3$) is widened as we take the TMD mass, m_2, to be a greater percentage of the original mass, m_1.

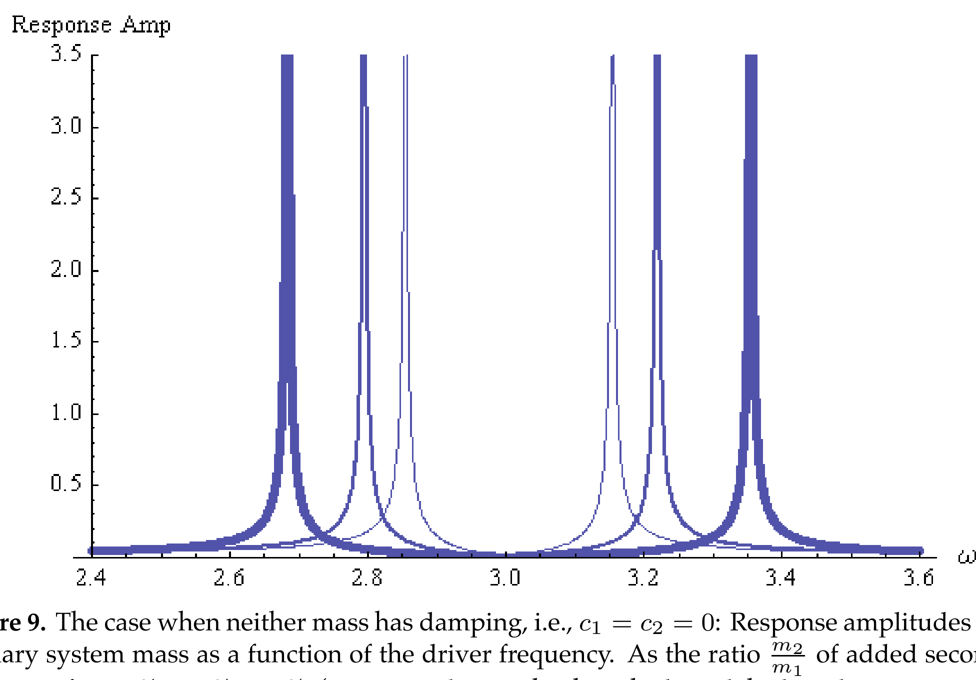


Figure 9. The case when neither mass has damping, i.e., $c_1 = c_2 = 0$: Response amplitudes of the primary system mass as a function of the driver frequency. As the ratio $\frac{m_2}{m_1}$ of added secondary mass goes from 1%, to 2%, to 5%, (corresponding to the thin, thick, and thicker plots, respectively), the frequency region of low responses to the driver frequency surrounding the natural frequency of the structure expands.

Only TMD Has Damping

Using **(3)** and **(4)** to model a system in which the structure is not damped while the TMD is damped (we use $c_2 = 0.01$), we conjecture a steady-state solution and proceed as in the previous subsection to find the amplitude of the structural mass's motion. This is shown in **Figure 10**. In this case, the maximum response for any input frequency ω is not infinite, as in the case of resonance ($c_1 = 0$); but the response does peak, and it is clear that at the structural mass's natural frequency of $\omega = 3$ the amplitude of the structural mass's displacement is lowest in the region of concern about the natural frequency of the structure. That is, the addition of a damped TMD to the structure helps reduce the oscillation of the structure considerably and never permits this oscillation to get too high in a region surrounding the natural frequency of the structure.

Here too we could alter m_2, the mass of the TMD, to be various percentages of m_1 and compare the response amplitudes; but we leave that for the reader to pursue.

What is presented in this case is not optimal tuning: See Koo et al. [2005] for directions on optimal tuning. Koo et al. [2005, 3], using the terminology "tuned vibration absorber (TVA)" rather than TMD, say:

> Adding damping in the TVA (damped TVA) and tuning it optimally reduces the two resonant peaks and broadens the isolation bandwidth. However, it occurs with a cost of sacrificing the isolation valley at the natural frequency.

We see the sacrifice in the isolation value in comparing **Figures 8** and **10**, for when the TMD with a damper is tuned less than optimally as we have offered, the structure does not have a zero response amplitude when the driver has the natural frequency of the structure.

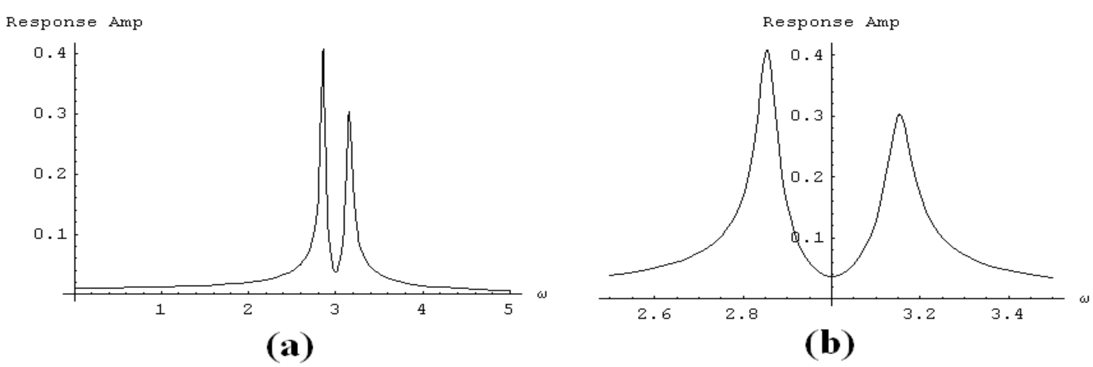


Figure 10. The case when only the TMD has a damper: Response amplitudes of the structural mass m_1 to driver frequencies $F(t) = F_0 \cos(\omega t)$. The two plots differ in horizontal scale.

Both Structure and TMD Have Damping

We do not consider the case where both structure and TMD are damped. Indeed, according to Koo et al. [2005, 3], "In order to tune a TVA in a damped system, one needs to apply a numerical technique to optimally tune the TVA or use a messy closed-formed solution [9] and [1]." (The citations are to Nishihara and Asami [2002] and Asami and Nishihara [2003].) This is beyond the scope of both our teaching intent and this paper.

Conclusion

Peak-frequency response is a damped system's counterpart to resonance and indicates just what frequency will excite a spring mass dashpot system the most and how much amplitude there can be due to that excitement. Tuned mass dampers are currently being used in new structural designs, as well as in retrofits to existing structures in place, to mitigate the vibrations imparted to the structure due to earthquakes and winds.

With little added material to traditional mathematics texts, the instructor can demonstrate the practicality and usefulness of differential equations in modeling real situations in structural design. We have shown how this can be done with theory and example. We encourage the reader to take the journey with students and explore these possibilities.

Reflection

The authors (one an engineer and one a mathematician) taught several sections of engineering mathematics together, several years ago at West Point. Each learned from the other—language and symbols, methods and rationale, and solution strategies used in the respective approaches. We gained a better understanding of each other's approaches because we taught together. We reaffirmed that teaching is learning.

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