Analytical Determination of Shock Response Spectra, for a Proportionally Damped System

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1 Executive Summary

Many military systems must be capable of sustained operation in the face of mechanical shocks due to projectile or other impacts. The most widely used method of quantifying a system's vibratory transient response to shock loading is called the shock response spectrum (SRS). The system response for which the SRS is to be determined can be due, physically, either to a collocated or to a noncollocated shock loading. Taking into account both possibilities, one can define the SRS as follows: the SRS presents graphically the maximum transient response (output) of an imaginary ideal mass-spring-damper system at one point on a flexible structure, to a particular mechanical shock (input) applied to an arbitrary (perhaps noncollocated) point on the structure, as a function of the natural frequency of the imaginary mass-spring-damper system. For a response point sufficiently distant from the impact area, many army platforms (such as vehicles) can be accurately treated as linear systems with proportional damping. In such cases the output due to an impulsive mechanical-shock input can be decomposed into exponentially decaying sinusoidal components, using normal-mode orthogonalization. Given a shock-induced loading comprising such components, this paper provides analytical expressions for the various common SRS forms. The analytical approach to SRS-determination can serve as a verification of, or an alternative to, the numerical approaches in current use for such systems. No numerical convolution is required, because the convolution integrals have already been accomplished analytically (and *exactly*), with the results incorporated into the algebraic expressions for the respective SRS forms.

2 Introduction

Modern warfare calls for many military systems to be capable of sustained operation under extreme environmental conditions. The mechanical shocks from such sources as blast-waves and projectile impacts, and even vehicular motion over rough terrain, make high demands on military equipment. For design and analysis purposes, the vibratory, transient response of systems (or of system models) subject to mechanical shock is typically captured using two frequency-dependent ("spectral") tools [1, 2]: (1) the Fourier spectrum; and (2) the shock response spectrum (SRS).

A two-dimensional SRS [1] (typically termed simply an SRS) represents graphically the frequency content of a specified shock input $d(t)$ in terms of the maximum response $x(t)$ it would induce in a hypothetical, single-degree-of-freedom (SDOF) mass-spring-damper (MSD) system, seismically subjected to the shock. (Refer to Figure 1.) The SRS plots a selected kinematic measure of the maximum time-domain motion-response

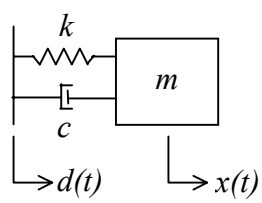

Figure 1. Hypothetical SDOF MSD system, for SRS determination

(of mass m) against the SDOF-system natural frequency, with the frequency varied over some range of interest. In producing the plot, both the shock input and the SDOF-system damping ratio are held constant across the entire range of natural frequencies. The shock input is typically a position input $[d(t)$ in Figure 1$]$. Some useful measures of the response include (1) relative displacement, $x(t)-d(t)$; (2) "pseudovelocity," $\omega_n[x(t)-d(t)]$; and (3) absolute acceleration, $\ddot{x}(t)$. As used in this report, the respective SRS's are designated and defined as follows [2]:

Spectral displacement (or relative-displacement SRS):

$$S_D(\omega_n) := |x(t)-d(t)|_{\max} . \tag{1}$$

Spectral velocity (or pseudovelocity SRS) [3, 4]:

$$S_V(\omega_n) := \omega_n \left| x(t) - d(t) \right|_{\max} = \omega_n S_D(\omega_n) . \tag{2}$$

Spectral acceleration (or absolute acceleration SRS):

$$S_A(\omega_n) := \left| \ddot{x}(t) \right|_{\max} . \tag{3}$$

Consider now the case of a mechanical shock $c(t)$ applied to an arbitrary point *C* of a generic system *S*. (Refer to Figure 2.) If a nonlinear finite-element model exists for *S*, the induced displacement $d(t)$ at some other point *D* on *S* can typically be determined numerically, and then used to calculate (again, numerically) the indicated spectral quantities at *D*. In general, though, *analytical* evaluation of S_D, S_V, or S_A is not possible for nonlinear systems. However, for a response point sufficiently distant from the impact area, many army platforms (e.g., vehicles) can be accurately treated as linear systems with proportional damping. In such cases analytical SRS determination proves possible.

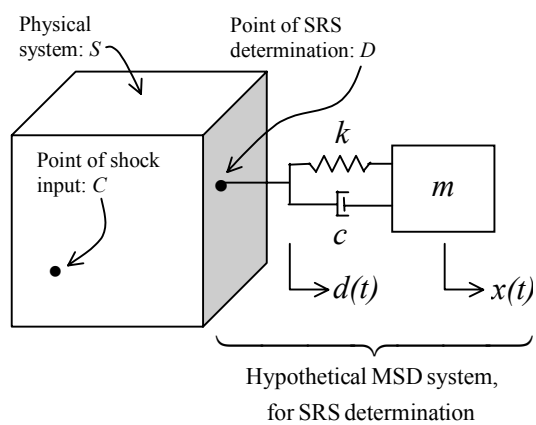

Figure 2. Shock-loaded system *S*, with (hypothetical) attached MSD system for SRS determination

Consider the typical case of an undamped or underdamped, linear system, subjected to a mechanical-shock disturbance, at point *C*. In such cases, analysis by the method of modal superposition yields an expression for the induced displacement $d(t)$ (at *D*) as a linear combination of modal coordinates [5, 6]. These coordinates are the solutions to their respective decoupled scalar differential equations. The inputs to these scalar equations are modal forces, which are themselves scalar multiples of the mechanical-shock disturbance. If the mechanical shock can be approximated as an impulse, the induced displacement $d(t)$ will be a linear combination of damped sinusoids. Denoting the ith sinusoid by $d_i(t)$, one can express the displacement at *D* in the following form:

$$d(t) = \sum_{i=1}^{\nu} d_i(t), \tag{4}$$

where
$$d_i(t) = D_i e^{-\alpha_i t} \sin(\omega_i t + \phi_i) u_{-1}(t). \tag{5}$$

This displacement $d(t)$ serves as the input to the SDOF MSD system (see Figure 2). For the ith component $d_i(t)$, the constants D_i, α_i, ω_i, and ϕ_i represent the input amplitude, rate of exponential decay, oscillation frequency, and phase shift, respectively. Each component begins at time $t = 0$ (the time of impact), as indicated by the unit step function $u_{-1}(t)$. As documented below, if one assumes the induced displacement $d(t)$ to be given by Equations (4) and (5), one can find an analytical expression for the relative displacement

$$\delta(t) = x(t) - d(t), \tag{6}$$

and for the absolute acceleration $\ddot{x}(t)$, to use in evaluating $S_D(\omega_n)$, $S_V(\omega_n)$, and $S_A(\omega_n)$ for any ς in the range $0 \le \varsigma < 1$. These analytical expressions can either be used to plot the respective spectral quantities, or to check the plots found using alternative evaluation methods.

3 Problem Statement

Consider the linear, SDOF MSD system shown in Figure 1. Assume the displacement $d(t)$ at point D to comprise a linear combination of ν exponentially decaying sinusoidal inputs (i.e., to the SDOF MSD system), as given by Equations (4) and (5). The objectives of this research effort are to develop analytical forms for the spectral quantities $S_D(\omega_n)$, $S_V(\omega_n)$, and $S_A(\omega_n)$, as defined by Equations (1), (2), and (3), respectively. The general case ($0 \le \varsigma < 1$) will be considered first (Section 3), and then the special case of $\varsigma = 0$ (Section 4).

4 Solution for the general case ($0 \le \varsigma < 1$)

4.1 Basic approach

The system differential equation of motion (DEOM) can be expressed by

$$m\ddot{x} + c\dot{x} + kx = c\dot{d} + kd\ . \tag{7}$$

In standard form (i.e., in terms of the damping ratio ς and the natural frequency ω_n), the DEOM is

$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = 2\varsigma\omega_n\dot{d} + \omega_n^2 d\,. \tag{8}$$

Since $d(t)$ is known [Eqs. (4), (5)], one can find the relative displacement by solving Equation (8) for $x(t)$. Then Equations (1), (2), and (3) can be used to formulate the respective shock response spectra.

4.2 Position response $x(t)$ to a generic induced displacement $d(t)$

Consider first the response $x(t)$ of the SDOF MSD system to an arbitrary (assumed Laplace-transformable) induced disturbance $d(t)$. Let the Laplace transforms of $x(t)$ and $d(t)$ be represented by $X(s)$ and $D(s)$, respectively. Then the Laplace transform of Equation (8) is

$$\begin{aligned} s^2X(s) - sx(0+) - \dot{x}(0+) + 2\varsigma\omega_n\left[sX(s) - x(0+)\right] + \omega_n^2 X(s) \\ = 2\varsigma\omega_n\left[sD(s) - d(0+)\right] + \omega_n^2 D(s)\,. \end{aligned} \tag{9}$$

Rearrangement yields

$$\begin{aligned} X(s) = \frac{2\varsigma\omega_n s + \omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2} D(s) + \frac{s + 2\varsigma\omega_n}{s^2 + 2\varsigma\omega_n s + \omega_n^2} x(0+) \\ + \frac{1}{s^2 + 2\varsigma\omega_n s + \omega_n^2}\dot{x}(0+) - \frac{2\varsigma\omega_n}{s^2 + 2\varsigma\omega_n s + \omega_n^2} d(0+)\,. \end{aligned} \tag{10}$$

Taking the inverse Laplace transform of Equation (10), one obtains the following expression for the displacement response, in terms of induced displacement input $d(t)$:

$$\begin{aligned} x(t) = e^{-\varsigma\omega_n t}\left(\cos\omega_d t + \frac{\varsigma}{\sqrt{1-\varsigma^2}}\sin\omega_d t\right) x(0+) \\ + e^{-\varsigma\omega_n t}\left(\frac{1}{\omega_d}\sin\omega_d t\right)\dot{x}(0+) - e^{-\varsigma\omega_n t}\left(\frac{2\varsigma}{\sqrt{1-\varsigma^2}}\sin\omega_d t\right) d(0+) \\ + \frac{\omega_n}{\sqrt{1-\varsigma^2}} e^{-\varsigma\omega_n t}\sin(\omega_d t + \phi) * d(t)\,, \end{aligned} \tag{11}$$

where
$$\phi = \tan^{-1}\left(\frac{2\varsigma\sqrt{1-\varsigma^2}}{1-2\varsigma^2}\right), \tag{12}$$

and
$$\omega_d = \omega_n\sqrt{1-\varsigma^2}\ . \tag{13}$$

The asterisk indicates convolution, defined for arbitrary functions f and g as follows:

$$f(t) * g(t) := \int_{0+}^{t} f(\tau)g(t-\tau)d\tau\ . \tag{14}$$

To simplify, recall the relationship

$$A\cos\theta + B\sin\theta = \sqrt{A^2+B^2}\ \sin\left(\theta + \tan^{-1}\frac{A}{B}\right). \tag{15}$$

Define now the following variables, for convenience:

$$A := x(0+), \tag{16}$$

$$B := \frac{\varsigma x(0+)}{\sqrt{1-\varsigma^2}} + \frac{\dot{x}(0+)}{\omega_d} - \frac{2\varsigma d(0+)}{\sqrt{1-\varsigma^2}}, \tag{17}$$

$$\beta = \sqrt{A^2+B^2}\ , \tag{18}$$

$$\psi = \tan^{-1}\left(\frac{A}{B}\right). \tag{19}$$

and
$$\gamma = \frac{\omega_n}{\sqrt{1-\varsigma^2}}\ . \tag{20}$$

Using Equation (15), Equation (11) can now be re-expressed as follows:

$$x(t) = \beta e^{-\varsigma\omega_n t}\ \sin(\omega_d t + \psi) + \gamma\, e^{-\varsigma\omega_n t}\ \sin(\omega_d t + \phi) * d(t)\ . \tag{21}$$

4.3 Convolution term, for modal (damped-sinusoid) induced-disturbance component $d_i(t)$

To proceed it will be necessary to evaluate the convolution term of Equation (21) for an exponentially decaying sinusoid $d_i(t)$ [Eq. (5)]. Upon expanding the integrand; integrating [7]; simplifying by means of the trigonometric addition formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{22}$$

and

$$\cos\ (\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta\,; \tag{23}$$

and making the substitutions

$$\mu_i = \alpha_i - \varsigma\omega_n\,, \tag{24}$$

$$\nu_i = \omega_d - \omega_i\,, \tag{25}$$

and

$$\rho_i = \omega_d + \omega_i\,; \tag{26}$$

one obtains the following expression for the convolution term:

$$\begin{aligned}
&\gamma\, e^{-\varsigma\omega_n t}\sin(\omega_d t + \phi) * d_i(t) \\
&\quad = \gamma D_i \int_{0+}^{t} e^{-\varsigma\omega_n \tau}\sin(\omega_d \tau + \phi)\, e^{-\alpha_i(t-\tau)}\sin[\omega_i(t-\tau) + \phi_i]\,d\tau \qquad (27a)\\
&\quad = \gamma D_i e^{-\alpha_i t}\Bigg\langle \sin(\omega_i t + \phi_i)\Bigg\{ \frac{e^{\mu_i t}[\mu_i \sin(\nu_i t + \phi) - \nu_i \cos(\nu_i t + \phi)] + \nu_i}{2(\mu_i^2 + \nu_i^2)} \\
&\qquad\qquad + \frac{e^{\mu_i t}[\mu_i \sin(\rho_i t + \phi) - \rho_i \cos(\rho_i t + \phi)] + \rho_i}{2(\mu_i^2 + \rho_i^2)}\Bigg\} \\
&\qquad - \cos(\omega_i t + \phi_i)\Bigg\{ \frac{e^{\mu_i t}[\mu_i \cos(\nu_i t + \phi) + \nu_i \sin(\nu_i t + \phi)] - \mu_i}{2(\mu_i^2 + \nu_i^2)} \\
&\qquad\qquad - \frac{e^{\mu_i t}[\mu_i \cos(\rho_i t + \phi) + \rho_i \cos(\rho_i t + \phi)] - \mu_i}{2(\mu_i^2 + \rho_i^2)}\Bigg\} \qquad (27b)
\end{aligned}$$

Define now the following variables, for convenience:

$$\kappa_{1i} = \frac{\mu_i}{2(\mu_i^2 + \nu_i^2)}\,, \tag{28}$$

$$\kappa_{2i} = \frac{\mu_i}{2(\mu_i^2 + \rho_i^2)}\,, \tag{29}$$

$$\kappa_{3i} = \frac{\nu_i}{2\left(\mu_i^2 + \nu_i^2\right)}, \tag{30}$$

and

$$\kappa_{4i} = \frac{\rho_i}{2\left(\mu_i^2 + \rho_i^2\right)}. \tag{31}$$

Upon substituting from Equations (28) through (31), Equation (27) can be simplified to the following form:

$$\begin{aligned}
&\gamma\, e^{-\varsigma\omega_n t} \sin\left(\omega_d t + \phi\right) * d_i\left(t\right) \\
&\quad = \gamma\, D_i e^{-\varsigma\omega_n t} \left[-\kappa_{1i} \cos\left(\omega_d\, t + \phi + \phi_i\right) + \kappa_{2i} \cos\left(\omega_d\, t + \phi - \phi_i\right)\right. \\
&\qquad\qquad \left. - \kappa_{3i} \sin\left(\omega_d t + \phi + \phi_i\right) + \kappa_{4i} \sin\left(\omega_d\, t + \phi - \phi_i\right)\right] \\
&\quad + \gamma\, D_i e^{-\alpha_i t} \left[\left(\kappa_{1i} - \kappa_{2i}\right) \cos\left(\omega_i t + \phi_i\right) + \left(\kappa_{3i} + \kappa_{4i}\right) \sin\left(\omega_i t + \phi_i\right)\right].
\end{aligned} \tag{32}$$

Define now the following:

$$\delta_{1i} = \sqrt{\kappa_{1i}^2 + \kappa_{3i}^2}\,, \tag{33}$$

$$\delta_{2i} = \sqrt{\kappa_{2i}^2 + \kappa_{4i}^2}\,, \tag{34}$$

$$\delta_{3i} = \sqrt{\left(\kappa_{1i} - \kappa_{2i}\right)^2 + \left(\kappa_{3i} + \kappa_{4i}\right)^2}\,, \tag{35}$$

$$\theta_{1i} = \tan^{-1} \frac{\kappa_{1i}}{\kappa_{3i}}, \tag{36}$$

$$\theta_{2i} = \tan^{-1} \frac{\kappa_{2i}}{\kappa_{4i}}, \tag{37}$$

and

$$\theta_{3i} = \tan^{-1}\left(\frac{\kappa_{1i} - \kappa_{2i}}{\kappa_{3i} + \kappa_{4i}}\right). \tag{38}$$

Equation (32) can now be expressed, with the help of Equation (15), in the following simplified form:

$$\begin{aligned}
&\gamma\, e^{-\varsigma\omega_n t} \sin\left(\omega_d t + \phi\right) * d_i\left(t\right) \\
&\quad = \gamma\, D_i e^{-\varsigma\omega_n t} \left[\delta_{1i} \sin\left(\omega_d\, t + \phi + \phi_i + \theta_{1i}\right) + \delta_{2i} \sin\left(\omega_d t + \phi - \phi_i + \theta_{2i}\right)\right] \\
&\quad + \gamma\, D_i e^{-\alpha_i t} \left[\delta_{3i} \sin\left(\omega_i t + \phi_i + \theta_{3i}\right)\right].
\end{aligned} \tag{39}$$

Simplified expressions for the various parameters δ_{ij} and θ_{ij}, appearing in Equation (39), can be obtained by substituting from Equations (25, (26), and (28) through (31) into Equations (33) through (38). The results are as follows:

$$\delta_{1i} = \frac{1}{2\sqrt{\mu_i^2 + \nu_i^2}}, \tag{40}$$

$$\delta_{2i} = \frac{1}{2\sqrt{\mu_i^2 + \rho_i^2}}, \tag{41}$$

$$\delta_{3i} = 4\omega_d \delta_{1i} \delta_{2i}, \tag{42}$$

$$\theta_{1i} = \tan^{-1}\left(\frac{\mu_i}{\nu_i}\right), \tag{43}$$

$$\theta_{2i} = \tan^{-1}\left(\frac{\mu_i}{\rho_i}\right), \tag{44}$$

and

$$\theta_{3i} = \tan^{-1}\left(\frac{2\mu_i \omega_i}{\omega_d^2 + \mu_i^2 - \omega_i^2}\right); \tag{45}$$

where γ, μ_i, ν_i, and ρ_i are given by Equations (20), (24), (25), and (26), respectively.

4.4 Position response $x(t)$ to modally componentiated $d(t)$

Substitution from Equation (4) into Equation (21), followed by application of Equation (39), yields the following equation for the SDOF MSD position $x(t)$:

$$\begin{aligned} x(t) = {} & \beta e^{-\varsigma\omega_n t} \sin(\omega_d t + \psi) \\ & + \gamma \sum_{i=1}^{\nu} D_i \Big\{ e^{-\varsigma\omega_n t} \left[\delta_{1i} \sin(\omega_d t + \phi + \phi_i + \theta_{1i}) + \delta_{2i} \sin(\omega_d t + \phi - \phi_i + \theta_{2i})\right] \\ & \qquad + e^{-\alpha_i t} \left[\delta_{3i} \sin(\omega_i t + \phi_i + \theta_{3i})\right] \Big\}, \end{aligned} \tag{46}$$

Equation (46) describes the response $x(t)$ to the total (composite) induced displacement given by Equations (4) and (5).

4.5 Relative displacement $\delta(t)$, for modally componentiated $d(t)$

Substitution from Equations (4), (5), and (6) into Equation (46), yields the following expression for relative displacement, of the SDOF MSD system mass:

$$\begin{aligned} \delta(t) = {} & \beta e^{-\varsigma\omega_n t} \sin(\omega_d t + \psi) \\ & + \gamma \sum_{i=1}^{\nu} D_i \Big\{ e^{-\varsigma\omega_n t} \left[\delta_{1i} \sin(\omega_d t + \phi + \phi_i + \theta_{1i}) + \delta_{2i} \sin(\omega_d t + \phi - \phi_i + \theta_{2i})\right] \\ & \qquad + e^{-\alpha_i t} \left[\delta_{3i} \sin(\omega_i t + \phi_i + \theta_{3i}) - \sin(\omega_i t + \phi_i)/\gamma\right] \Big\}. \end{aligned} \tag{47}$$

4.6 Velocity response $\dot{x}(t)$ to modally componentiated $d(t)$

Differentiation of Equation (46) yields the following expression for the velocity response $\dot{x}(t)$ to the induced displacement $d(t)$:

$$\begin{aligned}\dot{x}(t) = &-\varsigma\omega_n\beta e^{-\varsigma\omega_n t}\sin(\omega_d t+\psi)+\omega_d\beta e^{-\varsigma\omega_n t}\cos(\omega_d t+\psi)\\ &+\gamma\sum_{i=1}^{\nu}D_i\Big\{-\varsigma\omega_n e^{-\varsigma\omega_n t}\left[\delta_{1i}\sin(\omega_d t+\phi+\phi_i+\theta_{1i})+\delta_{2i}\sin(\omega_d t+\phi-\phi_i+\theta_{2i})\right]\\ &\qquad+\omega_d e^{-\varsigma\omega_n t}\left[\delta_{1i}\cos(\omega_d t+\phi+\phi_i+\theta_{1i})+\delta_{2i}\cos(\omega_d t+\phi-\phi_i+\theta_{2i})\right]\\ &\qquad-\alpha_i e^{-\alpha_i t}\delta_{3i}\sin(\omega_i t+\phi_i+\theta_{3i})+\omega_i e^{-\alpha_i t}\delta_{3i}\cos(\omega_i t+\phi_i+\theta_{3i})\Big\}. \qquad (48)\end{aligned}$$

4.7 Acceleration response $\ddot{x}(t)$ to modally componentiated $d(t)$

Differentiating Equation (48) with respect to time, and combining terms, yields an equation for the acceleration response $\ddot{x}(t)$:

$$\begin{aligned}\ddot{x}(t) = &\left(\varsigma^2\omega_n^2-\omega_d^2\right)\beta e^{-\varsigma\omega_n t}\sin(\omega_d t+\psi)-2\varsigma\omega_n\omega_d\beta e^{-\varsigma\omega_n t}\cos(\omega_d t+\psi)\\ &+\gamma\sum_{i=1}^{\nu}D_i\Big\{\left(\varsigma^2\omega_n^2-\omega_d^2\right)e^{-\varsigma\omega_n t}\left[\delta_{1i}\sin(\omega_d t+\phi+\phi_i+\theta_{1i})+\delta_{2i}\sin(\omega_d t+\phi-\phi_i+\theta_{2i})\right]\\ &\qquad-2\varsigma\omega_n\omega_d e^{-\varsigma\omega_n t}\left[\delta_{1i}\cos(\omega_d t+\phi+\phi_i+\theta_{1i})+\delta_{2i}\cos(\omega_d t+\phi-\phi_i+\theta_{2i})\right]\\ &\qquad+\left(\alpha_i^2-\omega_i^2\right)e^{-\alpha_i t}\delta_{3i}\sin(\omega_i t+\phi_i+\theta_{3i})\\ &\qquad-2\alpha_i\omega_i e^{-\alpha_i t}\delta_{3i}\cos(\omega_i t+\phi_i+\theta_{3i})\Big\}. \qquad (49)\end{aligned}$$

Note that the number of terms in Equation (49) could be reduced by a factor of two, for computer implementation, by use of Equation (15).

4.8 Shock response spectra

Now it is possible to express the various desired shock response spectra, in analytical form. The spectral displacement and spectral velocity are defined, respectively, by Equations (1) and (2), where the relative displacement $\delta(t)$ is given by Equation (47). The spectral acceleration is defined by Equation (3), where the absolute acceleration is given by Equation (49).

5 Solution for the undamped case ($\varsigma = 0$)

5.1 Position response $x(t)$, undamped case

For the special case where $\varsigma = 0$, the position response given by Equation (46) reduces to the following:

$$x(t) = \beta \sin(\omega_n t + \psi) + \omega_n \sum_{i=1}^{\nu} D_i \left\{ \left[\delta_{1i} \sin(\omega_n t + \phi + \phi_i + \theta_{1i}) + \delta_{2i} \sin(\omega_n t + \phi - \phi_i + \theta_{2i}) \right] + e^{-\alpha_i t} \delta_{3i} \sin(\omega_i t + \phi_i + \theta_{3i}) \right\}; \tag{50}$$

where

$$\beta = \sqrt{x^2(0+) + \frac{\dot{x}^2(0+)}{\omega_n^2}}, \tag{51}$$

$$\psi = \tan^{-1} \frac{\omega_n x(0+)}{\dot{x}(0+)}, \tag{52}$$

$$\delta_{1i} = \frac{1}{2\sqrt{\alpha_i^2 + (\omega_n - \omega_i)^2}}, \tag{53}$$

$$\delta_{2i} = \frac{1}{2\sqrt{\alpha_i^2 + (\omega_n + \omega_i)^2}}, \tag{54}$$

$$\delta_{3i} = 4\omega_n \delta_{1i} \delta_{2i}, \tag{55}$$

$$\theta_{1i} = \tan^{-1}\left(\frac{\alpha_i}{\omega_n - \omega_i} \right), \tag{56}$$

$$\theta_{2i} = \tan^{-1}\left(\frac{\alpha_i}{\omega_n + \omega_i} \right), \tag{57}$$

and

$$\theta_{3i} = \tan^{-1}\left(\frac{2\alpha_i \omega_i}{\omega_n^2 + \alpha_i^2 - \omega_i^2} \right). \tag{58}$$

The angle ϕ is zero for $\omega_i < \omega_n$, and π for $\omega_i > \omega_n$; and the response is infinite for $\omega_i = \omega_n$.

5.2 Relative displacement $\delta(t)$, undamped case

With no damping, the relative displacement given by Equation (47) reduces to the following:

$$\begin{aligned}\delta(t) &= \beta \sin(\omega_n t + \psi) \\ &+ \omega_n \sum_{i=1}^{\nu} D_i \{ [\delta_{1i} \sin(\omega_n t + \phi + \phi_i + \theta_{1i}) + \delta_{2i} \sin(\omega_n t + \phi - \phi_i + \theta_{2i})] \\ &\qquad + e^{-\alpha_i t} [\delta_{3i} \sin(\omega_i t + \phi_i + \theta_{3i}) - \sin(\omega_i t + \phi_i)/\omega_n] \}. \end{aligned} \tag{59}$$

where the constants are appropriately simplified, as given in Equations (51) through (58).

5.3 Acceleration response $\ddot{x}(t)$, undamped case

Without damping, the absolute acceleration given by Equation (49) reduces (again, as above, with simplified constants) to the following form:

$$\begin{aligned}\ddot{x}(t) &= -\omega_n^2 \beta \sin(\omega_n t + \psi) \\ &- \omega_n \sum_{i=1}^{\nu} D_i \{ \omega_n^2 [\delta_{1i} \sin(\omega_n t + \phi + \phi_i + \theta_{1i}) + \delta_{2i} \sin(\omega_n t + \phi - \phi_i + \theta_{2i})] \\ &\qquad + (\omega_i^2 - \alpha_i^2) e^{-\alpha_i t} \delta_{3i} \sin(\omega_i t + \phi_i + \theta_{3i}) \\ &\qquad + 2\alpha_i \omega_i e^{-\alpha_i t} \delta_{3i} \cos(\omega_i t + \phi_i + \theta_{3i}) \}. \end{aligned} \tag{60}$$

5.4 An algebraic check of Equations (59) and (60)

In this limiting case, where $\varsigma = 0$, Equation (8) can be expressed by

$$\ddot{x} + \omega_n^2 \delta = 0, \tag{61}$$

where δ is defined by Equation (6). The expressions for δ and $\ddot{x}$ from Equations (59) and (60), respectively, should satisfy Equation (61) identically. Substituting, one obtains

$$
\begin{aligned}
\ddot{x}+\omega_n^2\delta = & -\omega_n^2\,\beta\sin(\omega_n t+\psi) \\
& -\omega_n\sum_{i=1}^{\nu}D_i\Big\{\,\omega_n^2\,[\delta_{1i}\sin(\omega_n\,t+\phi+\phi_i+\theta_{1i})+\delta_{2i}\sin(\omega_n t+\phi-\phi_i+\theta_{2i})] \\
& \qquad +\left(\omega_i^2-\alpha_i^2\right)e^{-\alpha_i t}\delta_{3i}\sin(\,\omega_i t+\phi_i+\theta_{3i}) \\
& \qquad +2\alpha_i\omega_i e^{-\alpha_i t}\delta_{3i}\cos(\,\omega_i t+\phi_i+\theta_{3i})\,\Big\} \\
& +\omega_n^2\beta\sin(\omega_n t+\psi) \\
& +\omega_n\sum_{i=1}^{\nu}D_i\Big\{\omega_n^2\,[\delta_{1i}\sin(\omega_n\,t+\phi+\phi_i+\theta_{1i})+\delta_{2i}\sin(\omega_n t+\phi-\phi_i+\theta_{2i})] \\
& \qquad +\omega_n^2 e^{-\alpha_i t}\delta_{3i}\sin(\,\omega_i t+\phi_i+\theta_{3i})-\omega_n e^{-\alpha_i t}\sin(\omega_i t+\phi_i)\Big\},
\end{aligned}
\tag{62}
$$

which reduces readily to

$$
\begin{aligned}
\ddot{x}+\omega_n^2\delta = -\omega_n\sum_{i=1}^{\nu}D_i\Big\{ & \left(\omega_i^2-\alpha_i^2\right)e^{-\alpha_i t}\delta_{3i}\sin(\,\omega_i t+\phi_i+\theta_{3i}) \\
& +2\alpha_i\omega_i e^{-\alpha_i t}\delta_{3i}\cos(\,\omega_i t+\phi_i+\theta_{3i}) \\
& -\omega_n^2 e^{-\alpha_i t}\delta_{3i}\sin(\,\omega_i t+\phi_i+\theta_{3i})+\omega_n e^{-\alpha_i t}\sin(\omega_i t+\phi_i)\Big\}.
\end{aligned}
\tag{63}
$$

Combination of like trigonometric terms yields

$$
\begin{aligned}
\ddot{x}+\omega_n^2\delta = \omega_n\sum_{i=1}^{\nu}D_i\Big\{ & \left(\omega_n^2-\omega_i^2+\alpha_i^2\right)e^{-\alpha_i t}\delta_{3i}\sin(\,\omega_i t+\phi_i+\theta_{3i}) \\
& -2\alpha_i\omega_i e^{-\alpha_i t}\delta_{3i}\cos(\,\omega_i t+\phi_i+\theta_{3i})-\omega_n e^{-\alpha_i t}\sin(\omega_i t+\phi_i)\Big\}.
\end{aligned}
\tag{64}
$$

Further expansion of terms yields the following:

$$
\begin{aligned}
\ddot{x}+\omega_n^2\delta = \omega_n\sum_{i=1}^{\nu}D_i\;e^{-\alpha_i t}\Big\{ & \left(\omega_n^2-\omega_i^2+\alpha_i^2\right)[\delta_{3i}\cos\theta_{3i}\sin(\omega_i t+\phi_i) \\
& \qquad +\delta_{3i}\sin\theta_{3i}\cos(\omega_i t+\phi_i)] \\
& -2\alpha_i\omega_i[\delta_{3i}\cos\theta_{3i}\cos(\omega_i t+\phi_i) \\
& \qquad -\delta_{3i}\sin\theta_{3i}\sin(\omega_i t+\phi_i)] \\
& -\omega_n\sin(\omega_i t+\phi_i)\Big\}.
\end{aligned}
\tag{65}
$$

Combination of like trigonometric terms yields

$$\ddot{x} + \omega_n^2 \delta = \omega_n \sum_{i=1}^{\nu} D_i \; e^{-\alpha_i t} \Big\{ \sin\left(\omega_i t + \phi_i\right) \Big[\delta_{3i} \cos\theta_{3i} \left(\omega_n^2 - \omega_i^2 + \alpha_i^2\right)$$

$$+ \delta_{3i} \sin\theta_{3i} \left(2\alpha_i \omega_i\right) - \omega_n \Big]$$

$$+ \cos\left(\omega_i t + \phi_i\right) \Big[\delta_{3i} \sin\theta_{3i} \left(\omega_n^2 - \omega_i^2 + \alpha_i^2\right)$$

$$- \delta_{3i} \cos\theta_{3i} \left(2\alpha_i \omega_i\right) \Big] \Big\}. \tag{66}$$

In order for Equation (61) to hold nontrivially, it is necessary for the square-bracketed terms in Equation (66) to be zero, for all indices i. In particular, it is necessary that

$$\left[\delta_{3i} \cos\theta_{3i} \left(\omega_n^2 - \omega_i^2 + \alpha_i^2\right) + \delta_{3i} \sin\theta_{3i} \left(2\alpha_i \omega_i\right) - \omega_n \right] = 0, \tag{67}$$

and that

$$\left[\delta_{3i} \sin\theta_{3i} \left(\omega_n^2 - \omega_i^2 + \alpha_i^2\right) - \delta_{3i} \cos\theta_{3i} \left(2\alpha_i \omega_i\right) \right] = 0. \tag{68}$$

From Equation (58),

$$\tan\theta_{3i} = \frac{2\alpha_i \omega_i}{\omega_n^2 + \alpha_i^2 - \omega_i^2}. \tag{69}$$

Consequently,

$$\sin\theta_{3i} = \frac{2\alpha_i \omega_i}{\sqrt{\left(\omega_n^2 + \alpha_i^2 - \omega_i^2\right)^2 + \left(2\alpha_i \omega_i\right)^2}}, \tag{70}$$

and

$$\cos\theta_{3i} = \frac{\omega_n^2 + \alpha_i^2 - \omega_i^2}{\sqrt{\left(\omega_n^2 + \alpha_i^2 - \omega_i^2\right)^2 + \left(2\alpha_i \omega_i\right)^2}}. \tag{71}$$

Substitution from Equations (53) through (55) and (70) and (71), into Equations (67) and (68), proves the latter two equations to express valid identities. This in turn verifies Equation (61), completing the algebraic check of Equations (59) and (60).

6 Conclusion

This paper has presented analytical equations describing the spectral displacement (displacement SRS), the spectral velocity (pseudovelocity SRS), and the spectral acceleration (absolute acceleration SRS), for a proportionally damped, linear system. For such systems an impulsive mechanical-shock disturbance produces a vibratory response expressible analytically in terms of the system modes and modeshapes, using normal-mode orthogonalization. In particular, the response has the form of a linear combination of exponentially decaying sinusoids, of various amplitudes and phase shifts. A response

of such a form can be represented by SRS's for which this paper has provided analytical descriptions. The displacement and pseudovelocity SRS's are defined, respectively, by Equations (1) and (2), where the relative displacement $\delta(t)$ is given analytically by Equation (47). The spectral acceleration is defined by Equation (3), where the absolute acceleration is given analytically by Equation (49).

Having these analytical expressions for the various SRS's permits the SRS's to be computed exactly for impulsive shock disturbances, without necessitating numerical evaluation of the convolution integral. The analytical expressions can be used for other, non-impulsive input loads, even those for which there is no simple analytical description, provided they can be approximated as impulses. The physical system itself serves as a modal filter of the shock input, to produce a vibratory response, known at any desired point on the system in terms of its exponentially decaying sinusoidal components. Each of these components is known in terms of four values: its frequency, its decay rate (time constant), its phase angle, and its amplitude. From these values the desired SRS can be determined by evaluating, at each point of a discretized continuum of frequencies (i.e., those of the conceptualized SDOF SMD systems), the maximum (or minimum) of a time function consisting of simple *algebraic* expressions involving simple *trigonometric* operations. No numerical convolution is required, because the integrations have already been accomplished analytically (and *exactly*, for impulse loading), with the results incorporated into the algebraic expressions. This method can provide for accurate SRS computation irrespective of the input shock's exact shape, provided the input is approximately impulsive. For linear systems the described method can be used as a benchmark to evaluate the accuracy of other methods of SRS determination. It can also be used to determine the minimum number of modes required, in a system's finite-element model, to produce an SRS of specified accuracy.

7 References

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