Laplace Transform Inversion and Viscoelastic Wave Propagation

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Problem: Solve for viscoelastic response to suddenly applied stress.

$$\sigma_x = \rho u_{tt}$$

Equation of motion

$$P(\partial_t)\sigma = Q(\partial_t)\varepsilon$$

Constitutive law

$$\varepsilon = u_x$$

Strain-displacement relationship

$$deg\ P \leq deg\ Q$$

Property of viscoelastic materials

Wave Equations

Stress, velocity, displacement and strain all obey identical PDEs.

$$\boxed{P(\partial_t)\sigma_{tt} = \frac{1}{\rho} Q(\partial_t)\sigma_{xx}}$$

$$P(\partial_t)v_{tt} = \frac{1}{\rho} Q(\partial_t)v_{xx}$$

$$P(\partial_t)u_{tt} = \frac{1}{\rho} Q(\partial_t)u_{xx}$$

$$P(\partial_t)\varepsilon_{tt} = \frac{1}{\rho} Q(\partial_t)\varepsilon_{xx}$$

Solution

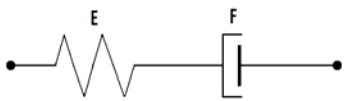
$$\hat{\sigma}(x) = \hat{\sigma}_{app} e^{-\alpha(s)x}$$

where $\alpha^2(s) = \rho s^2 \dfrac{P(s)}{Q(s)}$

Elastic

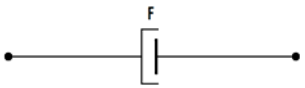
$$\alpha^2 = \frac{\rho s^2}{E}$$

Maxwell



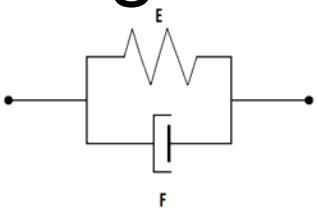
$$\alpha^2 = \rho s^2 \frac{E + Fs}{EFs}$$

Viscous



$$\alpha^2 = \frac{\rho s}{F}$$

Voight



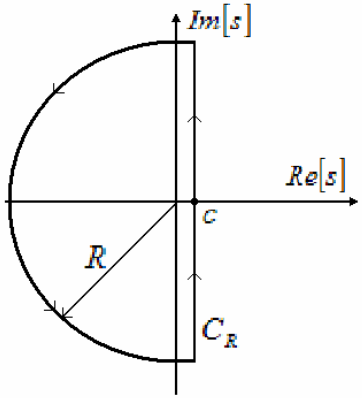
$$\alpha^2 = \frac{\rho s^2}{E + Fs}$$

Inverse Laplace Transform

$$\sigma(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\hat{\sigma}(s)\,ds$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\oint_{C_R} e^{st}\hat{\sigma}(s)\,ds$$

The Bromwich Contour



Not a practical method for many transforms.
Seek reliable, robust numerical algorithms.

Inversion Algorithms

There are **many** inversion algorithms:

MathSciNet (AMS Database)

"Inverse" and "Laplace" and "Transform" in Review Text

762 Articles since 1939

Compendex Database (more applied sciences and eng.)

"Inverse" and "Laplace" and "Transform" in Abstract

870 Articles since 1970

Bellman, Kalaba and Lockett (BKL) Method

A Nystrom Method (Integral Equation Language)

step 1: Approximate forward transform by quadrature.

$$\hat{\sigma}(s) \approx \sum_{i=1}^{N} w_i h_i(s) \sigma_i$$

step 2: Choose N values of transform variable.

$$\hat{\sigma}(s_k) \approx \sum_{i=1}^{N} \left[w_i h_i(s_k) \right] \sigma_i, \quad k = 1, \ldots, N$$

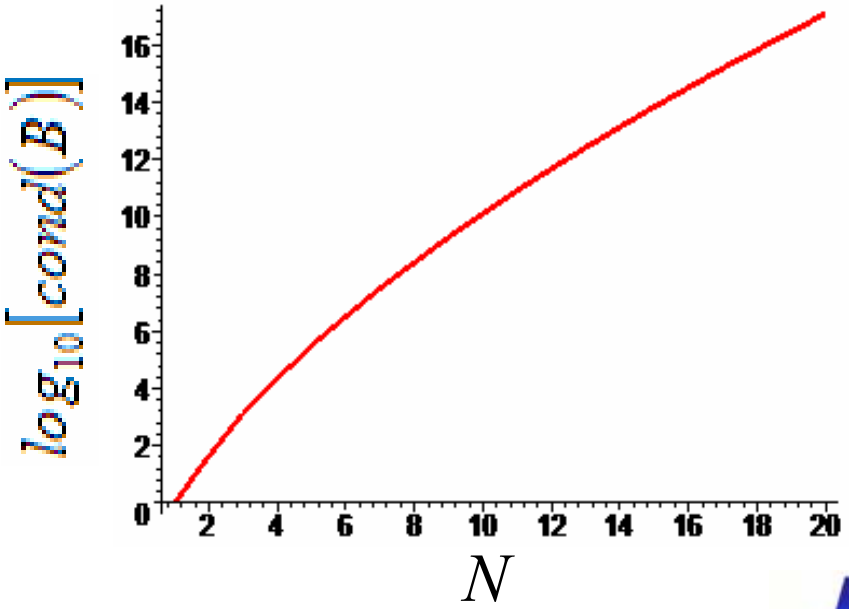
step 3: Solve linear system.

step 4: Invert change of variables.

Notes:

1. Discrete approximation vice continuous
2. Fixed time nodes
3. Poorly conditioned - **cond(B) grows exponentially**

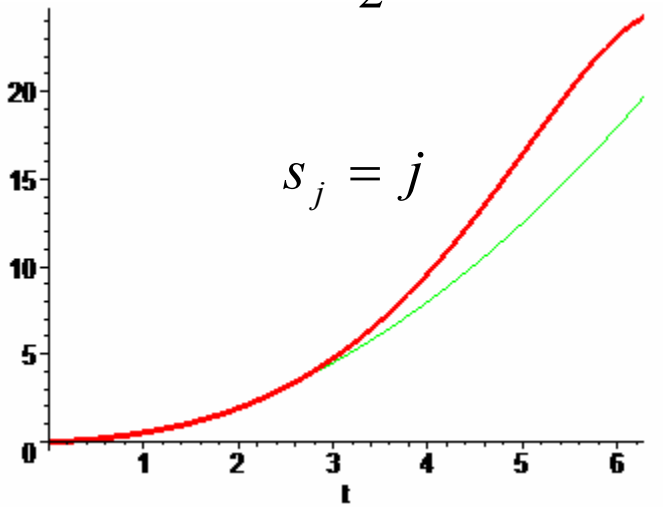
Exponential Growth of condition number for Nystrom method with Midpoint Rule



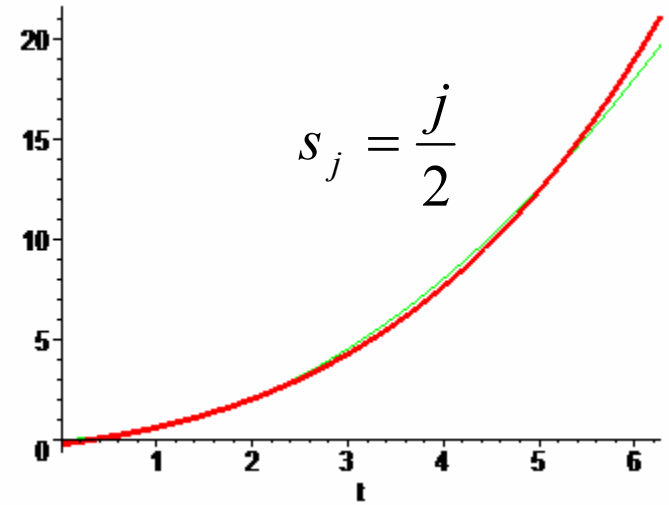
Heaviside Interpolation

step 1: Interpolate transform with rational function.
step 2: Partial fraction decomposition of rational function
step 3: Heaviside theorem to invert

Approximation of $\frac{t^2}{2}$, $N = 5$



Approximation of $\frac{t^2}{2}$, $N = 5$



Jacobi Polynomials

An Orthogonal Polynomial Method

step 1: Find transforms of basis functions.

$$\{\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \ldots\} \rightarrow \{\hat{\varphi}_1(s), \hat{\varphi}_2(s), \ldots, \hat{\varphi}_n(s), \ldots\}$$

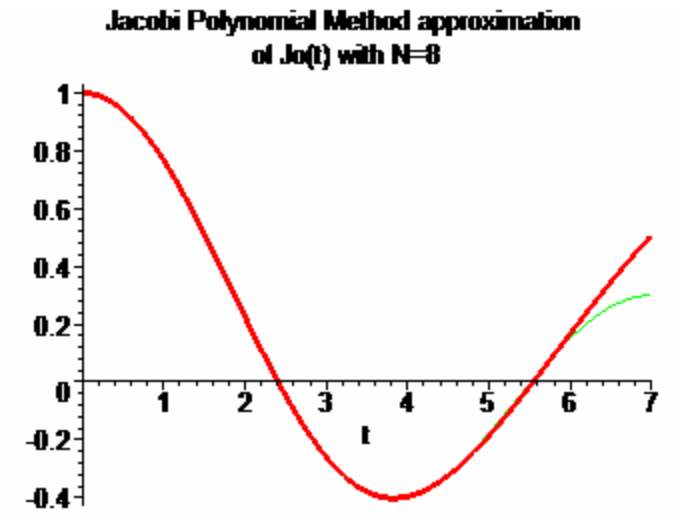
step 2: Approximate unknown transform.

$$\hat{\sigma}(s) \approx \sum_k c_k \hat{\varphi}_k(s)$$

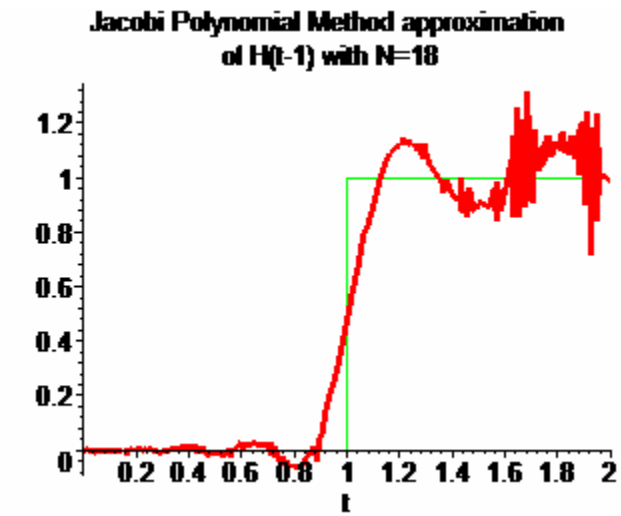
step 3: Time function is the pre-image (may need change of variables).

$$\boxed{\sigma(t) \approx \sum_k c_k \varphi_k(t)}$$

A reasonable approximation of a Bessel function.
Terms can be added to lengthen time interval.



A poor approximation of a step function.
Additional terms make approximation worse.



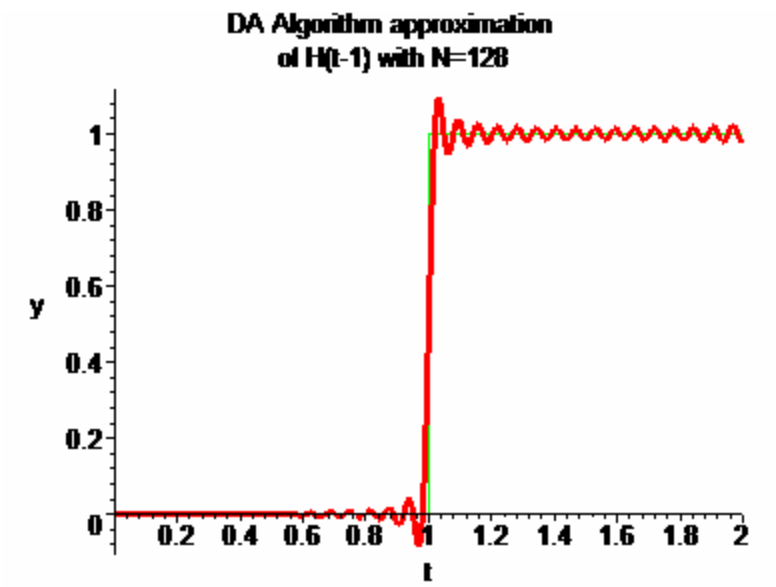
Dubner, Abate and Crump (DAC) Algorithm

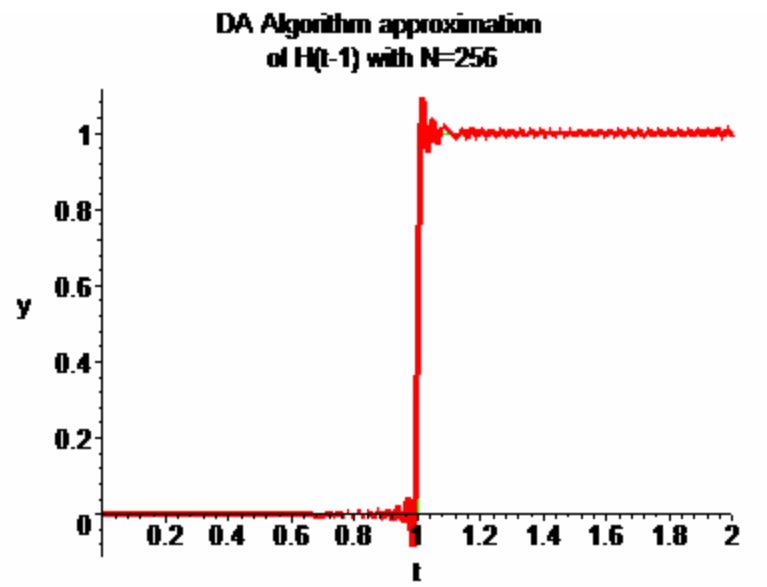
step 1: Apply Trapezoid Rule to Bromwich Integral.

$$\sigma(t) \approx \frac{e^{ct}}{T}\left[\frac{\hat{\sigma}(c)}{2} + \sum_{n=1}^{\infty}\left(Re\left[\hat{\sigma}\left(c + i\frac{n\pi}{T}\right)\right]cos\left(\frac{n\pi}{T}t\right) - Im\left[\hat{\sigma}\left(c + i\frac{n\pi}{T}\right)\right]sin\left(\frac{n\pi}{T}t\right)\right)\right]$$

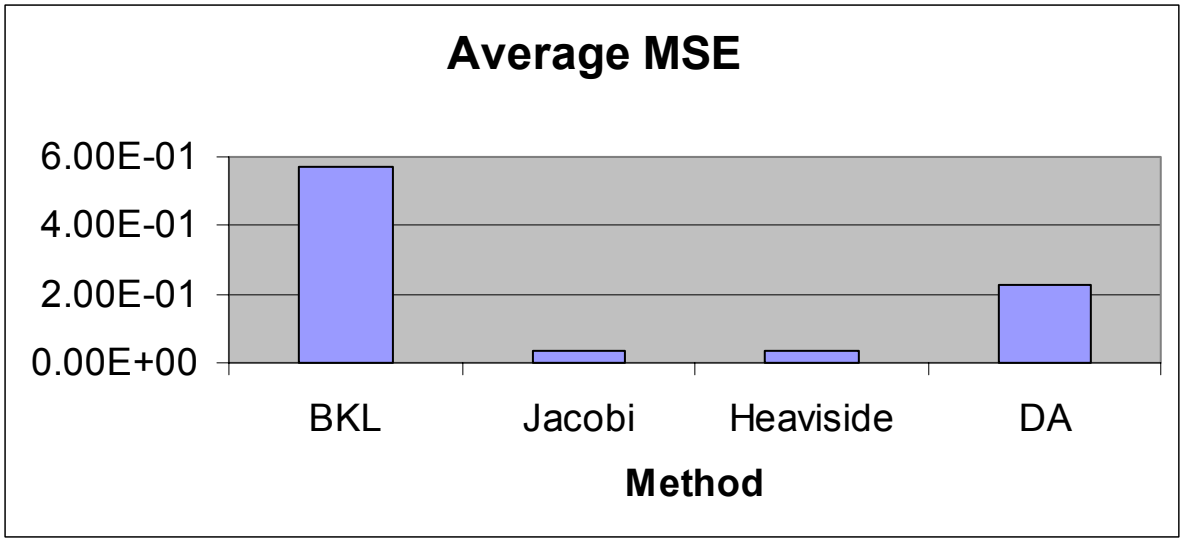
step 2: Truncate resulting series.

Handles discontinuities, but has Gibb's Phenomenon.





General Performance



Notes:

BKL average driven by high order polynomial.
DA average driven by logarithm (singularity at t=0).

Comments

BKL: Robust and consistent (with one exception), limited time domain.

Jacobi: Excellent with polynomials, terrible with discontinuities.

Heaviside: Perfect with complex exponentials (sines, cosines, exp), sensitive to node choice.

DA(C): Fourier Series (can be made faster with FFT), generally good with discontinuities, bad with singularities.

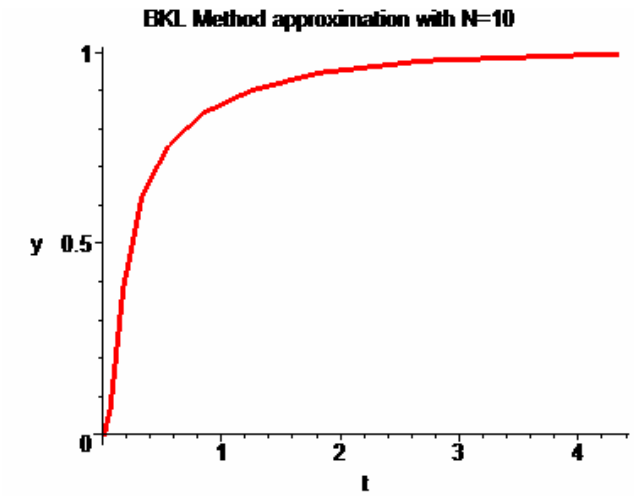
Viscoelastic Waves

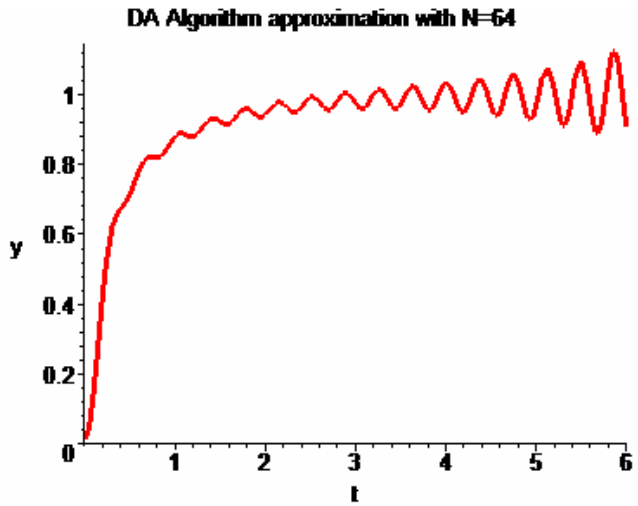
Material	Stress Transform	Nondimensional Stress
Elastic	$\hat{\Sigma}(\xi,s)=\dfrac{e^{-\xi s}}{s}$	$\Sigma(\xi,\tau)=H(\tau-\xi)$
Viscous	$\hat{\Sigma}(\xi,s)=\dfrac{e^{-\xi\sqrt{s}}}{\sqrt{s}}$	$\Sigma(\xi,\tau)=\dfrac{1}{\sqrt{\pi\tau}}e^{-\frac{\xi^2}{4\tau}}$
Maxwell	$\hat{\Sigma}(\xi,s)=\dfrac{e^{-\xi\sqrt{s^2+s}}}{\sqrt{s^2+s}}$	$\Sigma(\xi,\tau)=e^{-\frac{\tau}{2}}I_0\left(\dfrac{1}{2}\sqrt{\tau^2-\xi^2}\right)H(\tau-\xi)$
Voight	$\hat{\Sigma}(\xi,s)=\sqrt{\dfrac{1+s}{s^2}}e^{-\xi\sqrt{\frac{s^2}{1+s}}}$	$\Sigma(\xi,\tau)=\dfrac{e^{\tau}}{\sqrt{\pi}}\left[\dfrac{e^{-\frac{\xi^2}{4\tau}}}{\sqrt{\tau}}-\int_0^{\tau}\dfrac{J_1\left(2\sqrt{\lambda\tau-\lambda^2}\right)}{\sqrt{\tau-\lambda}}e^{-2(\tau-\lambda)-\frac{\xi^2}{4\lambda}}d\lambda\right]$

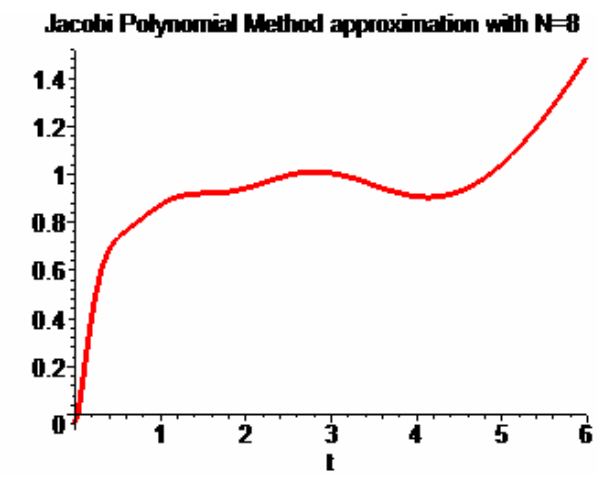
MSE

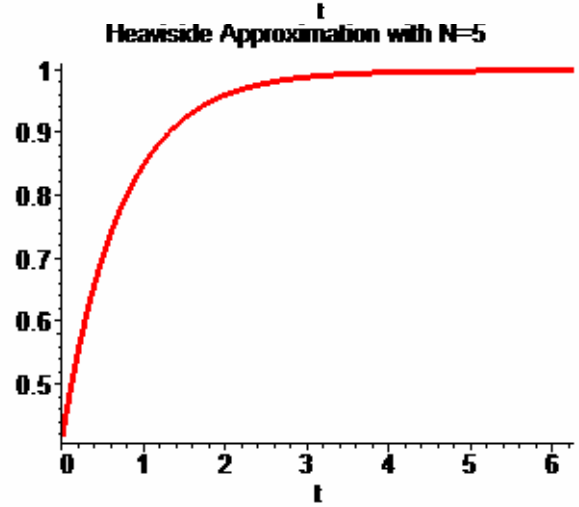
	BKL	Heaviside	Jacobi	DA(C)
Elastic	2.10E-3	5.28E-2	2.24E-2	2.26E-2
Viscous	1.03E-5	3.62E-3	2.33E-2	9.00E-4
Maxwell	4.01E-4	6.16E-2	??	1.42E-3

Voight Approximations









Conclusions

Jacobi: Doesn't work well with viscoelastic functions.

BKL: Works excellent on viscoelastic functions, but needs time extension.

DA(C): Robust enough, but takes many more terms than other methods. Could be useful if sped up (IFFT).

Heaviside: Robust, does well except with discontinuities and too sensitive to node choice.

Will continue with DA(C) and BKL methods.

Questions