Personal Research Enhances One's Own Mental Condition and Relationships with Students

PROF Brian J. Winkel, Department of Mathematical Sciences,
United States Military Academy, West Point NY 10996 USA

Mathematica Militaris (2004) 14(3): 14-18

INTRODUCTION

The topic for this issue is "Research vs Teaching: How Do They Reinforce Each Other?" I have one research topic I pick up every once in a while that accomplishes several goals in my personal and professional life. First it engages my thinking and second it interests students. I have used this topic some 5 or 6 times over a 20 year period to get students to discover and see how to do research for themselves. I have not yet published any results in the area. This is an easy topic to get into and to explore and I have no trouble "snaring" students into considering it, nor do they have trouble seeing patterns, asking questions, or getting results.

I call this personal research because the issues are on the order of "What is going on here?" "What are these things like?" "What might we do with them?" "How pretty are they?" I know of no applications, I have only seen one article [2] on this topic and even that author knows little about its history or applications.

I have found that students are immediately captivated by the topic and I have had great success in engaging their minds to discover patterns and principles in many individual directed (or non-directed!) studies over the years. I once had a team of three students buzzing over the material and just recently I had a bright student examining possibilities [1]. In this most recent round I discovered things I did not know about the phenomena.

So what is this stuff and what is so engaging about it? Well, it is geometric so it permits pictures. That helps! It is about familiar functions. That helps! It has a tactile sense. That helps! It is easy to explain and one can quickly get feedback and graphical outputs for stimulation. That really helps!

THE STUFF - BIPOLAR FUNCTION PLOTS

"Bipolar function plots" consists of a process to produce a plot. See Figure 1 (from [1]).

(1) Set two poles, $P_1 = (0, 0)$ and $P_2 = (1, 0)$.
(2) Consider a function $s = f(t)$.
(3) At pole P_1 construct a ray at the angle of t radians measured counterclockwise from the x-axis and at pole P_2 construct a ray at the angle of $s = f(t)$ radians measured clockwise from the x-axis.
(4) Now find the point of intersection $P(t) = (x(t), y(t))$ of these two rays. This point is the parametric value of the Bipolar Function and the collection of such points is the bipolar function plot for the function $s = f(t)$.

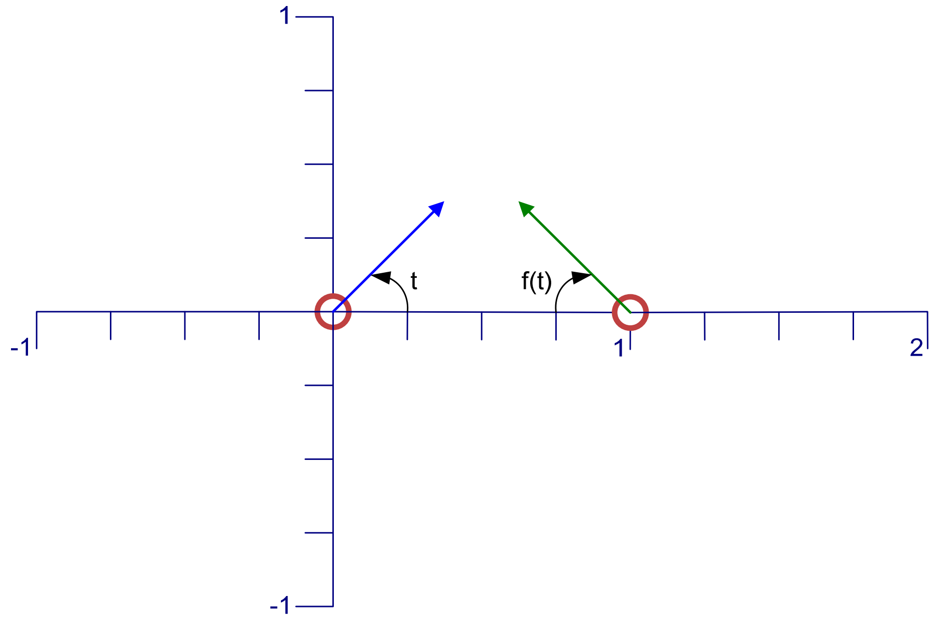

Figure 1. Generating Bipolar Function $s = f(t)$.

Immediately students wonder what obvious functions like $f(t) = t$ or $f(t) = m\,t + b$ or $f(t) = \sin(3t)$ will produce as a Bipolar function plot. Indeed, $f(t) = t$ is obvious – see Figure 2.

Over the years I let the students discover appropriate ways to represent this process and they come to the conclusion that the point $P(t) = (x(t), y(t))$ has a nice parametric equation representation. They do the geometry, they build the equations. I give them nothing. Here is what they obtain:

$$x(t) = \frac{\sin(f(t)\cos(t)}{\sin(t + f(t))} \quad y(t) = \frac{\sin(f(t)\sin(t)}{\sin(t + f(t))} \qquad (1)$$

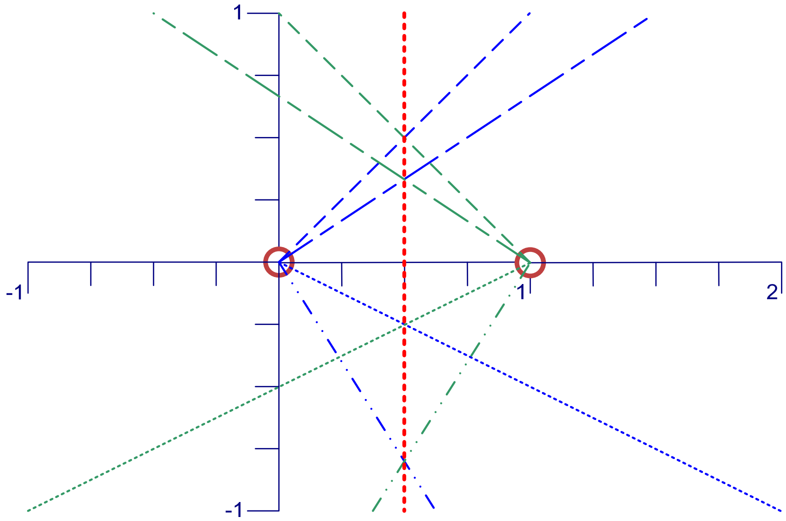

Figure 2. Bipolar function plot for f(t) = t. From [1].

Once they have the representation of the bipolar function as a set of parametric equations offered in Equations (1) then it is easy to use technology to render plots and begin experimenting. I have used both *MathCad* and *Mathematica* and the animation features in both to explore families of functions, e.g., f(t) = *a* t to study the effects of changing parameter *a* on the bipolar plots.

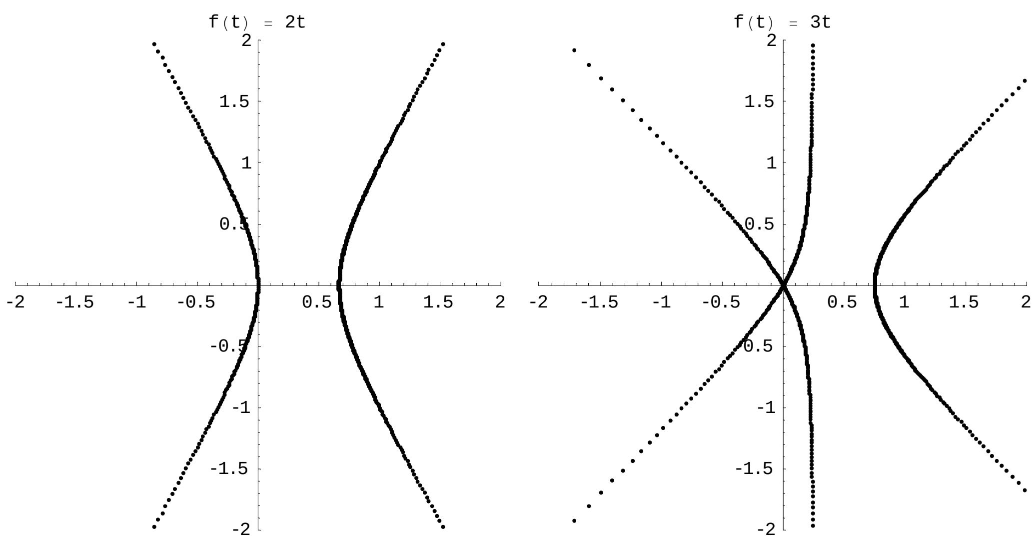

Figure 3. Bipolar plots (left) f(t) = 2t and (right) f(t) = 3t.

Figure 3 shows the plots using *Mathematica*'s ListPlot command in which "connect-a-dot" action is not in force, while Figure 4 shows *Mathematica*'s ParametricPlot routine in which the connect-a-dot feature indicates the asymptotes.

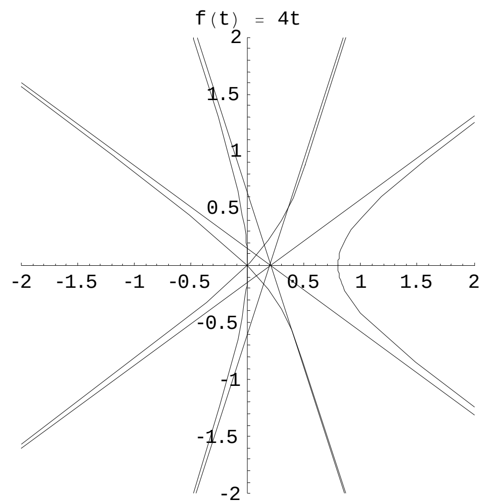

Figure 4. Bipolar plot f(t) = 4t.

You should be able to see some pattern emerging in just these three plots. Students do! Issues emerge right away in just the study of the family f(t) = n t, n = 1, 2, 3, ... Where does this asymptotic behavior come from? Can we predict the number of asymptotes and where they will occur? Do the asymptotes intersect – they appear to do so in Figure 4 – and if so where? Students often will move slowly but an occasional student will want to move to f(t) = m t + b right away. See Figure 5.

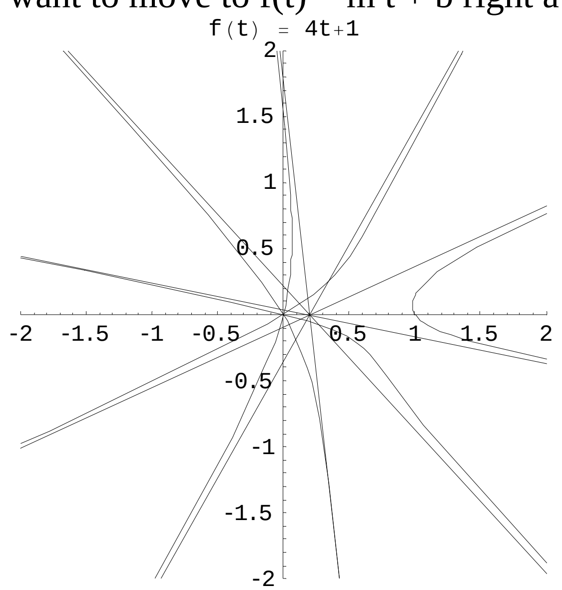

Figure 5. Bipolar plot f(t) = 4t+1.

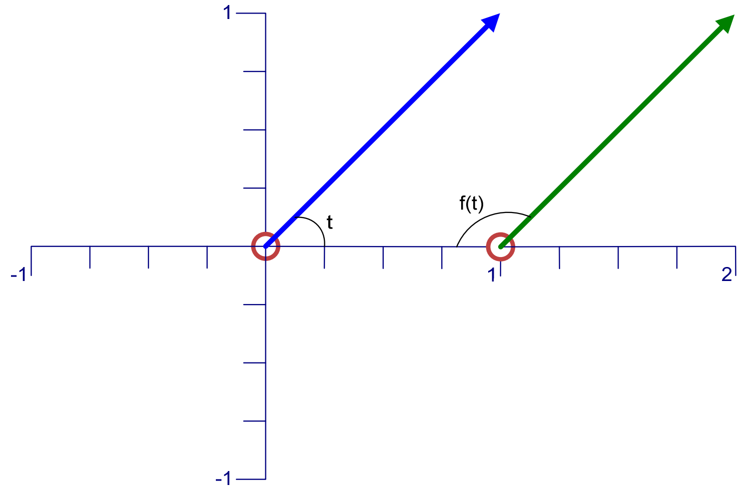

Figure 6. Diagram showing when asymptotes will occur,
i.e when angle t and angle f(t) are supplementary.

This research is good practice in visualization and geometry to note when asymptotes will occur. From Figure 6 we see that asymptotes will occur when we have f(t) + t = π, actually f(t) + t = n π, where n = 1, 2, 3, . . .

One can study entire families of functions, e.g., f(t) = cos(n t), n = 0, 2, 3, . . . ; f(t) = t^n; f(t) = m t + b; as well as parametric families F(t,s) = $s^2 + t^2$ =25.

INVERSE PROBLEMS

Another rich area of exploration is to “predict” what relationship between the bipolar angles, t and s = f(t), or more generally, what relationship F(s,t) = 0 would produce a given shape when used in the bipolar sense. I had never thought of this until one year a student simple asked, “I wonder what function would produce a circle as a bipolar function plot.” One can use solvers and implicit plot routines in *Mathematica* to produce the circle of radius 0.1 with center (0.5, 1) as a bipolar function plot (right in Figure 7) when the relationship between bipolar angles t and s is given by the implicit function seen in the left frame of Figure 7.

In Figure 7 the point (1.05, 1.09) as a parameter (t, s) will produce the point on the far upper left portion of the bipolar function plot image circle while the point (1.05 , 1.20) as a parameter (t, s) will produce the point on the far upper left portion of the image circle.

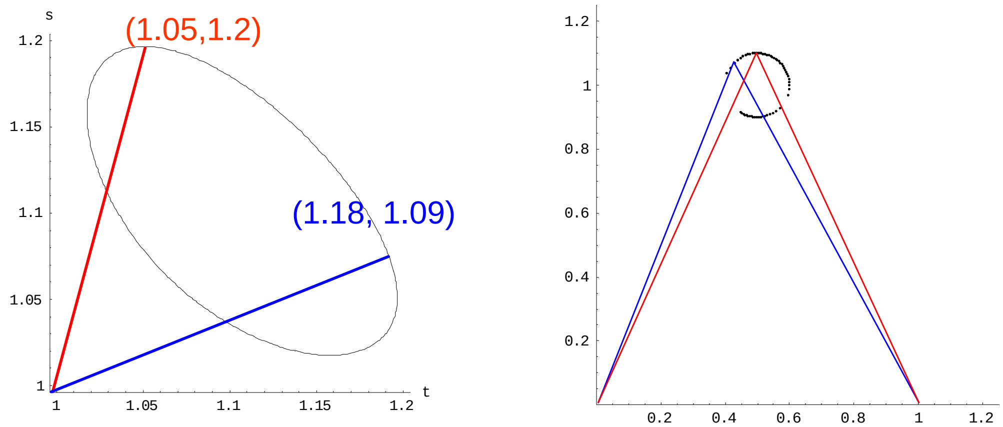

Figure 7. Angles (t,s) in the relation F(t,s) = 0 depicted on the left will produce the circle of radius 0.1 centered at (0.5, 1.0) as a bipolar plot.

RESEARCH VALUE

Students discover all these things and many more. They conjecture, they make lists of observations, they see the difference between (for n small or large) observations and a proof for all values of n, and they ask questions. For example, “What does the bipolar derivative, f’(t), mean?” “How many asymptotes, petals, relative max/mins, etc. are there for a given function?”

I am compiling a catalog of events, of results, of observations proved, of curves, and of questions and unsolved issues. Each time I introduce students to the subject I let them discover the elementary principles for themselves and in every instance they go in their own direction discovering new things. I produce tools to enable student discovery, some useful and some not so useful, e.g., projecting plots from the plane on to the sphere so that asymptotes will map to finite curves through the north pole of the sphere. See Figure 8.

This area of investigation, bipolar functions, gives students a taste of research as they wander the geometric, algebraic, and calculus landscape in search of interesting things to study, of principles and general statements to extract, and of families of functions to organize and relate.

If you have something you have always wondered about then share it with your students as a joint research effort. Both you and they will grow because of the sharing.

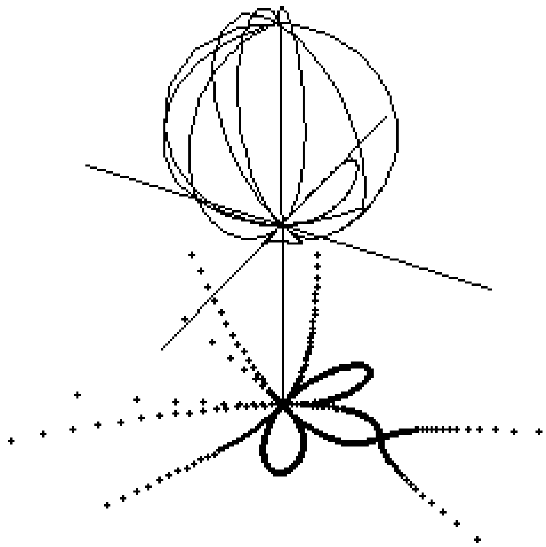

Figure 8. Depiction of the bipolar function f(t) = cos(3t) in on the x-y plane and its projection onto a sphere raised several units above the plane of f(t) = cos(3t) for clarity.

ACKNOWLEGEMENT

Figures 1 - 7 used in this article come from the final presentation of Ann Millen, a senior cadet at West Point, in her Senior Seminar MA491 directed effort which I supervised.

REFERENCES

1. Millen, Ann. 2004. Patterns in Bipolar Parametric Plots -- Projects Day Presentation, 5 May 2004. West Point NY: US Military Academy.
2. Nelson, David. 1994. Bipolar Coordinates and Plotters. *PRIMUS*. 14(1): 77-83.