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# Modelling mixing problems with differential equations gives rise to interesting questions

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# Modelling mixing problems with differential equations gives rise to interesting questions

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We discuss a variation of a routine salt mixing problem offered in a differential equations course. We further discuss problems which arise in the approach and solution, both professorial and student problems.

## 1. First differential equation model

Salt mixing problems form modelling experiences for students in their first differential equations. Most textbooks present these modelling opportunities with problems like the following [1, p. 48]:

> At time $t=0$ a tank contains $Q_0$ lb of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ lb of salt per gallon is entering the tank at a rate of 3 gal/min, and that the well-stirred solution is leaving the tank at the same rate. Find an expression for the amount of salt $Q(t)$ in the tank at time $t$.

Most students succeed in setting up the differential equation correctly using an 'input–output' model. Moreover, units need to be carefully observed, indeed, units make the modelling process very comfortable for students.

| rate of change of salt in lb/min | | rate of salt in in lb/min | | rate of salt out in lb/min |
|---|---|---|---|---|
| $\dfrac{dQ}{dt}\dfrac{\text{lb}}{\text{min}}$ | $=$ | $\dfrac{1}{4}\dfrac{\text{lb}}{\text{gal}}\times 3\dfrac{\text{gal}}{\text{min}}$ | $-$ | $\dfrac{Q(t)}{100}\dfrac{\text{lb}}{\text{gal}}\times 3\dfrac{\text{gal}}{\text{min}}$ |

The problem yields a separable first-order linear differential equation with initial condition

$$Q'(t)=\frac{3}{4}-\frac{3}{100}Q(t), \qquad Q(0)=Q_0$$

whose solution is

$$Q(t)=25(1-\exp(-0{\cdot}03t))+Q_0\exp(-0{\cdot}03t)$$

This problem serves to help students see the modelling process and to examine the reasonableness of the solution. If we know a value for $Q_0$, the amount of salt in the tank at time $t=0$, say $Q_0=4$, we can render a plot of $Q(t)$ (figure 1).

We can ask a number of questions concerning this model: What does the value of $Q(t)$ approach? Does this make sense? Why would we expect this? At what time will the tank contain 20 lb of salt?

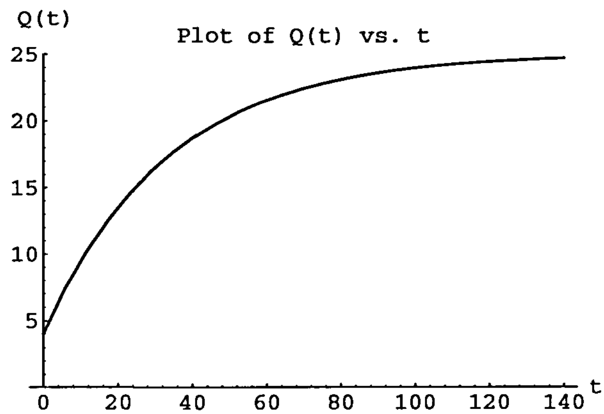


Figure 1. For original differential equation model with $Q_0 = 4$ we plot $Q(t)$, the amount of salt in the tank at time $t$.

## 2. Second model: Modification of first differential equation model

We can now extend this model to introduce the need for the integrating factor by having the volumetric flow 'in' differ from the volumetric flow 'out'. For example we consider the modified problem:

> At time $t = 0$ a tank contains $Q_0$ lb of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ lb of salt per gallon is entering the tank at a rate of 3 gal/min, and that the well-stirred solution is leaving the tank at the *rate of 3·5 gal/min*. Find an expression for the amount of salt $Q(t)$ in the tank at time $t$.

This then changes the modelling equation as follows:

$$\begin{array}{cccc} \text{rate of change} & \text{rate of salt in} & & \text{rate of salt out} \\ \text{of salt in lb/min} & \text{in lb/min} & & \text{in lb/min} \\ \dfrac{dQ}{dt}\dfrac{\text{lb}}{\text{min}} & = \dfrac{1}{4}\dfrac{\text{lb}}{\text{gal}} \times 3\,\dfrac{\text{gal}}{\text{min}} & - & \dfrac{Q(t)}{(100 - 0{\cdot}05t)}\dfrac{\text{lb}}{\text{gal}} \times 3\,\dfrac{\text{gal}}{\text{min}} \end{array}$$

Here we note that the volume of the tank does not stay constant, but loses 0·05 gal/min due to the differences in volumetric flow rates.

Again, if we offer a value for $Q_0$, say $Q_0 = 4$, we can solve this differential equation using an integrating factor or a package such as *Mathematica* to produce a solution:

$$Q(t) = 30 - 2{\cdot}6 \times 10^{-11}(-100 + 0{\cdot}5t)^6 - 0{\cdot}15t$$

Further we can render a plot of $Q(t)$ in this case (figure 2).

We ask a number of questions concerning this model: What does the value of $Q(t)$ approach? Does this make sense? Why would we expect this? When will the tank contain 20 lb of salt? 10 lb of salt? What is the maximum amount of salt the tank ever contains? What is the minimum amount of salt the tank ever contains? What happens after time $t = 200$ min?

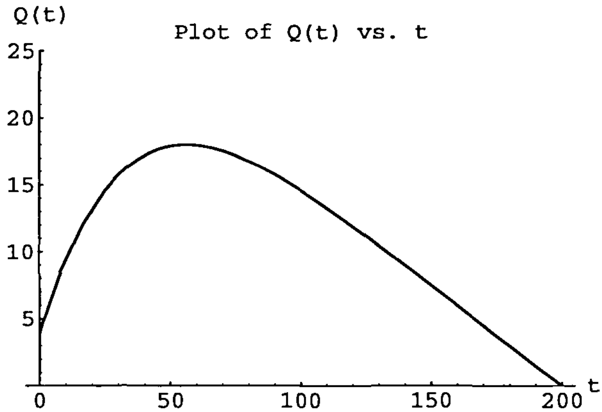


Figure 2. For modified differential equation model with $Q_0 = 4$ we plot $Q(t)$, the amount of salt in the tank at time $t$.

## 3. Third model: Modification of first differential equation model

On a recent examination in a differential equations course we gave the following modification of the mixture problem:

> A 1200 gal tank has 400 gal of water in it containing exactly 20 lb of salt at the start of the process.
> Salt water is being added to the tank. The concentration of the salt in the water being added is 0·3 lb/gal. This salt water mixture initially is added at a rate of 0·1 gal/s and this rate is steadily increased by 0·02 gal/s each second.
> The water in the tank is presumed to mix instantaneously and this mixture is being pumped out of the tank at the rate of 5 gal/s.
> Let $y(t)$ be the amount of salt in the tank at time $t$ (s).
>
> (*a*) Describe the rate at which the salt water is being added to the tank as a function of time $t$ in s.
> (*b*) Write out a differential equation with initial condition which describes the rate of change of salt in the tank, i.e. $y'(t)$.
> (*c*) Determine the amount of salt in the tank at time $t$ for the first 500 s of operation.
> (*d*) Plot the amount of salt in the tank as a function of time $t$.
> (*e*) What is the minimum amount of salt in the tank during the first 500 s of operation?

### 3.1. *Confession time*

In our key to the examination for the above problem we gave the following differential equation:

$$y'(t) = 3(0{\cdot}1 + 0{\cdot}02t) - \frac{5y(t)}{400 - 5t + (0{\cdot}1 + 0{\cdot}02t)t}$$

implying that the tank has volume $400 - 5t + (0{\cdot}1 + 0{\cdot}02t)t$ at time $t$—*which is wrong*! Over half the students in the course, a course laden with very sharp students, produced this error as well. The water flows out at a rate of 5 gal/s making the volume at time $t$ actually $5t$ gal less than the original 400 gal. Both the professor and a good

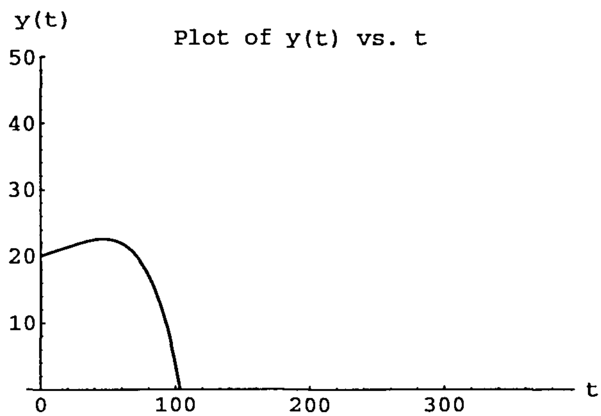


Figure 3. Plot of interpolating function solution to the third model. A plot of $y(t)$, the amount of salt in the tank at time $t$.

number of students missed the fact that to determine the gain in the volume due to the increasing rate of inflow of water, i.e. $0{\cdot}1+0{\cdot}02t$ gal/min, one *does not multiply* this rate by time, but rather one integrates this rate from 0 to $t$. We all learned a good lesson in this regard. About 1/3 of the students integrated as they should. This gives the correct differential equation using the appropriate volume of water at time $t$ expression:

$$y'(t)=3(0{\cdot}1+0{\cdot}02t)-\frac{5y(t)}{400-5t+(0{\cdot}1t+0{\cdot}01t^2)}$$

### 3.2. *Back to the third model*

The technique of using integrating factors does not lead to a solution and we use the numerical capabilities of the NDSolve command in *Mathematica*

```
sol=NDSolve[{y'[t]==.3(.1+.02t)-5y[t]/(400+(0.1t+.01t^2)-5t),
        y[0]==20}, y[t],{t,0,500}]
```

to obtain an InterpolatingFunction[1] solution:

```
{{y[t] -> InterpolatingFunction[{0., 386.51}, <>][t]}}
```

The fact that we asked for a solution for the first 500 s and got only a solution in the interval $[0, 386{\cdot}51]$ should clue us to the fact that something may be different about this variation.

*Mathematica* can plot such interpolated solutions and we show the plot the students gave on the examination (figure 3).

Some students said that somewhere around 100 s all the water ran out and 'the model then became unrealistic'. End of problem! Still others found the precise time where $y(t)$ became 0, i.e. $t=103{\cdot}49$ by utilizing the FindRoot command of *Mathematica* on the interpolating function solution for $y(t)$:

```
FindRoot[y[t]==0,{t,100}]
```

but then went no further except to fall back on the observation that 'the amount of salt cannot become negative after $t=103{\cdot}49$ sec so that is all there is to this problem'.

[1] *Mathematica* solves the differential equation numerically and offers an interpolating function which may be plotted, differentiated, evaluated, and used in further numerical work.

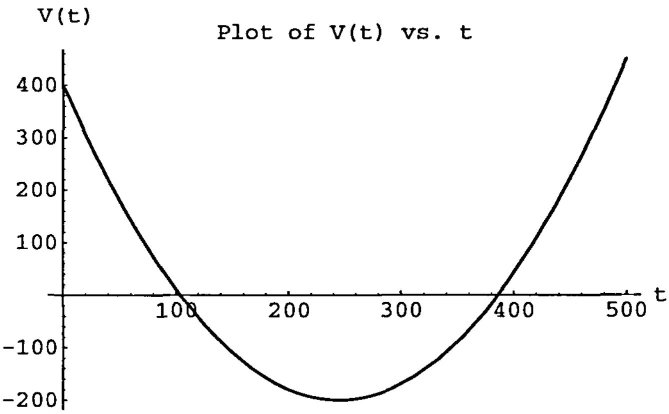


Figure 4. Plot of $V(t)$, the volume of water in the tank at time $t$.

A small number of students noted the volume of water in the tank, $V(t)=400+(0{\cdot}1t+0{\cdot}01t^2)-5t$, and plotted $V(t)$ (figure 4). From here a few students proceeded with a very nice analysis which we discuss below.

In view of our grading error and the difficulty of this problem, we gave the students a chance to do this problem for homework. A good number of them got the modification to the differential equation correct, and even tried to interpret the plot in fig. 3. But of those who plotted $V(t)$, many said that the tank would start filling up again at time $t=386{\cdot}51$. This means they interpreted the graph to say that the volume, $V(t)$, would first become 0 at time $t=103{\cdot}49$ and then again at $t=386{\cdot}51$. What did they think happened in the time interval $[103{\cdot}49, 386{\cdot}51]$? A number of them said that during the time interval $[103{\cdot}49, 386{\cdot}51]$ the volumetric flow *in* was less than the volumetric flow *out* and thus nothing, neither water nor salt, accumulated in the tank.

A good number of the students realized that from time $t=103{\cdot}49$ onward, *until the volumetric flow rate, $V'(t)$, became positive*, the tank would stay empty, but when $V'(t)$ became positive then the tank would start to fill up again. These students determined that $V'(t)=0$ when $t=245{\cdot}0$ s. This means that we now have another equation for the volume of water in the tank after 245 s and hence another differential equation to govern the amount of salt in the tank after 245 s. Now since the flow rate in is increasing at a rate of $0{\cdot}02(t-245)$ gal/s at $t>245$, the new volume of water at time $t>245$ is $0{\cdot}1(t-245)^2$ gal. Thus the new differential equation is

$$y'(t)=3(0{\cdot}1+0{\cdot}02t)-\frac{5y(t)}{0{\cdot}01(t-245)^2}\quad y(245)=0$$

Using *Mathematica*'s NDSolve command

```
sol=NDSolve[{y'[t]==.3 (.1+.02t)-5y[t]/(.01(t-245)^2),
        y[245.1]==0}, y[t],{t,245.1,500}]
```

we obtain a solution for $y(t)$, the amount of salt in the system in the time interval $[245{\cdot}1, 500]$.[2] Now when we combine the plot of the amount of salt in the tank in the

[2] Notice that at $t=245{\cdot}0$ the value of $y'(t)$ is infinite and thus the attempt to approximate this with an initial time of $t=245{\cdot}1$ instead of $t=245$.

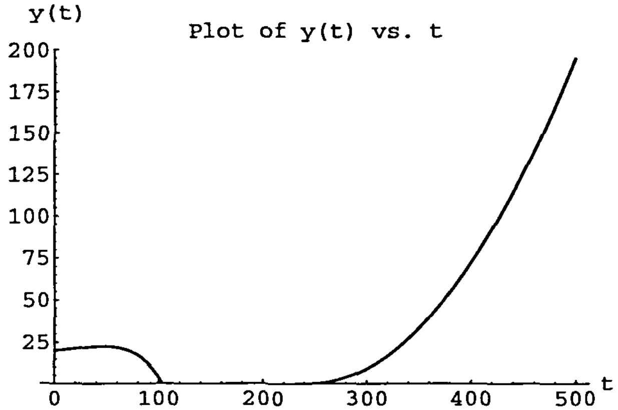


Figure 5. Combined plot of interpolating function solutions to the two parts of the third model. A plot of $y(t)$, the amount of salt in the tank at time $t$.

time intervals $[0, 103{\cdot}49]$ and $[245{\cdot}1, 500]$, noting that no water, and hence no salt, was in the tank in the time interval $[103{\cdot}49, 245{\cdot}1]$, we see a plot of the amount of salt in the tank over the entire time interval *[0, 500] in* figure 5.

From the interpolating solutions to the differential equations (first and second parts) and the graph in figure 5 we can answer all the parts in the original question.

Several students pointed out that we did not need a differential equation for $t > 245$. Indeed, the only thing which is happening is that water containing 0·3 lb of salt per gallon is coming in at a rate of $0{\cdot}01\ (t-245)$ and thus the actual amount of salt in the tank at time $t > 245$ is

$$\int_{245}^{t} (0{\cdot}3)0{\cdot}01(\tau-245)\mathrm{d}\tau = 0{\cdot}003(t-245)^2.$$

Again a plot of this function over the region yields the same result as figure 5.

## 4. Conclusion

We have offered up a review of the value of mixing problems for teaching modelling with differential equations and the increased complexities of models. We have indicated pitfalls and points of interest in the solution and interpretation process. Finally, we have indicated the power of *Mathematica* in enabling students to obtain solutions they can interpret.

## Reference

[1] BOYCE, W. E., and DIPRIMA, R. C., 1986, *Elementary Differential Equations and Boundary Value Problems*, Fourth Edition (New York: John Wiley & Sons)..